# NOTES ON CUBIC EQUATIONS

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For a cubic element (or a triangle) we define its "basic line". It leads to the normalization (4) of cubic equations. The method works in all characteristics for generic cubic elements. We describe for this 1-parameter family the discriminant extension and the corresponding variant of Cardano's formula.

1. **Basic invariants.** Let $F$ be a field and let $K$ be a cubic extension of $F$. We consider the functions

$$T,\ Q,\ N,\ A,\ B,\ D,\ M,\ f,\ g\colon K \to F, \qquad \delta,\ \varphi\colon K \to K$$

defined as follows: For $x \in K$ the polynomial

$$P_x(r) = r^3 - T(x)r^2 + Q(x)r - N(x)$$

is the characteristic polynomial of $x$. In other words, $T(x)$ is the trace of $x$, $N(x)$ is the norm of $x$, and, for invertible $x$,

$$Q(x) = T(x^{-1})N(x)$$

Moreover

$$\begin{aligned}
\delta(x) &= \frac{\mathrm{d}P_x(r)}{\mathrm{d}r}\bigg|_{r=x} = 3x^2 - 2T(x)x + Q(x)\\
\varphi(x) &= 3x - T(x)\\
\Delta(x) &= N\big(\varphi(x)^2 - 4\delta(x)\big)\\
A(x) &= N\big(\varphi(x)\big)\\
B &= \frac{\Delta - A^2}{4}\\
D(x) &= T\big(\delta(x)\big)\\
M &= QT - 9N\\
f &= \frac{TD - M}{D}\\
g &= -\frac{A}{D}
\end{aligned}$$

One finds

$$\begin{aligned}
D &= T^2 - 3Q\\
A &= -(3M - 2TD)\\
3f + g &= T\\
2f + g &= \frac{M}{D}
\end{aligned}$$

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The polynomial $\Delta(x)$ is the discriminant of $x$.

*Remark* 1. Suppose $K = F \times F \times F$. Then for $x = (x_0, x_1, x_2)$ one has

$$\begin{aligned} D(x) &= x_0^2 + x_1^2 + x_2^2 - x_0x_1 - x_1x_2 - x_2x_0 \\ &= (x_0 + \zeta x_1 + \zeta^2 x_2)(x_0 + \zeta^2 x_1 + \zeta x_2) \end{aligned}$$

where $\zeta$ is subject to

$$1 + \zeta + \zeta^2 = 0$$

*Remark* 2. Note that $1 + \zeta + \zeta^2 = 0$ means that $\zeta$ is a cube root of unity (and is primitive as long as $\operatorname{char} F \neq 3$). Thus, if $F = \mathbf{C}$ (complex numbers) in Remark 1, then $D(x) = 0$ if and only if the Euclidean triangle $x_0$, $x_1$, $x_2$ is equilateral.

*Remark* 3. Suppose $K = F \times F \times F$. Then for $x = (x_0, x_1, x_2)$ one has

$$A(x) = (2x_0 - x_1 - x_2)(2x_1 - x_2 - x_0)(2x_2 - x_0 - x_1)$$

**Lemma.** *For $x \in K$ with $D(x) \neq 0$ and $a$, $b \in F$ one has*

$$D(ax + b) = a^2 D(x) \tag{1}$$

$$g(ax + b) = ag(x) \tag{2}$$

$$f(ax + b) = af(x) + b \tag{3}$$

*Proof.* Claim (1) is easy to check, for instance using Remark 1, or by using a similar property for $\delta(x)$.

As for (2), note that $T(x) - 3x$ is invariant under translations $x \mapsto x + b$. The same is true for $A(x)$ and also for $D$ by (1). The claim is now clear from $\deg g = 1$.

Claim (3) follows easily from (2) and $3f + g = T$. □

## 2. **The basic line.**

**Definition.** For $x \in K$ with $D(x) \neq 0$ the function

$$\ell_x(s) = f(x) + sg(x), \qquad s \in F$$

is called the *basic line* of $x$.

Clearly one has

$$\ell_{ax+b}(s) = a\ell_x(s) + b$$

*Remark* 4. Let $x_0$, $x_1$, $x_2$ be an Euclidean triangle which is not equilateral. Then $\ell_{(x_0,x_1,x_2)}(s)$ (understanding $K = \mathbf{C} \times \mathbf{C} \times \mathbf{C}$) determines a natural line associated to the triangle.

Apart from the degenerate cases $D(x) = 0$ and $A(x) = 0$, the basic line is defined and non-degenerate. We thus can normalize it by means of an affine transformation $x \mapsto ax + b$, so that $f(x) = 0$ and $g(x) = 1$. This yields the following normalization for the basic parameters:

$$T = 1, \qquad 9N - 4Q + 1 = 0$$

This gives

$$(T, Q, N) = (1, 9t - 2, 4t - 1)$$

with $t \in F$ and we get the following normalized form of a cubic equation

$$x^3 - x^2 + (9t - 2)x - (4t - 1) = 0 \tag{4}$$

*Remark* 5. One finds

$$t = \frac{B}{A^2}$$

The invariant $t$ is the basic modulus for triangles up to affine transformations.

I think it can serve as a toy model for the $j$-invariant for elliptic curves.

*Remark* 6. For $t = 0$ equation (4) becomes $x^3 - x^2 - 2x + 1 = 0$ which has the complex roots $x = 2\cos\frac{\pi}{7}, 2\cos\frac{3\pi}{7}, 2\cos\frac{5\pi}{7}$.

*Remark* 7. After the change of variables

$$y = 3x - 1$$

one gets the family

$$y^3 - 3Dy + D = 0$$

with $D = 7 - 27t$.

3. **The discriminant.** The discriminant of the cubic equation (4) is

$$(1 - 4t)(27t - 7)^2$$

The discriminant algebra is after a normalization simply given by the quadratic equation

$$u^2 - u + t = 0$$

In fact, one may check that the linear fractional transformation

$$\Phi(x) = \frac{ux + 3t - 1}{x + u - 1}$$

defines an $F[u]$-automorphism of $F[u][x]$ of order 3. In particular, $\Phi(x)$ and $\Phi^2(x)$ are the conjugates of $x$.

4. **Cardano's formula.** Explicit solutions of (4) in terms of radicals are given by:

$$\begin{aligned} D &= 7 - 27t \\ w^2 + w + D &= 0 \\ w + \overline{w} &= -1 \\ w\overline{w} &= D \\ \alpha^3 &= w^2\overline{w} \\ \overline{\alpha}^3 &= \overline{w}^2 w \\ \alpha\overline{\alpha} &= D \\ x &= \frac{1 + \alpha + \overline{\alpha}}{3} \end{aligned}$$

This gives

$$x = \frac{1}{6}\left(2 - \sqrt[3]{(1-E)(1+\sqrt{E})} - \sqrt[3]{(1-E)(1-\sqrt{E})}\right)$$

with

$$E = 1 - 4D = 27(4t - 1)$$

Here are further related formulas:

$$X := 3x - 1$$
$$D\left(\frac{1}{X+w}\right) = D\left(\frac{1}{X+\overline{w}}\right) = 0$$
$$N(X+w) = 27(4t-1)w$$
$$Y := \frac{X+w}{X+\overline{w}}$$
$$Y^3 = \frac{w}{\overline{w}}$$

5. **Appendix.** (Added April 2004)

Here are more remarks:

Let

$$\phi_t \colon K \to K$$
$$\phi_t(x) = (1-3t)x + tT(x)$$

Then

$$D(x) = Q(x) - Q\big(\phi_1(x)\big)$$
$$D(x) = \frac{-1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q\big(\phi_t(x)\big)|_{t=0}$$
$$f(x) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}N\big(\phi_t(x)\big)|_{t=0}}{\frac{\mathrm{d}}{\mathrm{d}t}Q\big(\phi_t(x)\big)|_{t=0}}$$
$$f(x) = \frac{N(x) - N\big(\phi_1(x)\big)}{Q(x) - Q\big(\phi_1(x)\big)}$$

This is perhaps helpful to give a more geometric definition of $f$.

Moreover one has

$$f(x_1, x_2, x_3) = x_1 \Rightarrow x_2 = x_3$$

This is perhaps helpful to characterize $f$.

Can one draw any analogies between $x \mapsto f(x)$ and the orthocenter of a Euclidean triangle?

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