ESSENTIAL DIMENSION OF TWISTED C_4

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Introduction

In this note we show that over a field of characteristic different from 2 the essential dimension of any Galois module which is cyclic of order 4 is equal to 2, except for μ_4.

The method is consists of a (non-)ramification argument, which is also used in the text [1] for the computation of of the essential dimension of PGL_4.

For the essential dimension of $\mathbf{Z}/4$ with trivial Galois action see also [2].

1. Statement of result

Let k be a field of characteristic different from 2 and let G be a Galois module over k, which as an abstract group is cyclic of order 4. Let $d \in k^*$ be an element such that G is isomorphic to μ_4 twisted by the quadratic character given by the square class of d. Thus $G \simeq \mu_4$ if d is a square and $G \simeq \mathbf{Z}/4$ if $-d$ is a square.

One has $H^1(k, \mu_4) = k^*/(k^*)^4$ and it easy to see that $\mathrm{ed}(\mu_4) = 1$.

Proposition. *If d is not a square, then* $\mathrm{ed}(G) = 2$.

2. Galois cohomology

We assume in the following that d is not a square.

Then G can be identified with the elements of order 4 in the norm 1 sub group of the multiplicative group of $K = k(\sqrt{d})$.

Moreover there is an exact sequence

$$1 \to G \to (K \otimes_k \bar{k})^* \xrightarrow{\pi} \bar{k}^* \times \frac{(K \otimes_k \bar{k})^*}{\bar{k}^*} \to 1$$

where

$$\pi(\lambda) = \big(N_{K/k}(\lambda), [\lambda^2]\big)$$

This is easy to check.

By Hilbert's Theorem 90 one gets an exact sequence

$$K^* \xrightarrow{\pi} k^* \times \frac{K^*}{k^*} \xrightarrow{\delta_k} H^1(k, G) \to 0$$

Note that $H^1(k, G)$ classifies the extensions L/k of degree 4 together with a monomorphism (of algebraic groups!) $G \to \mathrm{Aut}(L/k)$. In fact, let $a \in k^*$ and $\mu \in K^*$ and let $\bar{\mu}$ be the conjugate of μ under the Galois group of K/k. Consider the Kummer extension

$$E = K[u], \qquad u^4 = a^2 \frac{\mu}{\bar{\mu}}$$

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It has a K-semilinear automorphism

$$\sigma\colon E \to E, \qquad \sigma(u) = \frac{a}{u}$$

Then

$$L = E^\sigma$$

is an extension of k of degree 4 with a natural G-action. This extension corresponds to $\delta_k(a, [\mu])$.

3. Invariants

We consider the cohomological invariants

$$\eta_1\colon H^1(k, G) \to H^1(k, \mu_2)$$
$$\eta_2\colon H^1(k, G) \to H^2(k, \mu_2)$$

defined as follows: The invariant η_1 is just the map induced from the projection $G \to \mu_2$. The invariant η_2 is given by the embedding

$$G \to \mathrm{PGL}(2)$$
$$\zeta \to \left[\begin{pmatrix} 1 & -\sqrt{-d} \\ \sqrt{-d}^{-1} & 1 \end{pmatrix}\right]$$

where $\zeta \in G(\bar{k})$ is a generator, followed by the standard invariant $H^1\big(k, \mathrm{PGL}(2)\big) \to H^2(k, \mu_2)$.

One has for $a \in k^*$ and $\mu \in K^*$:

$$\eta_1\big(\delta_k(a, [\mu])\big) = \big(N_{K/k}(\mu)\big)$$
$$\eta_2\big(\delta_k(a, [\mu])\big) = (a) \cup (d)$$

One may take these formula also for a definition of η_1 and η_2.

4. Versal elements

Let $F = k(x, y)$. Then

$$\alpha = \delta_F(x, [1 + y\sqrt{d}]) \in H^1(F, G)$$

is a versal element for $H^1(?, G)$.

Let v be the valuation on F which is trivial on $k(y)$ and with prime element x. Then $\kappa(v) = k(y)$. One has

$$\eta_1(\alpha) \text{ is unramified at } v.$$
$$\eta_2(\alpha) \text{ is ramified at } v.$$

5. Proof of the Proposition

Now let $\alpha' \in H^1(F', G)$ be some versal element for $H^1(?, G)$. We have to show that F' has transcendence degree at least 2.

Let

$$\varphi\colon F' \rightsquigarrow F$$

be a k-place with

$$\varphi_*(\alpha') = \alpha$$

(see [1] for places).

Further let $v' = \varphi^*(v)$. Since $\eta_2(\alpha) = \varphi_*\big(\eta_2(\alpha')\big)$ is ramified at v (here we use that d is not a square), we must have that $\eta_2(\alpha')$ is ramified at v'. Hence the ramification index of $v'|v$ is odd. In particular, v' is not trivial. Since $\eta_1(\alpha) = \varphi_*\big(\eta_1(\alpha')\big)$ is unramified at v, it follows then that $\eta_1(\alpha')$ is unramified at v'.

Next note that the specialization

$$\eta_1(\alpha)(v) \in H^1\big(k(y), \mu_2\big)$$

is a versal element for square classes which are norms from K/k.

Let

$$\bar{\varphi}\colon \kappa(v') \to \kappa(v) = k(y)$$

be the induced homomorphism on the residue class fields. Then

$$\bar{\varphi}_*\big(\eta_1(\alpha')(v')\big) = \eta_1(\alpha)(v)$$

Hence

$$\eta_1(\alpha')(v') \in H^1\big(\kappa(v'), \mu_2\big)$$

is also a versal element for square classes which are norms from K/k. It follows that $\kappa(v')$ has transcendence degree at least 1. Since v' is non-trivial it follows that F' has transcendence degree at least 2. This finishes the proof.

References

[1] M. Rost, *Computation of some essential dimensions*, notes, 2000, http://www.math.ohio-state.edu/~rost/ed.html.

[2] J.-P. Serre, *Topics in Galois theory*, Jones and Bartlett Publishers, Boston, MA, 1992, Lecture notes prepared by Henri Damon [Henri Darmon], With a foreword by Darmon and the author.

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