COMPUTATION OF SOME ESSENTIAL DIMENSIONS

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Abstract

In these notes we show that the essential dimension (in the sense of [5]) of PGL_4 is equal to 5.

Along the way we discuss (in a rather unsystematic manner) generalities on essential dimension and degree formulas.

Contents

Most of the text dates back to August 2000.

I am thankful to V. Chernousov for comments.

There is now the preprint "Essential p-dimension of $\mathrm{PGL}(p^2)$" by A. Merkurjev (Nov. 2008, http://www.math.uni-bielefeld.de/LAG/man/313.html).

Merkurjev also hinted to a serious gap in the proof of Lemma 11.3. In December 2008 I added Lemma 14.1 and complemented the proofs of Lemma 11.3 and Lemma 12.3.

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1. Notations and conventions

We work over a ground field k. A *k-variety* is a separated scheme of finite type over k. Let F/k be a finitely generated field extension. By a *model* of F/k we understand an irreducible k-variety X together with an isomorphism $k(X) \simeq F$.

From section 6 on we assume all fields to be of characteristic $\neq 2$. From section 11 on we assume all fields to contain a square root of -1.

2. Places

The natural frame work for many of our considerations is the category of fields over k with the k-places as morphisms. In this section we recall some basic notions.

For a valuation v on a field F we use the (mostly standard) notations

$$\mathfrak{m}_v \subset \mathcal{O}_v \subset F,\ \kappa_v = \mathcal{O}_v/\mathfrak{m}_v,\ U_v = \mathcal{O}_v^* \subset F^*,\ U_v^{[1]} = 1 + \mathfrak{m}_v \subset U_v$$

for the valuation ring and its maximal ideal, for the residue field, for the group of units, and for the group of 1-units, respectively. Valuations on F with the same valuation ring will be identified. If k is a subfield of F, then by a valuation on F/k (or by a k-valuation of F) we understand a valuation v with $k \subset \mathcal{O}_v$. We write $\mathcal{V}(F/k)$ for the set of all k-valuations on F. If F/k is a finitely generated field extension, then there is a natural identification

$$\mathcal{V}(F/k) = \varprojlim X$$

where X runs through the proper models of F/k.

Let E, F be field extensions of k. By a *k-place* $\varphi\colon F \rightsquigarrow E$ we understand a pair $(v_\varphi, \alpha_\varphi)$ where v_φ is a valuation on F/k and $\alpha_\varphi\colon \kappa_{v_\varphi} \to E$ is a k-homomorphism. We also use the more geometric notation $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ for k-places and write (v_f, α_f) for the corresponding pair. A k-place $f = (v_f, \alpha_f)$ is given by a (uniquely determined) family of k-morphisms

$$f_X\colon \operatorname{Spec} E \to X$$

with X running through the proper models of F/k and with $f_X = g \circ f_{X'}$ for every morphism $g\colon X' \to X$ of models of F/k. For any X there exist a proper model Y of E such that f_X extends to a (uniquely determined) k-morphism

$$f_{Y,X}\colon Y \to X.$$

Passing to the limits we obtain a map

$$f^*\colon \mathcal{V}(E/k) \to \mathcal{V}(F/k).$$

This map sends a valuation v on E to the composite valuation of the valuations v_f and $v|\kappa_{v_f}$.

Let $d \geq 0$ and let $\operatorname{tr.deg}(E/k) \leq d$, $\operatorname{tr.deg}(F/k) \leq d$. For a place $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ we define its *d-degree* $\deg_d(f)$ by $\deg_d(f) = [E : F]$ if f is an inclusion of fields of transcendece degre d, and put $\deg_d(f) = 0$ otherwise.

3. Picard groups

Let A be an abelian group. For a finitely generated field extension F/k we put

$$\mathbf{P}(F/k, A) = \varinjlim_X \big(\operatorname{Pic}(X) \otimes A\big)$$

where X runs through the proper models of F/k. For a k-place $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ the maps $f_{Y,X}$ define a pullback map $f^*\colon \mathbf{P}(F/k, A) \to \mathbf{P}(E/k, A)$.

One has

$$\mathbf{P}\big(k(t)/k, A\big) = \operatorname{Pic}(\mathbf{P}^1) \otimes A = A.$$

Let $d = \operatorname{tr.deg}(F/k)$ and let $u_i \in \mathbf{P}(F/k, \mathbf{Z}/n)$, $i = 1, \ldots, d$. Then we define

$$e(u_1, \ldots, u_d) \in \mathbf{Z}/n$$

as follows. Choose a proper model X of F/k and line bundles L_i on X which represent the u_i and consider the vector bundle $V = L_1 \oplus \cdots \oplus L_d$. Let

$$\varepsilon\colon \operatorname{CH}_d(V) \simeq \operatorname{CH}_0(X) \xrightarrow{\deg} \mathbf{Z}$$

where the first map is given by homotopy invariance and the second map is the degree map for 0-cycles. We put

$$e(u_1, \ldots, u_d) = \varepsilon([\text{zero section}]) \pmod{n}.$$

This number does not depend on the choices made and is multi-linear in the u_i. Reference ??? Probably in [2].

Remark: For smooth X, the numbers $e(u_1, \ldots, u_d)$ are just given by intersecting divisors. This is all we need in the current version of this text, where we make free use of resolution of singularities. In a future version we plan to work with arbitrary varieties and then it will be necessary to have the numbers $e(u_1, \ldots, u_d)$ also for non-smooth X.

4. Ramification, Specialization, and Essential Dimension

Let $x \in K_1F/n = F^*/(F^*)^n$.

If v is a valuation on F, we say that x is *unramified* in v if x is in the subgroup $U_v/(U_v)^n \subset F^*/(F^*)^n$. In this case we define the *specialization*

$$x(v) \in K_1\kappa_v/n$$

of x in v as the image of x under $U_v/(U_v)^n \to \kappa_v^*/(\kappa_v^*)^n$.

If $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ is a place, we say that x is unramified in f, if x is unramified in v_f. In this case we put

$$f^*(x) = (\alpha_f)_*\big(x(v_f)\big) \in K_1E/n.$$

We extend these standard considerations to the Milnor K-ring. If v is a valuation on F we define its Milnor K-ring by

$$K_*^M(v) = K_*^M F \;/\; (1 + \mathfrak{m}_v) \cdot K_*^M F.$$

In the case of discrete valuations of rank 1 this ring has been considered in [1], [3, remark at the end of p. 323], [6]. In any case there is a natural injection

$$K_*^M\kappa_v \to K_*^M(v),$$
$$\{\bar u_1, \ldots, \bar u_n\} \mapsto \overline{\{u_1, \ldots, u_n\}}.$$

Let A be an abelian group. For $x \in (K_*^M F) \otimes A$ we denote by $x(v)$ its image in $K_*^M(v)$. If $x(v)$ belongs to the subgroup $K_*^M\kappa_v$, we say that x is unramified in v and call $x(v)$ its specialization. These notions extend to places $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ in an obvious way.

In the following we consider various covariant functors $F \mapsto M(F)$ from the category of fields F/k to sets. These functors will be subfunctors of $F \mapsto (K_*^M F) \otimes A$

for an appropriate abelian group A. They have the following property: If $x \in M(F)$ is unramified in v (as an element of $(K_*^M F) \otimes A)$, then $x(v)$ is in $M(\kappa_v)$. A pair (F, x) with $x \in M(F)$ is called *versal* for M, if for any E/k and $y \in M(E)$ there exists a place $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ such that x is unramified in f and $y = f^*(x)$. (This definition is tentative.) The *essential dimension* of M is the minimum of the transcendence degrees $\operatorname{tr.deg}(F/k)$ for versal pairs (F, x).

Later we consider also functors of the form $M(F) = H^1(F, G)$ where G is linear algebraic group over k. For the notion of essential dimension of these functors, see [5].

5. Side remarks

The material of this section will not be used in later sections.

Problem. For a linear algebraic group G over k let $M_G(F) = H^1(F, G)$. Give a neat definition of $M_G(v)$, in analogy with $K_*^M(v)$. Describe $M_G(v)$ using Bruhat-Tits theory.

Here is a further type of functors for which the notion of essential dimension is meaningful (these will not be considered later). Let $u \in K_n^M k/p$ and define $M_u(F) \subset \{*\}$ to be nonempty if and only if $u_F = 0$. In this case $\operatorname{ed}(M_u)$ should be defined as the minimal transcendence degree of a generic splitting field of u. Recent considerations show that for a nontrivial symbol u one may expect $\operatorname{ed}(M_u) = p^n - 1$. This can be proven for $p = 2$ or $n \leq 3$. In general one does not even know whether $\operatorname{ed}(M_u) < \infty$.

I don't know a good definition of functors on fields which is appropriate for the notion of essential dimension and covers all known examples. One feature appearing in all examples is the existence of a pair of morphisms $X_1 \rightrightarrows X_0$ such that the set of all F-rational points $X_0(F)$ parametrizes all elements of $M(F)$ (let's say by a function $x \mapsto \alpha(x)$) and such that if $\alpha(x) = \alpha(x')$ then there exist $y \in X_1(F)$ mapping to (x, x'). Moreover, for any $z \in M(F)$ and any open subset $U \subset X_0$ one may find $x \in U(F)$ with $\alpha(x) = z$.

In some cases one can compute essential dimensions by ramification methods. For instance, one concludes $\operatorname{ed}(\mathrm{PGL}_2) \geq 2$ from the fact that the quaternion algebra $Q(s, t)$ over $k((s))((t))$ is doubly ramified. One may try to define a notion of "essential valuation dimension" of M related to ramifications over complete valuation rings. Here is a tentative definition. Let F/k be a field extension, let $F_n = F((t_1)) \cdots ((t_n))$, and let v_n be the valuation of F_n/F. Let us say that $x \in M(F_n)$ is totally ramified, if for any subfield $F \subset E \subset F_n$ such that x is in the image of $M(E) \to M(F_n)$, the rank of $v_n|E$ is n. Let us define $\operatorname{evd}(M)$ as the maximal n for which there exist F and a totally ramified element $x \in M(F_n)$.

Certainly one has $\operatorname{evd} \leq \operatorname{ed}$. Here is an example with $\operatorname{evd}(M) < \operatorname{ed}(M)$ (without proof): Let p be a prime with $\operatorname{char} k \neq p$, let l/k be a field extension of degree p and let

$$M(F) = N_{F \otimes l/F}\big(K_1(F \otimes l)/p\big) \subset K_1 F/p$$

be the "group of norms from l/k in K_1/p". One finds $\operatorname{ed}(M) = p - 1$ and $\operatorname{evd}(M) = 1$.

Other computations are $\operatorname{evd}(\mathrm{PGL}_2) = 2$ and, at least if $\operatorname{char} k \neq 2$ and -1 is a square, $\operatorname{evd}(\mathrm{PGL}_4) = 4$.

Problem. Give a neat definition of "essential valuation dimension" (or whatever you want to name it).

6. The class Θ

For a field F/k let

$$M_0(F) = \{ (x_1, x_2) \in K_1F/2 \oplus K_1F/2 \mid x_1x_2 = 0 \}.$$

Thus an element $x = (x_1, x_2)$ of $M_0(F)$ is given by a pair of elements a, $b \in F^*$ such that the quaternion algebra $Q(a, b)$ is split.

Elements in $K_1F/2$ will be denoted by $\{a\}$, $a \in F^*$ and $\{a, b\} \in K_2F/2$ denotes the product of $\{a\}$, $\{b\}$.

Proposition 6.1. *For finitely generated fields F/k and for elements $x \in M_0(F)$ there exist unique elements $\Theta(x) \in \mathbf{P}(F/k, \mathbf{Z}/2)$ such that:*

- *(Functoriality) If $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ is a k-place, and if $x \in M_0(F)$ is unramified in f, then*
$$f^*\big(\Theta(x)\big) = \Theta\big(f^*(x)\big).$$
- *(Normalization) Let $F_u = k(t)$ and $x_u = (\{t\}, \{1-t\})$. Then*
$$\Theta(x_u) \in \mathbf{P}(F_u/k, \mathbf{Z}/2) = \mathbf{Z}/2$$
is the nontrivial element.

We denote the generator of $\mathbf{P}(F_u/k, \mathbf{Z}/2)$ by $[*]$.

Proof of uniqueness of Θ. If F is finite, then $\mathbf{P}(F/k, \mathbf{Z}/2) = 0$. Hence $\Theta(x) = 0$ in this case and we may assume that F is infinite. Then for $x = (\{a\}, \{b\}) \in M_0(F)$ there exist u, $v \in F^*$ with $b = u^2(1 - av^2)$. Let $f\colon \operatorname{Spec} F \rightsquigarrow \operatorname{Spec} F_u$ be the place with $f^*(t) = av^2$. Then x_u is unramified in f and $f^*(x_u) = x$. By functoriality one must have $\Theta(x) = f^*([*])$. □

Along the way have proved that (F_u, k_u) is versal for M_0, at least for infinite k.

Lemma 6.2. *Let X be a smooth proper model of F/k and let*

$$f,\, f'\colon X \to \mathbf{P}^1$$

be morphisms with $f^*(x_u) = f'^*(x_u)$. Then the two maps

$$f^*, f'^*\colon \operatorname{Pic}(\mathbf{P}^1)/2 \to \operatorname{Pic}(X)/2$$

coincide.

Proof. Put $x = (x_1, x_2) = f^*(x_u) = f'^*(x_u)$. For the divisors of the components of x we have

$$\operatorname{div}(x_1) = f^*[0] - f^*[\infty]$$
$$\operatorname{div}(x_2) = f^*[1] - f^*[\infty]$$

in $\bigoplus_{z\in X^{(1)}} \mathbf{Z}/2$ and similarly for f'. Hence

$$f^*[\infty] = \sum_{\substack{z\in X^{(1)} \\ \partial_z(x_1)=\partial_z(x_2)\neq 0}} [z] \;\in\; \bigoplus_{z\in X^{(1)}} \mathbf{Z}/2$$

where $\partial_z\colon K_1F/2 \to K_0\kappa(z)/2$ is the residue map at z. This expresses $f^*[\infty]$ entirely in terms of x, and by the same argument for f' we get $f^*[\infty] = f'^*[\infty]$. □

To prove the existence of the class Θ, we have to show that for any F and $x=(x_1,x_2)\in M_0(F)$ and any two places f, $f'\colon \operatorname{Spec} F \rightsquigarrow \operatorname{Spec} F_u$ with $x=f^*(x_u)=f'^*(x_u)$ one has $f^*([*])=f'^*([*])$. Assuming resolution of singularities, this follows from Lemma 6.2, by extending f, f' to morphisms $X\to \mathbf{P}^1$ on a smooth model of F/k.

I am pretty sure that one can avoid here resolution of singularities by using instead canonical flatening [4]. Anyway, there is a simpler direct way by investigating the possible choices f, f' more closely.

Lemma 6.3. *Let t, $t'\in F^*$ with $t\neq t'$ and assume $\{t\}=\{t'\}$ and $\{1-t\}=\{1-t'\}$ in $K_1F/2$. Then there exist α, $\beta\in F^*$ with $1\neq\alpha^2\neq\beta^2\neq 1$ such that*

$$t=\frac{1-\beta^2}{\alpha^2-\beta^2},\quad t'=\alpha^2\frac{1-\beta^2}{\alpha^2-\beta^2}.$$

Proof. By assumption we have $t'=t\alpha^2$ and $1-t'=(1-t)\beta^2$ for some α, $\beta\in F^*$. Hence $1-t\alpha^2=(1-t)\beta^2$ and the claim is immediate. $\square$

Let $P\to\mathbf{P}^2$ be the blow up in the 4 points $[0,0,1]$, $[0,1,0]$, $[1,0,0]$, and $[1,1,1]$. Let further $\tilde P\to\mathbf{P}^2$ be the blow up in the 7 points $[0,0,1]$, $[0,1,0]$, $[1,0,0]$, and $[\pm1,\pm1,1]$.

Lemma 6.4. *The rational maps*

$$\mathbf{P}^2\xrightarrow{g}\mathbf{P}^2\xrightarrow{h}\mathbf{P}^1\times\mathbf{P}^1,$$
$$g([\alpha,\beta,1])=[\alpha^2,\beta^2,1],$$
$$h([a,b,1])=([1-b,a-b],[a(1-b),a-b])$$

extend to everywhere defined morphisms

$$\tilde P\xrightarrow{\bar g}P\xrightarrow{\bar h}\mathbf{P}^1\times\mathbf{P}^1.$$

Proof. The verification is left to the reader. $\square$

Let π, $\pi'\colon\tilde P\to\mathbf{P}^1$ be given by $\bar h\circ\bar g$ followed by the projections. Note that $\pi^*(x_u)=\pi'^*(x_u)\in M_0\big(k(\tilde P)\big)/2$. By Lemma 6.2 we find that the two maps

$$\pi^*,\pi'^*\colon\ \operatorname{Pic}(\mathbf{P}^1)/2\to\operatorname{Pic}(\tilde P)/2$$

coincide. (Of course one may check this also directly).

Proof of existence of Θ. We have to show that for any F and $x=(x_1,x_2)\in M_0(F)$ and any two places f, $f'\colon\operatorname{Spec}F\rightsquigarrow\operatorname{Spec}F_u$ with $x=f^*(x_u)=f'^*(x_u)$ one has $f^*([*])=f'^*([*])$.

By Lemma 6.3 there exist a morphism $\hat f\colon\operatorname{Spec}F\to\tilde P$ such that $f=\pi\circ\hat f$ and $f'=\pi'\circ\hat f$. The claim follows now from $\pi^*=\pi'^*$ on $\operatorname{Pic}(\mathbf{P}^1)/2$. $\square$

The proof of Proposition 6.1 is now complete. The functoriality of Θ can also be described in the ramified situation:

Lemma 6.5. *If $f\colon\operatorname{Spec}E\rightsquigarrow\operatorname{Spec}F$ is a k-place, and if $x\in M_0(F)$ is ramified in f, then $f^*\big(\Theta(x)\big)=0$.*

Proof. Indeed, let $g\colon\operatorname{Spec}F\rightsquigarrow\operatorname{Spec}F_u$ be a place with $x=g^*(x_u)$. If x is ramified in f, then x_u is ramified in $g\circ f$ and therefore $g\circ f$ must map to one of 0, 1, ∞. But then $(g\circ f)^*([*])=0$. $\square$

The functor M_0 can be described in a more symmetric way as follows. For a field F/k let

$$M'_0(F) = \{\, (x_1, x_2, x_3) \in (K_1F/2)^3 \mid x_1 + x_2 + x_3 = \{-1\}, x_i x_j = 0 \text{ for } i \neq j \,\}.$$

Then each of the projections $M'_0(F) \to M_0(F)$, $(x_1, x_2, x_3) \mapsto (x_i, x_j)$, $i \neq j$, is a bijection. If v is a valuation of rank 1 and if $x = (x_1, x_2, x_3)$ is ramified in v, then exactly one of the x_i is unramified in v and for this component one has $x_i(v) = 0$.

Let $\Sigma(F)$ be the set of all $(x_1, x_2, x_3) \in M'_0(F)$ with $x_i = 0$ for at least one i. If $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ is a k-place, and if $x \in \Sigma(F)$ is unramified in f, then $f^*(x) \in \Sigma(E)$.

These remarks and Lemma 6.5 suggest the following definition. Let $\bar{M}_0(F)$ be the quotient of $M'_0(F)$ by collapsing the set $\Sigma(F)$ to a point (denoted by 0). Define the map

$$f^*\colon \bar{M}_0(F) \to \bar{M}_0(E)$$

on the unramified elements of $M_0(F)$ as before (and passing to the quotient) and sending all other elements to 0. Then we have

Proposition 6.6. *For finitely generated fields F/k and for elements $x \in \bar{M}_0(F)$ there exist unique elements $\bar{\Theta}(x) \in \mathbf{P}(F/k, \mathbf{Z}/2)$ such that:*

- *(Functoriality) If $f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$ is a k-place, then*

$$f^*\bigl(\bar{\Theta}(x)\bigr) = \bar{\Theta}\bigl(f^*(x)\bigr).$$

- *(Normalization) Let $F_u = k(t)$ and $x_u = (\{t\}, \{1 - t\})$. Then*

$$\bar{\Theta}(x_u) \in \mathbf{P}(F_u/k, \mathbf{Z}/2) = \mathbf{Z}/2$$

  is the nontrivial element.

□

7. A side remark

In the later sections we will meet the following construction. Let $x \in K_1F/2$ and let X be a proper smooth model of F/k. We choose a function $a \in F^*$ with $x = \{a\}$ and write

$$\operatorname{div}(a) = A + 2V$$

where A is a divisor with odd multiplicities (the latter means $A \in \bigoplus_{z \in X^{(1)}} (1 + 2\mathbf{Z})$).

At this point we just want give some comments on this situation.

Let $\pi\colon Y \to X$ be the normal closure of X in $F[t]/(t^2 - a)$. Then π is etale of degree 2 outside its locus of ramification Δ. Since A has odd multiplicities, one has $\operatorname{supp}(A) \subset \Delta$. Further, π defines a μ_2-torsor over $X \setminus \Delta$ and therefore a line bundle L on $X \setminus \Delta$ via $\mu_2 \to \mathbf{G}_{\mathrm{m}}$. The class of this line bundle and the class of V in $\operatorname{Pic}(X \setminus \Delta) = \operatorname{CH}^1(X \setminus \Delta)$ coincide.

The situation can be made more clean as follows. Assume that A is a divisor with all multiplicities equal to 1 and that A is a smooth divisor with normal crossings (this can be arranged using resolution of singularities). After blowing up the crossings, we may even assume that A is a smooth subvariety of codimension 1 (with no crossings). Then π is flat and the class of V in $\operatorname{Pic}(X)$ is given by the class of the line bundle $\bar{L} = \pi_*(\mathcal{O}_Y)/\mathcal{O}_X$.

In the following we will often use resolution of singularities in order to talk about the divisor V. Very probably this can be replaced by using flatening theorems [4]. One arranges that π is flat and then works with the line bundle $\bar{L}$ instead of V.

8. The invariant ρ

Let z_1, $z_2 \in K_1k/2$ be fixed elements. We denote by k_1, k_2 the corresponding quadratic extensions of k. Further let $K = k_1 \otimes k_2$ and let k_3 be the third quadratic subextension of K/k. We define $I = I(z_1, z_2) \subset K_0k = \mathbf{Z}$ as the subgroup generated by the norms from the k_i. Thus $I = 2\mathbf{Z}$ if K is a field, and $I = \mathbf{Z}$ otherwise.

For a field F/k let

$$M_1(F) \subset (K_1F/2)^4,$$
$$M_1(F) = \{\, (x_1, y_1, x_2, y_2) \mid x_1x_2 = y_1y_2 = 0, x_i + y_i + z_i = 0 \text{ for } i = 1,\, 2 \,\}.$$

Our aim is to define for fields F/k with $\operatorname{tr.deg}(F/k) \leq 2$ and for $\omega \in M_1(F)$ an invariant $\rho(\omega) \in \mathbf{Z}/I$.

We study a versal parameter space for elements in M_1 in some detail. Let c, $d \in k^*$ with $z_1 = \{c\}$ and $z_2 = \{d\}$.

Let

$$T = T(z_1, z_2) \subset \mathbf{P}^1 \times \mathbf{P}^2,$$
$$T = \{\, ([s,t],[x,y,z]) \mid x^2s - y^2tc - z^2(s-t)d = 0 \,\}$$

Lemma 8.1. *T is a smooth proper irreducible surface. The tupel*

$$\omega_T = (\{t/s\}, \{ct/s\}, \{1 - (t/s)\}, \{d\bigl(1 - (t/s)\bigr)\})$$

is an element of $M_1\bigl(k(T)\bigr)$. For any F/k and any $\omega \in M_1(F)$ there is a k-place $f\colon \operatorname{Spec} F \rightsquigarrow \operatorname{Spec} k(T)$ with $x = f^*(\omega_T)$.

Proof. The verification is left to the reader. □

Let further $\tilde{T} = \tilde{T}(z_1, z_2) \to T$ be the blow up in the 3 points $P_1 = ([1,1],[0,0,1])$, $P_2 = ([1,0],[0,1,0])$, $P_3 = ([0,1],[1,0,0])$. Lemma 8.1 remains valid with T replaced by $\tilde{T}$.

Lemma 8.2. *There exist smooth 1-dimensional closed subvarieties D_1, D_2, $D_3 \subset \tilde{T}$ such that:*

- *There are the following equalities of $\pmod 2$-divisors*

$$\operatorname{div}(\{t/s\}) = D_2 + D_3,$$
$$\operatorname{div}(\{1 - (t/s)\}) = D_1 + D_3.$$

- *There is a k-morphism $D_i \to \operatorname{Spec} k_i$ for $i = 1,\, 2,\, 3$.*
- *The D_i are pairwise disjoint.*
- *For the self intersection number of D_i one has $D_i \cdot D_i \equiv 4 \mod 8$.*

Proof. First compute the divisors of $\{t/s\}$ and $\{1 - (t/s)\}$ on T. Consider the three divisors

$$\bar{D}_2 = \{t = 0\},$$
$$\bar{D}_3 = \{s = 0\},$$
$$\bar{D}_1 = \{t = s\}.$$

One has

$$\operatorname{div}_T(\{t/s\}) = \bar{D}_2 - \bar{D}_3,$$
$$\operatorname{div}_T(\{1 - (t/s)\}) = \bar{D}_1 - \bar{D}_3.$$

Each of the divisors $\bar{D}_i$ consists geometrically of two lines. Their intersection consists of one point P_i at which they meet transversally. The two lines of $\bar{D}_i$ are defined over k_i and permuted by the Galois action of k_i/k. Let $D_i \subset \tilde{T}$ be the proper transforms of the $\bar{D}_i$. After the blow up, the two lines will be separated, and the D_i are smooth. The preimage of $\bar{D}_i$ under the blow up is $D_i + 2E_i$ where E_i is the exceptional fiber over the intersection point P_i. To compute the self intersection number of D_i, note first that $\bar{D}_i \cdot \bar{D}_i = 0$, since $\bar{D}_i$ is the preimage of a point under the projection $T \to \mathbf{P}^1$. Thus $(D_i)^2 = (\bar{D}_i - 2E_i)^2 = \bar{D}_i^2 - 4\bar{D}_i \cdot E_i + 4E_i^2 = 0 - 0 - 4 = -4$. $\square$

In the following we make free use of resolution of singularities in dimension 2 (for simplicity).

Let $\operatorname{tr.deg}(F/k) = 2$, let $\omega = (x_1, y_1, x_2, y_2) \in M_1(F)$, and choose a_1, $a_2 \in F^*$ with $x_1 = \{a_1\}$ and $x_2 = \{a_2\}$. Let X be a smooth proper model of F/k. We say that X is ω-regular if there exist integral divisors C_i, V, $W \subset X$ such that:

$$\begin{aligned} \operatorname{div}(a_1) &= C_2 + C_3 + 2V \\ \operatorname{div}(a_2) &= C_1 + C_3 + 2W. \end{aligned}$$

and such that there exist morphisms $\operatorname{supp}(C_i) \to \operatorname{Spec} k_i$.

ω-regular models exist: By resolution of singularities we find X such that there exist a morphism $f\colon X \to \tilde{T}$ with $\omega = f^*(\omega_T)$. Then we may take $C_i = f^*(D_i)$.

Here we use the pull back maps for the cycle complexes as defined in [6]. For a morphism $f\colon X \to Y$ with Y smooth there exist in particular pull back maps fitting into a commutative diagram

$$\begin{array}{ccc} k(X)^* & \xrightarrow{\operatorname{div}} & \coprod_{x \in X^{(1)}} \mathbf{Z} \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ k(Y)^* & \xrightarrow{\operatorname{div}} & \coprod_{y \in Y^{(1)}} \mathbf{Z} \end{array}$$

The maps f^* depend in general on the choice of a coordination of the tangent bundle of Y, see [6, Section 12].

Note also that if $X' \to X$ is a smooth proper model F/k lying over an ω-regular model X, then X' is ω-regular as well. For that one may just take the preimages of the corresponding divisors.

Given an ω-regular model X we put

$$\rho(\omega) = V \cdot W \mod I$$

This class does not depend on the choice of the C_i, V, W. Namely let C_i', V', W' be another choice. Then V and V' differ by a sum of divisors which are defined over one of k_2, k_3. Hence every component of the intersection of $V' - V$ with any divisor will be defined over one of k_2, k_3 and therefore of even degree (if k_2, k_3 are fields). Similarly for W and W'.

It follows also that $\rho(\omega)$ does not depend on the choice of X. Namely using resolution of singularities, any two models are covered by a smooth model.

If the C_i are additionally pairwise disjoint, we have

$$2V \cdot 2W = (C_2 + C_3) \cdot (C_1 + C_3) = C_3^2$$

and therefore

$$\rho(\omega) = \frac{C_3^2}{4} \mod I$$

By Lemma 8.2 this shows that $\rho(\omega_T) = 1 \mod I$. Hence $\rho(\omega_T)$ is nontrivial if K is a field.

Proposition 8.3 (Degree formula)**.** *Let* $\operatorname{tr.deg}(E/k) \leq 2$*,* $\operatorname{tr.deg}(F/k) \leq 2$*, let*

$$f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$$

be a k-place, and let $\omega \in M_1(F)$ be unramified in f. Then

$$\rho(f^*\omega) = \deg_2(f)\rho(\omega) \mod I.$$

Proof. The intersection number of the pullback of divisors V_j under a generically finite map f is the intersection number of the V_j times the degree of f. □

From the nontriviality of $\rho(\omega_T)$ one finds:

Corollary 8.4. *If K is a field, then ω_T is not defined over a subfield of $k(T)$ of transcendence degree < 2.* □

Corollary 8.5. *If K is a field, then* $\operatorname{ed}(M_1) = 2$*.* □

The invariant ρ has the following symmetry:

Lemma 8.6. $\rho(x_1, y_1, x_2, y_2) = \rho(y_1, x_1, y_2, x_2)$*.*

Proof. Let

$$\tau\colon M_1(F) \to M_1(F),$$
$$(x_1, y_1, x_2, y_2) \mapsto (y_1, x_1, y_2, x_2).$$

τ is an automorphism. There exist a place

$$\bar\tau\colon \operatorname{Spec} k(T) \rightsquigarrow \operatorname{Spec} k(T)$$

with $\bar\tau^*\big(\tau(\omega_T)\big) = \omega_T$. Assume that K is a field. Since $\rho(\omega_T) \neq 0$, the degree formula shows that $\rho\big(\tau(\omega_T)\big) \neq 0$. Thus in $\mathbf{Z}/2$ we must have $\rho(\omega_T) = \rho\big(\tau(\omega_T)\big)$. □

The degree formula shows also that $\bar\tau$ is of odd degree. One may choose $\bar\tau$ as an automorphism of $k(T)$.

One may check the symmetry also directly: If $x_1 = \{a_1\}$ and $x_2 = \{a_2\}$, then $y_1 = \{b_1\}$ and $y_2 = \{b_2\}$ with $b_1 = ca_1$ and $b_2 = da_2$. With these choices one has $\operatorname{div}(b_i) = \operatorname{div}(a_i)$.

The following proposition means that $\rho(x_1, y_1, x_2, y_2)$ is already determined by $(y_1, x_1, y_2 + x_2)$.

Proposition 8.7. *Let $\omega = (x_1, y_1, x_2, y_2) \in M_1(F)$ and let $w \in K_1F/2$ with $x_1 w = y_1 w = 0$. Then $\omega_w = (x_1, y_1, x_2 + w, y_2 + w)$ is in $M_1(F)$ and $\rho(\omega_w) = \rho(\omega)$.*

Proof. We assume $\operatorname{tr.deg}(F/k) = 2$.

Again let $c \in k^*$ with $z_1 = \{c\}$.

We have $\omega \in M_1(F)$, $x_1 w = 0$, and $z_1 w = 0$. Therefore there exist a smooth proper model X of F/k such that there are morphisms $f\colon X \to \tilde T$, g, $h\colon X \to \mathbf{P}^1$ with $f^*(\omega_T) = \omega$, $g^*(x_u) = (x_1, w)$, $h^*(\{1 - ct^2\}) = w$.

Moreover we may assume that x_1, x_2, and w are unramified outside a smooth divisor H with normal crossings. For n, m, $l \in \mathbf{Z}/2$ let $H(n, m, l) \subset H$ be the subdivisor where x_1, x_2, w has ramification index n, m, l, repectively.

Lemma 8.8. *The* 5 *sets* $H(0,1,0) \cup H(0,0,1) \cup H(0,1,1)$*,* $H(1,0,0)$*,* $H(1,0,1)$*,* $H(1,1,0)$*,* $H(1,1,1)$ *are pairwise disjoint.*

Proof. We have

$$\begin{aligned} H(1,0,0) \cup H(1,0,1) &= f^*(D_2), \\ H(1,1,0) \cup H(1,1,1) &= f^*(D_3), \\ H(0,1,0) \cup H(0,1,1) &= f^*(D_1). \end{aligned}$$

Hence these three sets are pairwise disjoint (see Lemma 8.2).

We have

$$\begin{aligned} H(1,0,0) \cup H(1,1,0) &= g^*([0]), \\ H(1,0,1) \cup H(1,1,1) &= g^*([\infty]), \\ H(0,0,1) \cup H(0,1,1) &= g^*([1]). \end{aligned}$$

Hence these three sets are pairwise disjoint.

The claim is immediate. □

Lemma 8.9. *There exist morphisms* $H(1,0,1) \to \operatorname{Spec} K$*,* $H(1,1,1) \to \operatorname{Spec} K$*.*

Proof. $H(1,0,1) \subset f^*(D_2)$ maps to $\operatorname{Spec} k_2$ and $H(1,1,1) \subset f^*(D_3)$ maps to $\operatorname{Spec} k_3$ (see Lemma 8.2). Furthermore $\operatorname{div}(w) = h^*(\{1-ct^2 = 0\})$ maps to $\operatorname{Spec} k_1$. Thus any of $H(?,?,1)$ maps to $\operatorname{Spec} k_1$. □

To conclude let a_1, a_2, $b \in F^*$ with $x_1 = \{a_1\}$, $x_2 = \{a_2\}$, and $w = \{b\}$.

Then we have integrally

$$\operatorname{div}(a_1) = [H(1,0,0) + H(1,0,1)] + [H(1,1,0) + H(1,1,1)] + 2V \tag{1}$$

$$\operatorname{div}(a_2) = [H(0,1,0) + H(0,1,1)] + [H(1,1,0) + H(1,1,1)] + 2W. \tag{2}$$

$$\operatorname{div}(b) = H(1,0,1) + H(1,1,1) + H(0,1,1) + H(0,0,1) + 2U. \tag{3}$$

for some divisors V, W, U.

We have

$$\rho(\omega_w) - \rho(\omega) = V \cdot U \mod I.$$

Further, by Lemma 8.8, one has

$$2V \cdot 2U = H(1,0,1)^2 + H(1,1,1)^2.$$

Again by Lemma 8.8 and by Equation (3) one has

$$\begin{aligned} H(1,0,1)^2 &= -H(1,0,1) \cdot [H(1,1,1) + H(0,1,1) + H(0,0,1) + 2U] \\ &= -2H(1,0,1) \cdot U \\ &\equiv 0 \mod 8, \\ H(1,1,1)^2 &= -H(1,1,1) \cdot [H(1,0,1) + H(0,1,1) + H(0,0,1) + 2U] \\ &= -2H(1,1,1) \cdot U \\ &\equiv 0 \mod 8. \end{aligned}$$

For this note also that by Lemma 8.9 one has $H(1,?,1) \cdot Y \equiv 0 \mod 4$ for all divisors Y (if K is a field). □

9. The invariant Q

In the following we make use of resolution of singularities in dimension 3 (probably this can be avoided).

For a field F/k let

$$M_2(F) \subset (K_1F/2)^6,$$
$$M_2(F) = \{ (x_1, y_1, z_1, x_2, y_2, z_2) \mid x_1x_2 = y_1y_2 = z_1z_2 = 0, \\ x_i + y_i + z_i = 0 \text{ for } i = 1, 2 \}.$$

Let $\bar z_1 = \{t\}$, $\bar z_2 = \{1-t\} \in K_1k(t)/2$ and let $T = T(\bar z_1, \bar z_2)$ and $\tilde T = \tilde T(\bar z_1, \bar z_2)$. $\tilde T$ is a 2-dimensional variety over $k(t)$.

Lemma 9.1. $\mathrm{ed}(M_2) \le 3$.

Proof. Let $\bar F$ be the function field of the variety $\tilde T$. Then $\big(\bar F, \bar\sigma\big)$ with $\bar\sigma = (\omega_T, \bar z_1, \bar z_2)$ is versal. □

Let $\bar T \to \mathbf{P}^1$ be a proper variety with generic fibre $\tilde T$.

Let $\operatorname{tr.deg}(F/k) = 3$ and let $\sigma = (x_1, y_1, z_1, x_2, y_2, z_2) \in M_2(F)$.

Choose a_1, $a_2 \in F^*$ with $x_1 = \{a_1\}$ and $x_2 = \{a_2\}$.

Let X be a smooth proper model of F/k such that there exist a morphism $f\colon X \to \bar T$ with $f^*(\bar\sigma) = \sigma$. Write

$$\operatorname{div}(a_1) = A_1 + 2V$$
$$\operatorname{div}(a_2) = A_2 + 2W.$$

for the integral divisors on X. Here we assume that the A_i are divisors with odd multiplicities.

We define $Q(\sigma) \in \mathbf{Z}/2$ by

$$Q(\sigma) = Q(X, f, \sigma) = V \cdot W \cdot \bar f^*([*]) \mod 2$$

where $\bar f\colon X \xrightarrow{f} \bar T \to \mathbf{P}^1$.

If we represent $\bar f^*([*])$ by the generic fibre of $X \to \mathbf{P}^1$, we see that

$$Q(X, f, \sigma) = \rho\big((x_1, y_1, x_2, y_2)\big) \tag{4}$$

where ρ is defined with respect to the ground field $k(\mathbf{P}^1)$ and to $z_1 = \{t\}$, $z_2 = \{1-t\}$. Note that z_1, z_2 are linearly independent square classes and so $I(z_1, z_2) = 2\mathbf{Z}$.

Equation (4) shows that $Q(X', f, \sigma) = Q(X, f, \sigma)$ for any $X' \to X$. Thus $Q(X, f, \sigma)$ does not depend on the choice of X. It does not depend on the choice of f as well, since for X large enough we have $\bar f^*([*]) = \bar f'^*([*])$ in $\operatorname{Pic}(X)/2$, see section 6.

Proposition 9.2 (Degree formula)**.** *Let* $\operatorname{tr.deg}(E/k) \le 3$, $\operatorname{tr.deg}(F/k) \le 3$, *let*

$$f\colon \operatorname{Spec} E \rightsquigarrow \operatorname{Spec} F$$

be a k-place, and let $\sigma \in M_2(F)$ be unramified in f. Then

$$Q(f^*\sigma) = \deg_3(f) Q(\sigma).$$

□

Lemma 9.3. $Q(\bar\sigma) \ne 0$

Proof. This follows from $\rho(\omega_T) \ne 0$. □

Corollary 9.4. *$\bar{\sigma}$ is not defined over a subfield of $\bar{F}$ of transcendence degree $<$ 3.* □

Corollary 9.5. $\mathrm{ed}(M_2) = 3$. □

Lemma 9.6. *$Q(\sigma)$ is invariant under the permutations $x_1 \leftrightarrow x_2$, $y_1 \leftrightarrow y_2$, $z_1 \leftrightarrow z_2$ and $x_i \mapsto y_i \mapsto z_i \mapsto x_i$.*

Proof. Use the same argument as in the proof of Lemma 8.6. □

10. The invariant $\hat{Q}$

For a field F/k let

$$M_3(F) \subset (K_1F/2)^3 \oplus K_2F/2,$$
$$M_3(F) = \{\, (x_1, x_2, x_3, u) \mid x_1 + x_2 + x_3 = 0,\, u \in x_i \cdot K_1F/2 \text{ for } i = 1,\, 2,\, 3 \,\}.$$

We have a map

$$\varphi\colon M_2(F) \to M_3(F),$$
$$\varphi(x_1, y_1, z_1, x_2, y_2, z_2) = (x_1, y_1, z_1, y_1x_2).$$

Lemma 10.1. *The map φ is surjective. One has*

$$\varphi(x_1, y_1, z_1, x_2, y_2, z_2) = \varphi(x_1, y_1, z_1, x_2', y_2', z_2')$$

if and only if there exist $w \in K_1F/2$ and $u \in K_1F/2$ with $x_1w = y_1w = 0$, $y_1u = z_1u = 0$ and $x_2' = x_2 + w$, $y_2' = y_2 + w + u$, $z_2' = z_2 + u$.

Proof. Usual biquadratic games. □

Corollary 10.2. *The pair $\bigl(\bar{F}, \varphi(\bar{\sigma})\bigr)$ is versal for M_3.* □

Let $\mathrm{tr.deg}(F/k) = 3$ and $\hat{\sigma} \in M_3(F)$. We put

$$\hat{Q}(\hat{\sigma}) = Q(\sigma) \in \mathbf{Z}/2$$

where $\sigma \in M_3(F)$ is any element with $\varphi(\sigma) = \hat{\sigma}$. By Proposition 8.7, Equation (4), Lemma 9.6, and Lemma 10.1 this gives a welldefined invariant.

It is nontrivial on the generic element and obeys a degree formula. From that we may conclude $\mathrm{ed}(M_3) = 3$ and

Corollary 10.3. *$\varphi(\bar{\sigma})$ is not defined over a subfield of $\bar{F}$ of transcendence degree $<$ 3.* □

11. The functor M_4

We consider triples $\Phi = (D, \varphi, \psi)$ where D is a quaternion algebra, and where φ, ψ are skew-hermitian forms over D of dimension 2 and 1, respectively, with $\det(\varphi \perp \psi) = 1$. We say that two such triples (D, φ, ψ), (D', φ', ψ') are similar, if there exist an isomorphism $\alpha\colon D \to D'$ such that φ is similar to $\alpha^*\varphi'$ and ψ is similar to $\alpha^*\psi'$.

For a field F/k let $M_4(F)$ be the set of similarity classes of such triples over F.

Let $(\hat{F}, \hat{\sigma})$ be a versal pair for M_3 with $\mathrm{tr.deg}(\hat{F}/k) = 3$. Write $\hat{\sigma} = (x_1, x_2, x_3, u)$. Let D be a quaternion algebra representing u and choose $d_i \in D$ with $\mathrm{Trd}(d_i) = 0$ and $\{\mathrm{Nrd}(d_i)\} = x_i$. Then $\hat{\Phi} = (D, \langle d_1, sd_2\rangle, \langle d_3\rangle)$ defines an element $[\hat{\Phi}]$ of $M_4\bigl(\hat{F}(s)\bigr)$.

Lemma 11.1. $\big(\hat{F}(s),[\hat{\Phi}]\big)$ *is a versal pair for* M_4.

Proof. First note in general that, if d, $d'\in D$ are trace zero elements with the property $\{\operatorname{Nrd}(d)\}=\{\operatorname{Nrd}(d')\}$, then the skew-hermitian forms $\langle d\rangle$, $\langle d'\rangle$ are similar. (This follows from Skolem-Noether).

By diagonalization, any Φ' over any F' can be written as $(D',\langle d_1',d_2'\rangle,\langle d_3'\rangle)$ with $d_i'\in D'$, $\operatorname{Trd}(d_i')=0$, $\operatorname{Nrd}(d_1')\operatorname{Nrd}(d_2')\operatorname{Nrd}(d_3')=1$. Then

$$\sigma_{\Phi'}=(\{\operatorname{Nrd}(d_1')\},\{\operatorname{Nrd}(d_2')\},\{\operatorname{Nrd}(d_3')\},[D])$$

is an element of $M_3(F')$. It follows that there exist a place $f\colon \operatorname{Spec}F'\rightsquigarrow\operatorname{Spec}\hat{F}$ with $f^*(\bar{\sigma})=\sigma_{\Phi'}$. Then $f^*D=D'$, and there exist $c_i\in F'^*$ with $\langle c_if^*d_i\rangle\simeq\langle d_i\rangle$ ($\simeq$ denoting isomorphism). Extend the place f to $f\colon\operatorname{Spec}F'\rightsquigarrow\operatorname{Spec}\hat{F}(s)$ by $f^*(s)=c_1^{-1}c_2$. Then ($\sim$ denoting similarity)

$$f^*(\hat{\Phi})=(f^*D,\langle f^*d_1,c_1^{-1}c_2f^*d_2\rangle,\langle f^*d_3\rangle)\sim(f^*D,\langle c_1f^*d_1,c_2f^*d_2\rangle,\langle c_3f^*d_3\rangle)$$

is similar to Φ'. $\square$

Corollary 11.2. $\operatorname{ed}(M_4)\le 4$.

Lemma 11.3. $[\hat{\Phi}]$ *is not defined over a subfield of* $\hat{F}(s)$ *of transcendence degree* 3.

Proof. Let $F'\subset\hat{F}(s)$ be of trancendence degree 3 and let $\Phi'=(D',\langle d_1',d_2'\rangle,\langle d_3'\rangle)$ be a triple defined over F' with $\Phi'_{\hat{F}(s)}\sim\hat{\Phi}$. Let v be the valuation on $\hat{F}((s))/\hat{F}$. Since $\langle d_1,sd_2\rangle$ is ramified in v (because the $\operatorname{Nrd}(d_i)$ are not squares), the valuation v cannot be trivial on F'. Then the residue class field κ' of $v|F'$ is a subfield of $\hat{F}$ of transcendence degree (at most) 2. Note that D is unramified.

The proof of the following claim (added in Dec. 2008) had been missing in the version from 2000.

Claim: D' is unramified.
Proof of the claim. We have

$$D_{\hat{F}(s)}\simeq D'_{\hat{F}(s)}$$

(with D defined over $\hat{F}$ and D' defined over F') and with respect to an isomorphism

$$f\colon D'_{\hat{F}(s)}\to D_{\hat{F}(s)}$$

one has

$$\langle d_1,sd_2\rangle_{\hat{F}(s)}\sim\langle f(d_1'),f(d_2')\rangle_{\hat{F}(s)}$$

The elements d_1, d_2 are defined over $\hat{F}$. Write

$$f(d_i')=s^{n_i}d_i''$$

with invertible $d_i''\in D_{\hat{F}[[s]]}$. The form $\langle d_1,sd_2\rangle$ is ramified. Therefore the exponents n_1, n_2 can't have the same parity and it follows that the residue forms $\langle d_1\rangle$, $\langle d_2\rangle$ coincide with the residue forms $\langle\bar{d}_1''\rangle$, $\langle\bar{d}_2''\rangle$, up to permutation and similarity. The similiarity class of a 1-dimensional D-skew-hermitian form $\langle x\rangle$ is determined by the square class of $\operatorname{Nrd}(x)$. Note that $\operatorname{Nrd}(d_i')$ has in $\hat{F}(s)$ the same square class as $\operatorname{Nrd}(d_i'')$. It follows that in $\hat{F}((s))$ the square classes of $\operatorname{Nrd}(d_1)$, $\operatorname{Nrd}(d_2)$ coincide with the square classes $\operatorname{Nrd}(d_1')$, $\operatorname{Nrd}(d_2')$, up to permutation.

Now, if D' would be ramified, one would have by Lemma 14.1 over $\hat{F}((s))$:

$$\begin{aligned}[D] &= \big(\mathrm{Nrd}(d_1'), \mathrm{Nrd}(d_2')\big)\\ &= \big(\mathrm{Nrd}(d_1), \mathrm{Nrd}(d_2)\big)\end{aligned}$$

Hence

$$[D] = \big(\mathrm{Nrd}(d_1), \mathrm{Nrd}(d_2)\big)$$

over $\hat{F}$.

This would mean that the versal pair $(\hat{F}, \hat{\sigma})$ for M_3 would have the form

$$\hat{\sigma} = (x_1, x_2, x_3, x_1x_2)$$

But for $K = k(u,v)$ the element

$$\sigma = (\{u\}, \{v\}, \{uv\}, 0) \in M_3(K)$$

is not of this form.

This ends the proof of the claim.

By standard ramification theory for quadratic forms, the residues of a form up to similarity are well defined, up to a permutation of the first and second residue form. It follows that

$$\Phi' \sim \tilde{\Phi}' = (\tilde{D}', \langle \tilde{d}_1', s\tilde{d}_2'\rangle, \langle \tilde{d}_3'\rangle)$$

with $\tilde{D}'$ and $\tilde{d}_i'$ defined and regular over the ring of $v|F'$. Taking residues for $\hat{\Phi}$ and $\tilde{\Phi}'$, we see that the quadruple $(D, \langle d_1\rangle, \langle d_2\rangle, \langle d_3\rangle)$ is similar to $\overline{(\tilde{D}', \langle \tilde{d}_1'\rangle, \langle \tilde{d}_2'\rangle, \langle \tilde{d}_3'\rangle)}$ or to $\overline{(\tilde{D}', \langle \tilde{d}_2'\rangle, \langle \tilde{d}_1'\rangle, \langle \tilde{d}_3'\rangle)}$.

Since these quadruples are defined over κ', we have a contradiction to Corollary 10.3. □

12. Computation of $\mathrm{ed}(\mathrm{PSO}_6)$

Finally let $M_5(F) = H^1(F, \mathrm{PSO}_6)$. Then $M_5(F)$ consists of similarity classes of pairs (D, ρ), where D is a quaternion algebra, and where ρ is a skew-hermitian forms over D of dimension 3 with $\det(\rho) = 1$.

Let $\big(E, [\hat{\Phi}]\big)$ be a versal pair for M_4 with $\hat{\Phi} = (D, \langle d_1, d_2\rangle, \langle d_3\rangle)$ and with $\mathrm{tr.deg}(E/k) = 4$, see Lemma 11.1. Then $x = [(D, \langle d_1, d_2, sd_3\rangle)]$ is an element of $M_5\big(E(s)\big)$.

Lemma 12.1. *$\big(E(s), x\big)$ is a versal pair for M_5.*

Proof. Similar as for Lemma 11.1. □

Corollary 12.2. $\mathrm{ed}(M_5) \leq 5$.

Lemma 12.3. *x is not defined over a subfield of $E(s)$ of transcendence degree 4.*

Proof. Similar as for Lemma 11.3, now using Lemma 11.3 instead of Corollary 10.3.

Added in Dec. 2008:

Consider the versal pair for M_5 in Lemma 12.1 given by

$$\Psi = (D, \langle d_1, d_2, sd_3\rangle)$$

over $E(s)$.

Let $E' \subset E(s)$ be of trancendence degree 4 and let

$$\Psi' = (D', \langle d_1', d_2', d_3'\rangle)$$

be a triple defined over E' with

$$\Psi'_{E(s)} \sim \Psi$$

Let v be the valuation on $E((s))/E$. Since $\langle d_1, d_2, sd_3\rangle$ is ramified in v (because the $\mathrm{Nrd}(d_i)$ are not squares), the valuation v cannot be trivial on E'. Then the residue class field κ' of $v|E'$ is a subfield of E of transcendence degree (at most) 3. Note that D is unramified.

Claim: D' is unramified.

Proof of the claim. We have

$$D_{E(s)} \simeq D'_{E(s)}$$

(with D defined over E and D' defined over E') and with respect to an isomorphism

$$f\colon D'_{E(s)} \to D_{E(s)}$$

one has

$$\langle d_1, d_2, sd_3\rangle_{E(s)} \sim \langle f(d_1'), f(d_2'), f(d_3')\rangle_{E(s)}$$

The elements d_1, d_2, d_3 are defined over E. Write

$$f(d_i') = s^{n_i} d_i''$$

with invertible $d_i'' \in D_{E[[s]]}$. The form $\langle d_1, d_2, sd_3\rangle$ is ramified. Therefore the exponents n_1, n_2, n_3 can't have the same parity. Suppose that n_1 and n_2 have the same parity. Then the residue form $\langle d_1, d_2\rangle$ coincides with the residue form $\langle \bar{d}_1'', \bar{d}_2''\rangle$ up to similarity:

$$\langle d_1, d_2\rangle \sim \langle \bar{d}_1'', \bar{d}_2''\rangle_E$$

Now, if D' would be ramified, one would have by Lemma 14.1 over $E((s))$:

$$\begin{aligned}[D] &= \big(\mathrm{Nrd}(d_1'), \mathrm{Nrd}(d_2')\big)\\ &= \big(\mathrm{Nrd}(d_1''), \mathrm{Nrd}(d_2'')\big)\end{aligned}$$

Taking residues one gets

$$[D] = \big(\mathrm{Nrd}(\bar{d}_1''), \mathrm{Nrd}(\bar{d}_2'')\big)$$

over E.

Therefore the versal pair $\big(E, [\hat{\Phi}]\big)$ for M_4 would have the form

$$\hat{\Phi} = \big(D, \langle e_1, e_2\rangle, \langle e_3\rangle\big)$$

with $a_i = e_i^2$ and $a_1 a_2 a_3 = 1$ and

$$[D] = (a_1, a_3)$$

This would mean that for any field K and any element of $M_4(K)$ given by

$$\Phi = (D, \varphi, \psi)$$

there exist a 1-dimensional subform ρ of φ such that

$$[D] = \big(\det(\rho), \det(\varphi)\big)$$

If D is split, this would mean that for any 4-dimensional (usual) quadratic form φ there exist a 2-dimensional quadratic subform ρ of φ such that $\det(\varphi)$ is a norm from the quadratic extension given by ρ. But then $\det(\varphi)$ would be a similarity factor of φ.

However for a 4-dimensional quadratic form of the form

$$\varphi = \langle w, u, v, uv \rangle$$

the determinant is a similarity factor if and only if the Pfister form

$$\langle\langle u, v, w \rangle\rangle$$

is split. This is not the case over the field $k(u, v, w)$.

This ends the proof of the claim.

The rest of the proof is similar as for Lemma 11.3 □

Corollary 12.4. $\mathrm{ed}(\mathrm{PGL}_4) = \mathrm{ed}(PSO_6) = \mathrm{ed}(M_5) = 5.$ □

13. Presentations of M and degree formulas

In the following we discuss some general aspects about essential dimensions and "degree formulas".

Definition 13.1 (tentative). A *presentation* of M consists of a pair of morphisms

$$X_1 \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X_0$$

of k-varieties and a function α on X_0 with $\alpha(x) \in M\big(\kappa(x)\big)$ such that:

- Let $x \in X_0$ and let v be a valuation on $\kappa(x)$ with center $y \in X_0$. Then $\alpha(x)$ is unramified in v and for its specialization one has $\alpha(v) = \alpha(y)_{\kappa(y)}$.
- For any F/k and $\beta \in M(F)$ and any open dense subvariety $U \subset X_0$ there exists $f\colon \operatorname{Spec} F \to U$ with $\beta = f^*(\alpha)$.
- For every $y \in X_1$ one has $\pi_0^*\big(\alpha(\pi_0(y)\big) = \pi_1^*\big(\alpha(\pi_1(y)\big)$ in $M\big(\kappa(y)\big)$.
- For any F/k and any two morphisms f_0, $f_1\colon \operatorname{Spec} F \to X_0$ with $f_0^*(\alpha) = f_1^*(\alpha)$ there exists $f\colon \operatorname{Spec} F \to X_1$ with $f_i = \pi_i \circ f$.

Example. Let $G \subset \mathrm{GL}_n$ be a linear algebraic group over k and let $M_G(F) = H^1(F, G)$. There is natural presentation of M_G with $X_0 = \mathrm{GL}_n / G$ and $X_1 = \mathrm{GL}_n \times \mathrm{GL}_n / G$.

Example. In section 6 we have seen that π, $\pi'\colon \tilde{P} \to \mathbf{P}^1$ is a presentation of M_0.

Exercise. Describe presentations of the functors M_1, M_2, ... of the preceding sections.

Let π_0, $\pi_1\colon X_1 \rightrightarrows X_0$, α be a presentation of M with X_0 irreducible of dimension d. Choose a completion $\bar\pi_0$, $\bar\pi_1\colon \bar X_1 \rightrightarrows \bar X_0$ and consider

$$\delta = (\bar\pi_1)_* - (\bar\pi_0)_*\colon \mathrm{CH}_d(\bar X_1) \to \mathrm{CH}_d(\bar X_0) = \mathbf{Z}.$$

Suppose that $\operatorname{im}\delta \subset n\mathbf{Z}$. Then for F/k with $\operatorname{tr.deg}(F/k) \le d$ and $\beta \in M(F)$ we have a invariant

$$Q(\beta) \in \mathbf{Z}/n$$

defined by $Q(\beta) = 0$ if $\operatorname{tr.deg}(F/k) < d$ and otherwise by $Q(\beta) = f_*([X])$ if X is a proper modell of F/k and $f\colon X \to X_0$ is a morphism with $\beta = f^*(\alpha)$. This invariant obeys the degree formula

$$Q(f^*\beta) = \deg_d(f) Q(\beta).$$

These considerations seem to provide a natural frame work for a systematic treatment of degree formulas in the context of these notes.

Example. For the presentation π, $\pi'\colon \tilde P \to \mathbf{P}^1$ of M_0 in section 6 one finds $n = 2$.

14. Complements

Recall that we work in characteristic different from 2 and that -1 is a square.

Lemma 14.1. *Let R be a complete discrete valuation ring with fraction field K. Let E be a quaternion algebra over K which is* ramified *with respect to R. Let $e_i \in E$ ($i = 1, 2, 3$) with* $\mathrm{Trd}(e_i) = 0$ *and*

$$\mathrm{Nrd}(e_1)\,\mathrm{Nrd}(e_2)\,\mathrm{Nrd}(e_3) = 1$$

Then

$$[E] = \big(\mathrm{Nrd}(e_i), \mathrm{Nrd}(e_j)\big)$$

for $i \neq j$.

Proof. First note that

$$\big(\mathrm{Nrd}(e_i), \mathrm{Nrd}(e_j)\big)$$

is independent of the choices of i, j. This follows from the product relation and from $(a, a) = 0$.

Let π be a prime element of R and denote by $\kappa = R/\pi R$ the residue class field of R. For $a \in R$ denote by $\bar a \in \kappa$ its residue.

Since E is ramfied there exists a, $b \in R^\times$ such that

$$[E] = (a, \pi b)$$

and such that the square class $(\bar a)$ is nontrivial.

Let 1, X, Y, XY be a basis of E with $X^2 = a$, $Y^2 = \pi b$ and $XY + YX = 0$. Then

$$e_i = \pi^{n_i}(X\alpha_i + Y\beta_i + XY\gamma_i)$$

with $n_i \in \mathbf{Z}$ and α_i, β_i, $\gamma_i \in R$ such that

$$(\bar\alpha_i, \bar\beta_i, \bar\gamma_i) \neq 0$$

in κ^3 for $i = 1, 2, 3$.

One now analyzes the product relation $\mathrm{Nrd}(e_1)\,\mathrm{Nrd}(e_2)\,\mathrm{Nrd}(e_3) = 1$. One has

$$-\mathrm{Nrd}(e_i) = \pi^{2n_i}\big(a\alpha_i^2 + \pi b(\beta_i^2 - a\gamma_i^2)\big) \tag{5}$$

Suppose $\bar\alpha_i = 0$ for some i. Then $\bar\beta_i \neq 0$ or $\bar\gamma_i \neq 0$ and since $\bar a$ is not a square, it follows that $(\beta_i^2 - a\gamma_i^2)$ is a unit of R. Write $\alpha_i = \pi\alpha_i'$ with $\alpha_i' \in R$. Then one has

$$-\mathrm{Nrd}(e_i) = \pi^{2n_i+1}\big(\pi a\alpha_i'^2 + b(\beta_i^2 - a\gamma_i^2)\big) \tag{6}$$

with the second factor a R-unit.

Suppose $\bar\alpha_i = 0$ for exactly one or for all 3 of the indices $i = 1, 2, 3$. Then (5) and (6) show that

$$1 = \prod_{i=1}^{3}\mathrm{Nrd}(e_i) = \pi^m \cdot \mathrm{unit}$$

with m odd, a contradiction.

Suppose $\bar\alpha_i \neq 0$ for $i = 1, 2, 3$. Then $n_1 + n_2 + n_3 = 0$ and

$$-1 = -\overline{\mathrm{Nrd}(e_1)\,\mathrm{Nrd}(e_2)\,\mathrm{Nrd}(e_3)} = \prod_{i=1}^{3}(\bar\alpha_i)^2\bar a = \bar a^3\prod_{i=1}^{3}(\bar\alpha_i)^2$$

Hence $\bar a$ would be a square, a contradiction.

Suppose $\bar{\alpha}_1 \neq 0$ and $\bar{\alpha}_i = 0$ for $i = 2, 3$. Then

$$-\operatorname{Nrd}(e_1) = \pi^{2n_i} a\alpha_1^2 U$$
$$-\operatorname{Nrd}(e_2) = \pi^{2n_2+1} b(\beta_2^2 - a\gamma_2^2)V$$

with U, $V \in R$ such that $\bar{U} = \bar{V} = 1$. Since R is complete, U and V are squares. One finds

$$\begin{aligned}\big(\operatorname{Nrd}(e_1), \operatorname{Nrd}(e_2)\big) &= \big(a, \pi b(\beta_2^2 - a\gamma_2^2)\big) \\ &= [E] + (a, \beta_2^2 - a\gamma_2^2) \\ &= [E]\end{aligned}$$

□

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