# NOTES ON THE DEGREE FORMULA

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The major aim of this text is to provide a proof of Remark 10.4 in [1]. I am indebted to A. Suslin for helpful and encouraging comments.

## 1. The invariants $\rho^i$

Let $p$ be a prime. We work over a ground field $k$ of characteristic different from $p$. Let $G = \mu_p$.

Let $X$ be a smooth variety over $k$, equidimensional of dimension $d$, together with a $G$-action. Let $Y \subset X$ be the fix-point subscheme. We assume that $Y$ is smooth (this is always the case, but I don't know a reference at the moment). Let $U = X \setminus Y$.

Let $\overline{X} = X/G$ be the quotient, let $\overline{U} \subset \overline{X}$ be the image of $U$ and let $\overline{Y} = \overline{X} \setminus \overline{U}$ with the reduced subscheme structure. Certainly $\overline{U}$ is smooth. Since $\operatorname{char} k \neq p$, the projection $\pi\colon X \to \overline{X}$ induces an isomorphism $Y \to \overline{Y}$.

In the following we will identify $\overline{Y}$ with $Y$.

The morphism $\pi' = \pi|U\colon U \to \overline{U}$ is a $\mu_p$-torsor. Let

$$\alpha \in A^0(\overline{U}, K_1/p)$$

be the corresponding unramified $p$-th power class (locally $\pi'$ is presented as $R = \overline{R}[t]/(t^p - f)$ for some regular function $f$, and $\alpha$ is then the $p$-th power class of $f$). Let further $L$ be the line bundle on $\overline{U}$ induced from $\pi'$ via the inclusion $\mu_p \to \mathbf{G}_{\mathrm{m}}$ and let

$$\beta \in \mathrm{CH}^1(\overline{U}) = A^1(\overline{U}, K_1)$$

be the first Chern class of $L$.

We define a series of classes $\rho^i \in \mathrm{CH}_{d-i}(Y)/p$ by

$$\rho^i(X) = \rho^i = \partial_Y^{\overline{U}}(\alpha\beta^{i-1})$$

where

$$\partial_Y^{\overline{U}}\colon A^{i-1}(\overline{U}, K_i/p) \to \mathrm{CH}_{d-i}(Y)/p$$

is the boundary map, cf. [2].

**Proposition 1.** *(Degree formula) Let $\varphi\colon X' \to X$ be a morphism of smooth $G$-varieties of the same dimension with $X/G$ irreducible. Let $Y$, $Y'$ be the $G$-fix-point subschemes and let $\psi = \varphi|Y'\colon Y' \to Y$. Then*

$$\psi_*\big(\rho^i(X')\big) = (\deg\varphi)\rho^i(X)$$

*Proof.* Let $Z = \varphi^{-1}(Y)$. The restriction of $\alpha'$ resp. $\beta'$ to $(X' \setminus Z)/G$ is equal to the pull back of $\alpha$ resp. $\beta$. Therefore

$$\varphi^*\big(\rho^i(X)\big) = i_*\big(\rho^i(X')\big)$$

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where $i\colon Z \to X'$ is the inclusion. Thus

$$\varphi_*\big(\rho^i(X')\big) = \varphi_*\big(\varphi^*\big(\rho^i(X)\big)\big) = (\deg\varphi)\rho^i(X).$$

□

*Remark* 1. It is very likely that one can extend the considerations to not necessarily smooth varieties. More generally one may consider also arbitrary branched coverings $X \to \overline{X}$ of degree $p$ which are not necessarily cyclic. Everything should work in arbitrary characteristic (this means essentially to establish the Steenrod operations in all characteristics) . It would be interesting to understand the situation also for arbitrary finite morphisms $X \to \overline{X}$.

## 2. The invariant $\rho$

Let us assume that $Y$ is proper. We write $I(Y)$ for the image of the degree map $\deg_Y\colon \mathrm{CH}_0(Y) \to \mathbf{Z}$. We have then a degree map

$$\widetilde{\deg}_Y\colon \mathrm{CH}_0(Y)/p \to \mathbf{Z}/pI(Y)$$

We define

$$\rho = \widetilde{\deg}_Y(\rho^d)$$

Note that the image of $\rho^i$ in $\mathrm{CH}_{d-i}(\overline{X})/p$ is trivial. Therefore, if $X$ is proper, then the image of $\rho$ in $\mathbf{Z}/p$ is trivial.

## 3. The invariant $\eta$

In this section we assume that $X$ is proper.

The degree map $\deg_{\overline{X}}$ defines a degree map

$$\deg_{\overline{U}}\colon \mathrm{CH}_0(\overline{U}) = \mathrm{CH}_0(\overline{X})/\mathrm{CH}_0(Y) \to \mathbf{Z}/I(Y)$$

We define

$$\eta = \deg(\beta^d) \in \mathbf{Z}/I(Y).$$

**Lemma 1.** *The image of $\eta$ under the inclusion*

$$\mathbf{Z}/I(Y) \xrightarrow{p} \mathbf{Z}/pI(Y)$$

*is equal to $-\rho$.*

*Proof.* Suppose first that $X$ is a proper smooth curve. Let $f \in k(\overline{X})^*$ be a function representing $\alpha$. Then

$$d(f) = pA + B$$

for a zero cycle $A$ on $\overline{X}$ and a zero cycle $B$ on $\overline{Y}$. Since $\sqrt[p]{f}$ is a section of the line bundle $L$, on has

$$\beta = [A]$$

On the other hand

$$\partial^{\overline{U}}_{\overline{Y}}(\alpha) = [B]$$

by the very definition of $\partial^{\overline{U}}_{\overline{Y}}$. Since

$$0 = \deg_{\overline{X}}\big(d(f)\big) = p\deg_{\overline{X}}(A) + \deg_{\overline{X}}(B)$$

one has

$$p\deg_{\overline{X}}(\beta) = p\deg_{\overline{X}}(A) = -\deg_{\overline{X}}(B) = -\deg_{\overline{Y}}(B) = -\deg_{\overline{Y}}\big(\partial^{\overline{U}}_{\overline{Y}}(\alpha)\big)$$

This proves the claim for $d = 1$.

For the general case one chooses a proper smooth 1-dimensional variety $\overline{C}$ and a morphism $f\colon \overline{C} \to \overline{X}$ such that $\beta^{d-1} \in \mathrm{CH}_1(\overline{U})$ is represented by the image of the fundamental cycle of $\overline{C}$. This way one easily reduces the problem to the case of a curve. □

## 4. Relations with $\eta_p$

Let $Y$ be a smooth proper equidimensional variety, let $X = Y^p$ and let $G = \mathbf{Z}/p = \mu_p$ act on $X$ by cyclic permutations. We then have the invariant

$$\eta \in \mathbf{Z}/I(Y)$$

as above. This class is denoted by $\eta_p$ in [1].

The computation of $\eta_p$ stated in Remark 10.4 in [1] will follow from Lemma 1 and the computations of the classes $\rho^i$ below. The technical advantage of the more general setting of the classes $\rho^i$ is that one does not need to assume that $X$ is proper.

## 5. Reduction to the case of $G$-vector bundles

**Lemma 2.** *Let $V$ be the normal bundle of $Y$ in $X$. Then*

$$\rho^i(X) = \rho^i(V)$$

*Proof.* This follows easily by applying our construction to the deformation to the normal cone. To be a bit more specific, let $M \to \mathbf{A}^1$ be the deformation to the normal cone of $Y$ in $X$ with fibre $V$ over $0 \in \mathbf{A}^1$ and fibre $X$ otherwise. Consider the class

$$\rho^i(M) \in \mathrm{CH}_{d+1-i}(Y \times \mathbf{A}^1)/p$$

It specializes to $\rho^i(V)$ at $Y \times \{0\}$ and to $\rho^i(X)$ at $Y \times \{1\}$. □

## 6. Preliminaries on Chern classes

Let $R = \mathbf{Z}/p[[c_1, c_2, \ldots]]$ be the completed ring of mod $p$ Chern classes.

For each $r \in \mathbf{Z}/p$, $r \neq 0$ define the series

$$b^{[r]} = 1 + b_1^{[r]} + b_2^{[r]} + \cdots \in R$$

as follows. Write formally

$$1 + c_1 + \cdots + c_n = \prod_{i=1}^{n} (1 + x_i)$$

and then define

$$b^{[r]} = \prod_{i=1}^{n} (1 + r^{-1} x_i) \quad \text{mod terms of degree} > n$$

*Remark* 2. Note that $b^{[1]} = 1 + c_1 + c_2 + \cdots$.

*Remark* 3. Put

$$b = \prod_{r} b^{[r]}$$

Since

$$b = \prod_{i=1}^{n} (1 + x_i^{p-1})^{-1} \quad \text{mod terms of degree} > n$$

the series $b$ lies in the subring of $R$ of elements of degree divisible by $p-1$. Thus $b$ is of the form
$$b = 1 + b_{p-1} + b_{2(p-1)} + \cdots$$
with $b_n$ of degree $n$. The classes $b_{k(p-1)}$ are closely related to the Steenrod powers $P^{2k(p-1)}$.

We have to consider also integral representatives of the $b^{[r]}$. For each $r \in \mathbf{Z}/p$, $r \neq 0$ choose an integer $r'$ with $r'r = 1 \mod p$. Then define
$$\tilde{b}^{[r]} \in \mathbf{Z}[[c_1, c_2, \ldots]]$$
as follows. Write formally
$$1 + c_1 + \cdots + c_n = \prod_{i=1}^{n} (1 + x_i)$$
and then define
$$\tilde{b}^{[r]} = \prod_{i=1}^{n} (1 + r'x_i) \quad \text{mod terms of degree} > n$$

## 7. The fundamental identity for the class $\beta$

Let $X = V$ be a $G$-vector bundle of rank $e$ over a smooth variety $Y$. We assume that $V^G = 0$ and identify $Y$ with the zero section. Write
$$V = \bigoplus_r V_r$$
for the decomposition of $V$ into $G$-eigenspaces with $G$ acting on $V_r$ by $(\zeta, v) \mapsto \zeta^r v$ with $r \in \mathbf{Z}/p$, $r \neq 0$. Let $q \colon \overline{U} = (V \setminus 0)/G \to Y$ be the projection.

Note that the class
$$\beta \in \mathrm{CH}^1(\overline{U})$$
is $p$-torsion, since the line bundle $L$ becomes trivial after pull back to $U$ and since $U \to \overline{U}$ is of degree $p$.

Put
$$\tilde{b}(V) = \prod_r \tilde{b}^{[r]}(V_r) \in \mathrm{CH}^*(Y)$$
and write
$$\tilde{b}(V) = 1 + \tilde{b}_1(V) + \tilde{b}_2(V) + \cdots$$
with $\tilde{b}_i(V) \in \mathrm{CH}^i(Y)$. One has $\tilde{b}_e(V) = Nc_e(V)$ for some integer $N$ with $N \mod p$ not depending on the choices of the $r'$.

Further let
$$b(V) = q^*\big(\tilde{b}(V)\big) \in \mathrm{CH}^*(\overline{U})$$
and write
$$b(V) = 1 + b_1(V) + b_2(V) + \cdots$$
with $b_i(V) = q^*\big(\tilde{b}_i(V)\big) \in \mathrm{CH}^i(\overline{U})$. Note that the vector bundle $V$ admits a non-vanishing section after pull back to $U$. Since $U \to \overline{U}$ is of degree $p$, the Euler class $q^*\big(c_e(V)\big)$ is $p$-torsion. Therefore the classes
$$\beta^e,\ b_1(V)\beta^{e-1},\ \ldots,\ b_e(V)$$
do not depend on the choices of the $r'$.

**Theorem 1.** *One has*

$$\beta^e + b_1(V)\beta^{e-1} + \cdots + b_e(V) = 0 \tag{1}$$

*in* $\mathrm{CH}^e(\overline{U})$.

*Proof.* Consider first the case $V = V_1$. Let $\overline{U} \to \mathbf{P}(V) = (V \setminus 0)/\mathbf{G}_\mathrm{m}$ be the projection. Then $\beta \in \mathrm{CH}^1(\overline{U})$ is the pull back of the first Chern class $\xi$ of the tautological line bundle on $\mathbf{P}(V)$ (up to a sign, about which I don't care in this exposition). Equation (1) follows then from the standard identity for $\xi$.

Consider next the case $V = V_r$. This case follows similarly as for $r = 1$. Let $\overline{U} \to \mathbf{P}(V) = (V \setminus 0)/\mathbf{G}_\mathrm{m}$ be the projection. Then $\beta \in \mathrm{CH}^1(\overline{U})$ is $r$-times the pull back of the first Chern class $\xi$ of the tautological line bundle on $\mathbf{P}(V)$. Equation (1) follows then again from the standard identity for $\xi$.

For the general case one uses a standard Mayer-Vietoris argument. Write

$$\tilde{b}_V(t) = t^e + \tilde{b}_1(V)t^{e-1} + \cdots + \tilde{b}_e(V) \in \mathrm{CH}^*(Y)[t]$$

and put

$$b_V(t) = q^*(\tilde{b}_V)(t) \in \mathrm{CH}^*(\overline{U})[t]$$

We further write $\beta = \beta_V$ to indicate the dependence on the bundle $V$.

By induction we may suppose that there exists a decomposition

$$V = W_1 \oplus W_2$$

of $G$-vector bundles such that equation (1) holds for the bundles $W_1$, $W_2$. One has

$$\tilde{b}_V(t) = \tilde{b}_{W_1}(t)\tilde{b}_{W_2}(t)$$

Consider the exact sequence

$$\mathrm{CH}^*\big((W_1 \setminus 0)/G\big) \xrightarrow{i_*} \mathrm{CH}^*\big((V \setminus 0)/G\big) \xrightarrow{j_*} \mathrm{CH}^*\big((V \setminus W_1)/G\big) \to 0$$

of Chow groups induced from the closed immersion $i\colon (W_1 \setminus 0)/G \to (V \setminus 0)/G$ and the open immersion $j\colon (V \setminus W_1)/G \to (V \setminus 0)/G$ of its complement. The projection $s\colon (V \setminus W_1)/G \to (W_2 \setminus 0)/G$ is a vector bundle. One has

$$j^*(\beta_V) = s^*(\beta_{W_2})$$

Hence

$$j^*\big(q^*\big(\tilde{b}_{W_2}\big)(\beta_V)\big) = s^*\big(b_{W_2}(\beta_{W_2})\big) = 0$$

Therefore there exist $T \in \mathrm{CH}^0\big((W_1 \setminus 0)/G\big)$ with

$$i_*(T) = q^*\big(\tilde{b}_{W_2}\big)(\beta_V)$$

Thus

$$b_V(\beta_V) = q^*\big(\tilde{b}_{W_1}\big)(\beta_V)q^*\big(\tilde{b}_{W_2}\big)(\beta_V) = q^*\big(\tilde{b}_{W_1}\big)(\beta_V)i_*(T) = i_*\big(i^*\big(q^*\big(\tilde{b}_{W_1}\big)(\beta_V)\big)T\big)$$

But $i^*(\beta_V) = \beta_{W_1}$ and therefore $i^*\big(q^*\big(\tilde{b}_{W_1}\big)(\beta_V)\big) = b_{W_1}(\beta_{W_1}) = 0$.

□

## 8. Computation of $\rho^i$

We are now ready to proceed similary as in [1, Section 5]. Let $f = \dim Y$, so that $d = \dim X = f + e$.

**Lemma 3.** *One has*

$$\partial(\alpha\beta^h) = \begin{cases} 0 & \text{for } h < e-1 \\ u[Y] & \text{for } h = e-1 \end{cases}$$

*for some* $u \in \mathbf{Z}/p$, $u \neq 0$.

*Remark* 4. The number $u$ can be easily determined from the ranks of the bundles $V_r$. We omit this.

*Proof.* One has $\partial(\alpha\beta^h) \in \mathrm{CH}_{f+e-(h+1)}(Y)$. Therefore $\partial(\alpha\beta^h) = 0$ for $h < e-1$ by dimension reasons. For $h = e-1$ we have $\partial(\alpha\beta^h) \in \mathrm{CH}^0(Y)$. This case follows from the case when $\dim Y = 0$. We may therefore assume that $Y = \operatorname{Spec} k$.

There exists a decomposition

$$V = W_1 \oplus W_2$$

of $G$-vector modules over $k$ such that $\operatorname{rank} W_2 = 1$. Let $r$ be the weight of $W_2$ as a $G$-module and choose $r'$ with $rr' = 1 \mod p$. The line bundle $L$ over $\overline{U} = (V \setminus 0)/G$ is given by

$$\big((W_1 \times W_2) \setminus 0\big) \times_G \mathbf{A}^1 \to \big((W_1 \times W_2) \setminus 0\big)/G$$

It has the section

$$[w_1, w_2] \mapsto [w_1, w_2, s(w_2)^{r'}]$$

where $s$ is a trivialization of the 1-dimensional vector space $W_2$. Hence

$$\beta = c_1(L) = r' i_*([(W_1 \setminus 0)/G])$$

By induction we are reduced to the case $\operatorname{rank} V = 1$. In this case $V = \mathbf{A}^1$ and $\overline{U} = (V \setminus 0)/G \simeq \mathbf{A}^1 \setminus 0$ with $\alpha$ represented by $t^r$ where $t$ is the standard coordinate of $\mathbf{A}^1$ and where $r$ is an integer representing the weight of $V$. Obviously

$$\partial_0^{\overline{U}}(\alpha) = r \mod p$$

and the claim follows. □

By equation (1) we have

$$\beta^{e-1+i} = \sum_{h=0}^{e-1} \beta^{e-1-h} q^*\big(g_{i,h}\big)$$

with $g_{i,h} \in \mathrm{CH}^{i+h}(Y)$. The $g_{i,h}$ are universal polynomials in the Chern classes of the bundles $V_r$. One finds in particular

$$g_{0,0} + g_{1,0} + g_{2,0} + \cdots = \tilde{b}(V)^{-1}$$

One has by Lemma 3

$$\rho^{e+i} = \partial(\alpha\beta^{e+i-1}) = \partial(\alpha\beta^{e-1}) g_{i,0} = u g_{i,0} \mod p$$

Thus we have computed

$$\rho = u \deg(g_{\dim Y,0}) \in \mathbf{Z}/pI(Y)$$

in terms of characteristic numbers of the bundles $V_r$. It is easy to check that this yields the description of $\eta_p$ in Remark 10.4 in [1], perhaps up to a sign.

## References

[1] A. S. Merkurjev, *Degree formula*, notes, May 2000, http://www.math.ohio-state.edu/~rost/chain-lemma.html.
[2] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).

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