THE S_6-SYMMETRY OF QUADRANGLES

Diplomarbeit

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Contents

Introduction

Given a triangle x in the Euclidean plane, we associate certain complex numbers to x, namely "algebraic sides", "moduli" and "double angles". In turn, these complex numbers determine any triangle up to similarity (cf. Remark 1.6 and Corollary 1.10). We proceed in the same way with quadrangles, where we consider the set $\overline{X}_Q$ of quadrangles with ordered vertices up to affine transformations. Besides the canonical operation of S_4 on this set, given by the permutations of the vertices of a quadrangle, there is another action on a suitable subset $\overline{Y}_Q \subset \overline{X}_Q$. This action is induced by the so-called pedal triangle construction (cf. Corollary 1.31). Given a triangle x and an additional point x_0, one derives another triangle by dropping perpendiculars from the point x_0 to the sides of x. The points of intersection of the perpendiculars and the sides of x are called pedal points. We thus get three pedal points and we take these pedal points as the pedal triangle x' of x. The three points of x together with x_0 form a quadrangle y. From this quadrangle we derive a new quadrangle consisting of x_0 and of the points of x'. We call it the pedal quadrangle of y with respect to x_0. This action cannot be defined on all quadrangles since the points of the pedal triangle are collinear if x_0 lies on the circumcircle of x (cf. Lemma 1.25).

The aim of this thesis is to show that these two operations generate a group isomorphic to S_6 (Theorem 3.3). For this purpose, we firstly define a map

$$v\colon \overline{X}_Q \longrightarrow V$$

in terms of the algebraic sides, where

$$V = \left(S^1\right)^6 / \Delta\left(S^1\right)$$

and Δ is the diagonal map. The map v is injective, whence the group V contains the set $\overline{X}_Q$.

In Section 2, we establish an action of $S_6 \times \{\pm 1\}$ on V. Consider the action of S_4 on $\overline{X}_Q$ given by the permutations of the vertices of a quadrangle. There is an injective group homomorphism

$$\delta\colon S_4 \longrightarrow S_6 \times \{\pm 1\}$$

such that v is δ-equivariant (cf. Lemmas 2.8 and 2.9). In section 3 we furthermore find an element $\widehat{\phi}_0$ of order 3 in $S_6 \times \{\pm 1\}$ that corresponds to the action induced by the pedal triangle construction, meaning the following. Given an element $[y] \in \overline{Y}_Q$, the image of the pedal quadrangle of $[y]$ under the map v is equal to $\widehat{\phi}_0\,(v([y]))$ (cf. Corollary 3.2). We conclude by proving that the element $\widehat{\phi}_0$ and the group $\delta(S_4)$ generate a group isomorphic to S_6.

The scope of this thesis is based on the text "The variety of angles" by M. Rost (cf. [3]).

At this point I would like to thank Prof. Markus Rost for his support and advice during the preparation of this thesis.

1. Triangles and Quadrangles

1.1. **Triangle Relations.** In the following, we identify the Euclidean plane with the complex numbers $\mathbb{C}$ and denote by $\overline{}\colon \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto \overline{z}$ the complex conjugation. Define

$$S^1 = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$$

where $|\cdot|$ denotes the complex norm.

Definition 1.1. Let D be a set with $|D| = 3$ and define X_D to be the set of maps $x\colon D \longrightarrow \mathbb{C}$ for which the following conditions hold, where we set $x_i := x(i)$.

- x is injective
- x_i, x_j, x_k are *not* collinear for i, j, $k \in D$, pairwise distinct

We call an element $x \in X_D$ a D-labeled triangle or simply a triangle with points x_i, $i \in D$.

Definition 1.2. Let v and w be two distinct complex numbers. We define

$$\hat{u}(v, w) := \frac{v - w}{\overline{v - w}} \in S^1$$

and call it the *algebraic side* given by v and w.

Consider the group $\mathrm{Aff}(1, \mathbb{C})$ of affine automorphisms of $\mathbb{C}$ and an element g of it. Then, for $z \in \mathbb{C}$, g has the form $g(z) = az + b$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. One has

$$\begin{aligned} \hat{u}\big(g(v), g(w)\big) &= \frac{av + b - aw - b}{\overline{av + b - aw - b}} \\ &= \frac{a}{\overline{a}}\, \hat{u}(v, w) \end{aligned} \tag{1}$$

Hence $\hat{u}(v, w)$ is invariant under translations and under multiplication with an element in $\mathbb{R} \setminus \{0\}$.

Definition 1.3. Let v, y, $w \in \mathbb{C}$ with $v \neq y$, $w \neq y$. Then define

$$\hat{\tau}(v, y, w) := \frac{w - y}{v - y} \in \mathbb{C}$$

We call $\hat{\tau}(v, y, w)$ the *modulus* at y given by v, y, w.

We easily observe that $\hat{\tau}(v, y, w)$ is $\mathrm{Aff}(1, \mathbb{C})$-invariant. Namely, let again $g \in \mathrm{Aff}(1, \mathbb{C})$ as above. Then

$$\begin{aligned} \hat{\tau}\big(g(v), g(y), g(w)\big) &= \frac{aw + b - ay - b}{av + b - ay - b} \\ &= \frac{w - y}{v - y} \\ &= \hat{\tau}(v, y, w) \end{aligned}$$

In order to interpret $\hat{u}(v, w)$ and $\hat{\tau}(v, y, w)$ geometrically, we consider polar coordinates. Let $z \in \mathbb{C} \setminus \{0\}$. Then z can be uniquely written as $z = r_z e^{i\varphi_z}$

with $r_z = |z|$, $\varphi_z \in [0, 2\pi)$ the argument of z, and i the imaginary unit. Dividing two complex numbers z, $z' \neq 0$ we get

$$\frac{z}{z'} = \frac{r_z e^{i\varphi_z}}{r_{z'} e^{i\varphi_{z'}}} = \frac{r_z}{r_{z'}} e^{i(\varphi_z - \varphi_{z'})}$$

Thus

$$\frac{z}{\overline{z}} = \frac{r_z e^{i\varphi_z}}{r_z e^{i(-\varphi_z)}} = e^{i \cdot 2\varphi_z}$$

Therefore $\hat{u}(v, w)$ is the element of S^1 with argument $\varphi_{\hat{u}(v,w)} = 2\varphi_{v-w}$, see Figure 1.

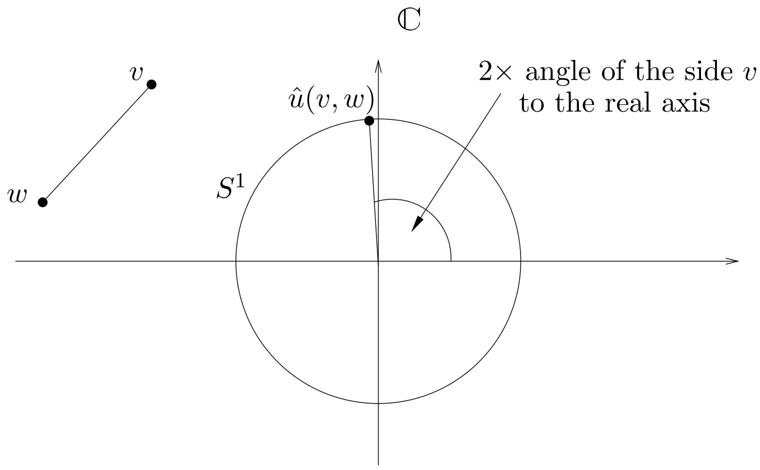

FIGURE 1

The argument of $\hat{\tau}(v, y, w)$ corresponds to the angle at y in the triangle with points v, y, w, confer Figure 2.

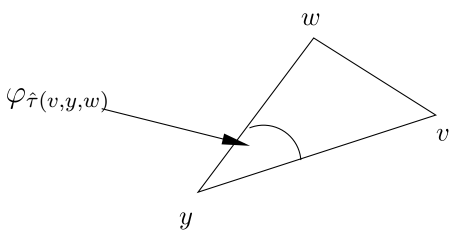

FIGURE 2

Definition 1.4. Two D-labeled triangles x, x' are called similar, if there exists an element $g \in \mathrm{Aff}(1, \mathbb{C})$ with $g(x_i) = x'_i$ for all $i \in D$. In that case we write $g(x) = x'$.

Lemma 1.5. *Let x, x' be two D-labeled triangles. Then the following statements are equivalent.*

(1) *For all $l \in D$, the angle at x_l in the triangle x is equal to the angle at x'_l in the triangle x'.*

(2) *The triangles x and x' are similar, i.e., there exists an element $g \in \mathrm{Aff}(1, \mathbb{C})$ with $g(x) = x'$.*

Proof. Easy. $\square$

Remark 1.6. Let x be a D-labeled triangle. Then each element $\hat{\tau}(x_i, x_j, x_k)$, where i, j, $k \in D$ are pairwise distinct, determines the triangle up to similarity. Namely, up to similarity we may assume $x_i = 0$ and $x_j = 1$ and $x_k = z$. Then one has

$$\hat{\tau}(1, 0, z) = \frac{z - 0}{1 - 0} = z$$

which shows that $\hat{\tau}(1, 0, z)$ determines x.

Definition 1.7. For complex numbers v, w, y with $v \neq y$, $w \neq y$ we define the *algebraic angle at y*

$$\hat{\alpha}(v, y, w) := \frac{\hat{u}(w, y)}{\hat{u}(v, y)} \in S^1$$

Note that

$$\hat{\alpha}(v, y, w) = \frac{\hat{\tau}(v, y, w)}{\overline{\hat{\tau}(v, y, w)}} \tag{2}$$

as one easily deduces from the definitions of the algebraic side and the modulus. Since $\hat{\tau}(v, y, w)$ is $\mathrm{Aff}(1, \mathbb{C})$-invariant, the same holds for $\hat{\alpha}(v, y, w)$.

We interpret $\hat{\alpha}(v, y, w)$ geometrically. Using equation (2), one observes that $\hat{\alpha}(v, y, w)$ is the element of S^1 whose argument is twice the angle at y in the triangle with points v, y, w. For this reason we call an algebraic angle $\hat{\alpha}(v, y, w)$ also a *double angle*. As seen in Remark 1.6, it is sufficient to consider only triangles x with points $x_i = 0$ and $x_j = 1$. Set $x_k = z$, then Figure 3 shows the geometric interpretation of $\hat{\alpha}(1, 0, z)$.

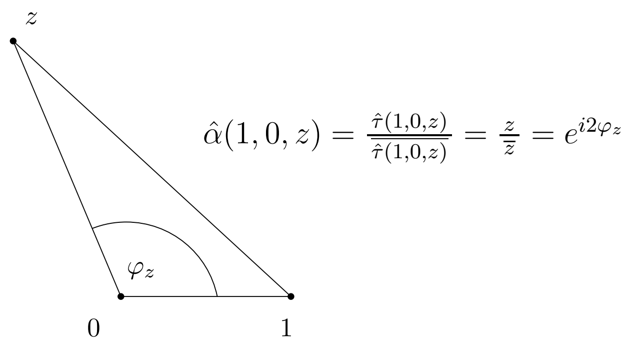

FIGURE 3

Definition 1.8. Define $\overline{X}_D$ to be the set of D-labeled triangles up to affine automorphisms, i.e., $\overline{X}_D = X_D / \mathrm{Aff}(1, \mathbb{C})$. Denote by $[\,\cdot\,]$ the residue class in $\overline{X}_D$.

We can associate to any D-labeled triangle x its *algebraic sides*, its *moduli* and its *double angles*. Among them certain relations hold which will be considered below. For simplicity of notation, we will write

$$\hat{\alpha}_{ijk}, \quad \hat{u}_{ij}, \quad \hat{\tau}_{ijk}$$

for

$$\hat{\alpha}(x_i, x_j, x_k), \quad \hat{u}(x_i, x_j), \quad \hat{\tau}(x_i, x_j, x_k),$$

respectively, if it is clear from the context which triangle is considered.

Lemma 1.9. *Let x be a D-labeled triangle. Then the following relations hold*

(1) $\hat{u}_{ij} = \hat{u}_{ji}, \quad i, j \in D,\ i \neq j$
(2) $\hat{\alpha}_{ijk} \cdot \hat{\alpha}_{jki} \cdot \hat{\alpha}_{kij} = 1,$
(3) $\hat{\alpha}_{ijk} = \hat{\alpha}_{kji}^{-1},$
(4) $\hat{\tau}_{jik} = \dfrac{1 - \hat{\alpha}_{ijk}}{1 - \hat{\alpha}_{ikj}}$

where in (2)–(4) *$i, j, k \in D$ are pairwise distinct.*

Proof. (1) is obvious. (2) follows immediately from the fact that the sum of the double angles of a triangle is 2π. The identity

$$\hat{\alpha}_{ijk} \cdot \hat{\alpha}_{kji} = \frac{\hat{u}_{kj}}{\hat{u}_{ij}} \cdot \frac{\hat{u}_{ij}}{\hat{u}_{kj}} = 1$$

shows (3). For (4) we may assume $x_i = 0$, $x_j = 1$ and $x_k = z$. Then $\hat{\tau}(1, 0, z) = z$. Furthermore one has

$$\begin{aligned}
\frac{1 - \hat{\alpha}(0,1,z)}{1 - \hat{\alpha}(0,z,1)} &= \frac{1 - \frac{\hat{u}(z,1)}{\hat{u}(0,1)}}{1 - \frac{\hat{u}(1,z)}{\hat{u}(0,z)}} \\
&= \frac{1 - \frac{z-1}{\overline{z}-1}}{1 - \frac{1-z}{1-\overline{z}} \cdot \frac{-\overline{z}}{-z}} \\
&= \frac{(\overline{z} - 1) - (z - 1)}{\overline{z} - 1} \cdot \frac{(1 - \overline{z})z}{(1 - \overline{z})z - (1 - z)\overline{z}} \\
&= \frac{\overline{z} - z}{\overline{z} - 1} \cdot \frac{(1 - \overline{z})z}{z - \overline{z}} = z = \hat{\tau}(1, 0, z)
\end{aligned}$$

□

Corollary 1.10. *Any D-labeled triangle x is, up to similarity, determined by the double angles $\hat{\alpha}_{ijk}$, $i, j, k \in D$, pairwise distinct.*

Proof. Due to Remark 1.6 the $\hat{\tau}_{ijk}$ determine the triangle and they can be expressed in terms of the double angles $\hat{\alpha}_{ijk}$ by 1.9(4). □

The following remark states an alternative proof of the corollary. Whereas the first proof is algebraic, this second one is geometrical and uses a well-known fact from Euclidean geometry, namely the Central Angle Theorem.

Remark 1.11. Let x be a triangle with points A, B, C, and let β and γ be two angles of x. Denote by M the center of its circumcircle. By the Central Angle Theorem the central angle is twice the peripherical angle, see Figure 4 on the left. Thus we receive a triangle similar to x when inscribing 2β and

2γ as central angles in an arbitrary circle as shown in Figure 4 on the right. Consequently, the *double angles* determine any triangle up to similarity.

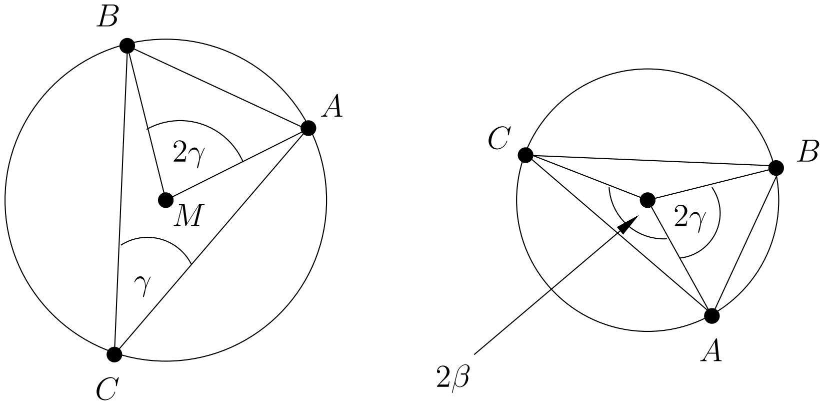

FIGURE 4

Example 1.12. Ceva's Theorem. This well-known theorem was first published in 1678 by Giovanni Ceva and states the following. Let $x = (x_1, x_2, x_3)$ be a triangle and let y_1, y_2, y_3 be three points lying on the sides of x as depicted in Figure 5. Denote by t_i the line through x_i and y_i for $i = 1, 2, 3$.

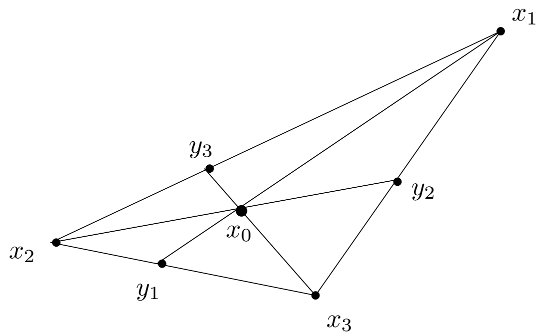

FIGURE 5

If t_1, t_2 and t_3 meet in a point x_0, then one has

$$\frac{|x_1 - y_3|}{|x_2 - y_3|} \cdot \frac{|x_2 - y_1|}{|x_3 - y_1|} \cdot \frac{|x_3 - y_2|}{|x_1 - y_2|} = 1$$

where $|\cdot|$ denotes the length of a side.

Here we do not give the converse statement, which also holds and which is usually considered as part of Ceva's Theorem.

We will prove the following lemma about moduli that obviously yields Ceva's Theorem.

Lemma 1.13. *One has the identity*

$$\hat{\tau}(x_2, y_3, x_1) \cdot \hat{\tau}(x_3, y_1, x_2) \cdot \hat{\tau}(x_1, y_2, x_3) = -1$$

Proof. First, we consider the three factors on the left hand side separately. Clearly,

$$\hat{\tau}(x_2, y_3, x_1) = \frac{\hat{\tau}(x_3, y_3, x_1)}{\hat{\tau}(x_3, y_3, x_2)} \tag{3}$$

From Lemma 1.9(4) we deduce for the numerator

$$\hat{\tau}(x_3, y_3, x_1) = \frac{1 - \hat{\alpha}(y_3, x_3, x_1)}{1 - \hat{\alpha}(y_3, x_1, x_3)} \tag{4}$$

Now consider the triangle consisting of the points y_3, x_3, x_1 in Figure 5. The angle at x_3 in this triangle is obviously the same as in the triangle consisting of x_0, x_3, x_1. Thence

$$\hat{\alpha}(y_3, x_3, x_1) = \hat{\alpha}(x_0, x_3, x_1)$$

Analogously, one has

$$\hat{\alpha}(y_3, x_1, x_3) = \hat{\alpha}(x_2, x_1, x_3)$$

Inserting these results in (4) yields

$$\hat{\tau}(x_3, y_3, x_1) = \frac{1 - \hat{\alpha}(x_0, x_3, x_1)}{1 - \hat{\alpha}(x_2, x_1, x_3)}$$

For the denominator in (3) we analogously get (by interchanging the indices 1 and 2)

$$\hat{\tau}(x_3, y_3, x_2) = \frac{1 - \hat{\alpha}(x_0, x_3, x_2)}{1 - \hat{\alpha}(x_1, x_2, x_3)}$$

Thus

$$\hat{\tau}(x_2, y_3, x_1) = \frac{1 - \hat{\alpha}(x_0, x_3, x_1)}{1 - \hat{\alpha}(x_2, x_1, x_3)} \cdot \frac{1 - \hat{\alpha}(x_1, x_2, x_3)}{1 - \hat{\alpha}(x_0, x_3, x_2)}$$

Since the situation at hand is completely symmetric, we receive analogous results for the other two factors of the claim. Namely,

$$\hat{\tau}(x_3, y_1, x_2) = \frac{1 - \hat{\alpha}(x_0, x_1, x_2)}{1 - \hat{\alpha}(x_0, x_1, x_3)} \cdot \frac{1 - \hat{\alpha}(x_2, x_3, x_1)}{1 - \hat{\alpha}(x_3, x_2, x_1)}$$

And

$$\hat{\tau}(x_1, y_2, x_3) = \frac{1 - \hat{\alpha}(x_0, x_2, x_3)}{1 - \hat{\alpha}(x_0, x_2, x_1)} \cdot \frac{1 - \hat{\alpha}(x_3, x_1, x_2)}{1 - \hat{\alpha}(x_1, x_3, x_2)}$$

From now on we write $\hat{\alpha}_{ijk}$ for $\hat{\alpha}(x_i, x_j, x_k)$ and $\hat{\tau}_{ijk}$ for $\hat{\tau}(x_i, x_j, x_k)$, where $0 \leqslant i, j, k \leqslant 0$. Consequently, using Lemma 1.9(4) one has

$$\begin{aligned}
& \hat{\tau}(x_2, y_3, x_1) \cdot \hat{\tau}(x_3, y_1, x_2) \cdot \hat{\tau}(x_1, y_2, x_3) \\
= \; & \frac{1 - \hat{\alpha}_{031}}{1 - \hat{\alpha}_{032}} \cdot \frac{1 - \hat{\alpha}_{123}}{1 - \hat{\alpha}_{213}} \cdot \frac{1 - \hat{\alpha}_{012}}{1 - \hat{\alpha}_{013}} \cdot \frac{1 - \hat{\alpha}_{231}}{1 - \hat{\alpha}_{321}} \cdot \frac{1 - \hat{\alpha}_{023}}{1 - \hat{\alpha}_{021}} \cdot \frac{1 - \hat{\alpha}_{312}}{1 - \hat{\alpha}_{132}} \\
= \; & \frac{1 - \hat{\alpha}_{031}}{1 - \hat{\alpha}_{013}} \cdot \frac{1 - \hat{\alpha}_{123}}{1 - \hat{\alpha}_{132}} \cdot \frac{1 - \hat{\alpha}_{012}}{1 - \hat{\alpha}_{021}} \cdot \frac{1 - \hat{\alpha}_{231}}{1 - \hat{\alpha}_{213}} \cdot \frac{1 - \hat{\alpha}_{023}}{1 - \hat{\alpha}_{032}} \cdot \frac{1 - \hat{\alpha}_{312}}{1 - \hat{\alpha}_{321}} \\
= \; & \hat{\tau}_{102} \cdot \hat{\tau}_{203} \cdot \hat{\tau}_{301} \cdot \hat{\tau}_{213} \cdot \hat{\tau}_{321} \cdot \hat{\tau}_{132}
\end{aligned}$$

Furthermore, one easily checks that

$$\hat{\tau}_{102} \cdot \hat{\tau}_{203} \cdot \hat{\tau}_{301} = 1$$

and

$$\hat{\tau}_{213} \cdot \hat{\tau}_{321} \cdot \hat{\tau}_{132} = -1$$

This finally yields the claim. $\square$

1.2. **Quadrangle Relations.**

Definition 1.14. Let Q be a set with $|Q| = 4$ and define X_Q to be the set of maps $x\colon Q \longrightarrow \mathbb{C}$ for which the following conditions hold, where we set $x_i := x(i)$.

- x is injective
- x_i, x_j, x_k are *not* collinear for i, j, $k \in Q$, pairwise distinct

We call an element $x \in X_Q$ a Q-labeled quadrangle or simply a quadrangle with points x_i, $i \in Q$.

Definition 1.15. Let $\overline{X}_Q = X_Q / \operatorname{Aff}(1, \mathbb{C})$ be the set of Q-labeled quadrangles up to affine automorphisms.

For a quadrangle $x \in X_Q$ there are again assigned $\hat{u}_{ij}$, $\hat{\tau}_{ijk}$ and $\hat{\alpha}_{ijk}$, where now i, j, k are pairwise distinct elements of Q.

Remark 1.16. We call a triangle consisting of 3 of the points of a Q-labeled quadrangle x a partial triangle of x. Then any Q-labeled quadrangle x is, up to similarity, determined by the partial triangles (given up to similarity). Namely, we may assume $x_i = 0$ and $x_j = 1$. Now knowledge of the partial triangle of x consisting of the points x_i, x_j, x_k determines x_k for $k \in Q$, $k \neq i, j$. One easily observes that already 2 partial triangles determine any Q-labeled quadrangle up to similarity.

So far, we have assigned to a triangle (quadrangle, respectively) several complex numbers. Our aim is now to use the algebraic sides to embed the set $\overline{X}_Q$ of quadrangles up to affine automorphisms into a group which is isomorphic to $\left(S^1\right)^n$, where $n \in \mathbb{N}$, $n \geqslant 1$.

For this purpose let R be the set of 2-element subsets of Q. For each $s \in R$ define a map

$$\begin{aligned} &\hat{u}_s\colon X_Q \longrightarrow S^1, \quad \text{by} \\ &\hat{u}_s(x) = \frac{x_i - x_j}{\overline{x_i - x_j}} = \hat{u}_{ij}(x), \qquad \text{for } s = \{i, j\}. \end{aligned}$$

This map is well-defined since $\hat{u}_{ij}(x) = \hat{u}_{ji}(x)$ for all $x \in X_Q$ and for all $i, j \in Q$, $i \neq j$.

Now define

$$U := S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^R$$

Then one has

$$U = S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^R = \left(S^1\right)^R$$

under the isomorphism

$$\begin{aligned} U \cong S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^6 &\longrightarrow \left(S^1\right)^6 \cong \left(S^1\right)^R \\ z \otimes (t_1, \ldots, t_6) &\longmapsto (z_1^{t_1}, \ldots, z_6^{t_6}) \end{aligned}$$

We combine the algebraic sides $\hat{u}_s$, $s \in R$, and get a map

$$\begin{aligned} u\colon X_Q &\longrightarrow U \\ x &\longmapsto u(x) = \left(\hat{u}_s(x)\right)_{s\in R} \end{aligned}$$

Now consider the following two diagonal maps

$$\begin{aligned} \Delta_{\mathbb{Z}}\colon \mathbb{Z} &\longrightarrow \mathbb{Z}^R \\ 1 &\longmapsto \sum_{s\in R} s \end{aligned}$$

where we use $\mathbb{Z}^R = \bigoplus_{s\in R} \mathbb{Z}\cdot s$, and

$$\begin{aligned} \Delta_{S^1}\colon S^1 &\longmapsto \left(S^1\right)^R \\ z &\longmapsto f_z \end{aligned}$$

where $f_z(s) = z$ for all $s \in R$. Obviously, both maps are group homomorphisms. We will denote both by Δ, if the context makes clear which diagonal map is considered. Define

$$V := S^1 \otimes_{\mathbb{Z}} \left(\mathbb{Z}^R/\Delta(\mathbb{Z})\right)$$

One checks that, for each $n \geqslant 1$, the following map is an isomorphism.

$$\begin{aligned} \left(S^1\right)^n/\Delta(S^1) &\longrightarrow S^1 \otimes_{\mathbb{Z}} \left(\mathbb{Z}^n/\Delta(\mathbb{Z})\right) \\ [(s_1,\ldots,s_n)] &\longmapsto \sum_{i=1}^{n} s_i \otimes [e_i] \end{aligned}$$

where $[\,\cdot\,]$, $[\,\cdot\,]$ denotes the residue class in $\left(S^1\right)^n/\Delta(S^1)$ and in $\mathbb{Z}^n/\Delta(\mathbb{Z})$, respectively. Thus one has

$$V = \left(S^1\right)^R/\Delta(S^1)$$

We wish to embed the set of Q-labeled quadrangles up to affine automorphisms $\overline{X}_Q$ into the group V. Therefore consider the map

$$\begin{aligned} v\colon \overline{X}_Q &\longrightarrow V \\ [x] &\longmapsto [u(x)] \end{aligned}$$

where $[\,\cdot\,]$, $[\,\cdot\,]$ denote the residue classes in $\overline{X}_Q$ and in V, respectively. We show that v is well-defined. If $[x] = [x']$ in $\overline{X}_Q$, then there exists an affine automorphism $g \in \operatorname{Aff}(1,\mathbb{C})$, say $g(z) = az + b$ where $a \in \mathbb{C}\setminus\{0\}$ and $b, z \in \mathbb{C}$, such that $g(x) = x'$. Then one has by equation (1)

$$\begin{aligned} \left(\hat{u}_s\big(g(x)\big)\right)_{s\in R} &= \left(\frac{a}{\overline{a}}\cdot \hat{u}_s(x)\right)_{s\in R} \\ &= \Delta_{S^1}\left(\frac{a}{\overline{a}}\right)\cdot\left(\hat{u}_s(x)\right)_{s\in R} \end{aligned}$$

Hence $\left[\left(\hat{u}_{s\in R}\left(g(x)\right)\right)\right] = \left[\left(\hat{u}_s(x)\right)_{s\in R}\right]$ in V. This proves that v is well-defined.

Lemma 1.17. *The map v is injective.*

Proof. Assume $v([x]) = v([x'])$ for two quadrangles $x, x' \in X_Q$. Then one has

$$\left[(\hat{u}_s(x))_{s\in R}\right] = v([x]) = v\left([x']\right) = \left[\left(\hat{u}_s\left(x'\right)\right)_{s\in R}\right]$$

in $V = \left(S^1\right)^R / \Delta\left(S^1\right)$. Hence there exists an element $t \in S^1$ such that $\hat{u}_s(x) = t \cdot \hat{u}_s(x')$ for all $s \in R$. This yields

$$\frac{\hat{u}_s(x)}{\hat{u}_{s'}(x)} = \frac{\hat{u}_s(x')}{\hat{u}_{s'}(x')} \qquad \text{for all } s,\ s' \in R$$

Therefore all double angles of x and x' coincide. According to Corollary 1.10, the double angles determine any triangle up to similarity. Hence all partial triangles of the quadrangles x and x' are similar. Due to Remark 1.16, this shows that x and x' are similar, which proves $[x] = [x']$ in $\overline{X}_Q$. □

For each ordered triple (ijk) with $i, j, k \in Q$, pairwise distinct, there is the function

$$\begin{aligned} \hat{\alpha}_{ijk} \colon X_Q &\longrightarrow S^1, \\ x &\longmapsto \hat{\alpha}_{ijk}(x) = \hat{\alpha}(x_i, x_j, x_k) \end{aligned}$$

Let $M_{\hat{\alpha}}$ be the set consisting of these functions for all ordered triples (ijk) as above.

Lemma 1.18. *Mark an element* $0 \in Q$. *The set* $M_{\hat{\alpha}}$ *is determined by the* 6 *elements* $\hat{\alpha}_{0ij}$, $i, j \in Q$, $i \neq j$. *Among these elements the following relations hold:*

$$\hat{\alpha}_{0ij} \cdot \hat{\alpha}_{0jk} \cdot \hat{\alpha}_{0ki} = \hat{\alpha}_{0ji} \cdot \hat{\alpha}_{0kj} \cdot \hat{\alpha}_{0ik} \tag{5}$$

$$(1 - \hat{\alpha}_{0ij}) \cdot (1 - \hat{\alpha}_{0jk}) \cdot (1 - \hat{\alpha}_{0ki}) = (1 - \hat{\alpha}_{0ji}) \cdot (1 - \hat{\alpha}_{0kj}) \cdot (1 - \hat{\alpha}_{0ik}) \tag{6}$$

Proof. Using 1.9(3) one has for $i, j \in Q$, $i \neq j$

$$\begin{aligned} \hat{\alpha}_{ij0} &= \hat{\alpha}_{0ji}^{-1} \\ \hat{\alpha}_{i0j} &= \frac{\hat{u}_{0j}}{\hat{u}_{i0}} \cdot \frac{\hat{u}_{ij}}{\hat{u}_{ij}} = \frac{\hat{u}_{ij}}{\hat{u}_{0i}} \cdot \frac{\hat{u}_{0j}}{\hat{u}_{ij}} = \hat{\alpha}_{0ij} \cdot \hat{\alpha}_{0ji}^{-1} \\ \hat{\alpha}_{ijk} &= \frac{\hat{u}_{j0}}{\hat{u}_{ij}} \cdot \frac{\hat{u}_{jk}}{\hat{u}_{0j}} = \hat{\alpha}_{0ji}^{-1} \cdot \hat{\alpha}_{0jk}^{-1}, \quad (i, j, k \in Q \setminus \{0\}) \end{aligned}$$

Therefore $M_{\hat{\alpha}}$ is determined by the $\hat{\alpha}_{0ij}$, $i, j \in Q$, $i \neq j$. Equation (5) follows directly from the definition of the double angles. One has

$$1 = \hat{\tau}_{102} \cdot \hat{\tau}_{203} \cdot \hat{\tau}_{301} = \frac{1 - \hat{\alpha}_{012}}{1 - \hat{\alpha}_{021}} \cdot \frac{1 - \hat{\alpha}_{023}}{1 - \hat{\alpha}_{032}} \cdot \frac{1 - \hat{\alpha}_{031}}{1 - \hat{\alpha}_{013}}$$

by the definition of the modulus and by Lemma 1.9(4). This shows (6). □

Let x be a Q-labeled quadrangle. Due to relation (5) already five of the double angles $\hat{\alpha}_{0ij}(x)$, $i, j \in Q$, $i \neq j$, determine all double angles of x.

Definition 1.19. For a quadrangle $x \in X_Q$ define the *cross ratios of* x to be

$$\hat{\gamma}(x_i, x_j, x_k, x_l) := \hat{\gamma}_{ijkl} := \hat{\alpha}_{ijk} \cdot \hat{\alpha}_{kli} = \frac{\hat{u}_{jk} \cdot \hat{u}_{il}}{\hat{u}_{ij} \cdot \hat{u}_{kl}}$$

where i, j, k, l are pairwise distinct elements of Q.

It follows directly from the definition that $\hat{\gamma}(x_i, x_j, x_k, x_l)$ is $\mathrm{Aff}(1,\mathbb{C})$-invariant, because the double angles are invariant under $\mathrm{Aff}(1,\mathbb{C})$. For each ordered quadruple $(ijkl)$ with $i, j, k, l \in Q$, pairwise distinct, there is the function

$$\begin{aligned}\hat{\gamma}_{ijkl}\colon X_Q &\longrightarrow S^1,\\ x &\longmapsto \hat{\gamma}_{ijkl}(x) = \hat{\gamma}(x_i, x_j, x_k, x_l)\end{aligned}$$

Denote by $M_{\hat{\gamma}}$ the set of these functions for all ordered quadruples $(ijkl)$ as above.

For each triple (ijk) with $i, j, k \in Q$, pairwise distinct, we do not only have the function $\hat{\alpha}_{ijk}\colon X_Q \longrightarrow S^1$, but also the function $\hat{\tau}_{ijk}\colon X_Q \longrightarrow S^1$, $\hat{\tau}_{ijk}(x) = \hat{\tau}(x_i, x_j, x_k)$. Since $\hat{\alpha}_{ijk}$ and $\hat{\tau}_{ijk}$ are $\mathrm{Aff}(1,\mathbb{C})$-invariant (cp. p. 2), we get functions $\hat{\tau}_{ijk}$, $\hat{\alpha}_{ijk}\colon \overline{X}_Q \longrightarrow S^1$. The same holds for $\hat{\gamma}_{ijkl}\colon \overline{X}_Q \longrightarrow S^1$.

For $s \in R$ let $u_s \in \mathrm{Hom}\left(\mathbb{Z}^R, \mathbb{Z}\right)$ be the projection, namely

$$\begin{aligned}u_s\colon \mathbb{Z}^R &\longrightarrow \mathbb{Z}\\ f &\longmapsto f(s)\end{aligned}$$

Define

$$\alpha_{ijk} := u_{\{j,k\}} - u_{\{i,j\}}$$

where i, j, k are pairwise distinct elements of Q. Obviously, one has $\Delta(\mathbb{Z}) \subset \mathrm{Ker}(\alpha_{ijk})$. Hence $\alpha_{ijk} \in \mathrm{Hom}\left(\mathbb{Z}^R/\Delta(\mathbb{Z}), \mathbb{Z}\right)$.

Set

$$M_\alpha := \{\, \alpha_{ijk} \mid i, j, k \in Q \text{ pairwise distinct} \,\}$$

Then M_α has 24 elements.

Furthermore, for pairwise distinct elements $i, j, k, l \in Q$ define

$$\gamma_{ijkl} := \alpha_{ijk} + \alpha_{kli} \quad \in \mathrm{Hom}\left(\mathbb{Z}^R/\Delta(\mathbb{Z}), \mathbb{Z}\right)$$

Set

$$M_\gamma = \{\, \gamma_{ijkl} \mid i, j, k, l \in Q \text{ pairwise distinct} \,\}$$

One easily finds

$$\gamma_{ijkl} = \gamma_{jilk} = \gamma_{klij} = \gamma_{lkji}$$

for all pairwise distinct $i, j, k, l \in Q$. Marking again an element $0 \in Q$, we thus get

$$M_\gamma = \{\, \gamma_{0ijk} \mid \text{pairwise distinct}\, i, j, k \in Q \setminus \{0\} \,\}$$

Hence M_γ has 6 elements.

The double angles $\hat{\alpha}_{ijk}$ are closely related to the maps α_{ijk} as the next lemma shows.

Lemma 1.20. *For pairwise distinct elements $i, j, k \in Q$ consider the composition of maps*

$$\left(\mathrm{id}_{S^1} \otimes \alpha_{ijk}\right) \circ v\colon \overline{X}_Q \longrightarrow S^1 \otimes_{\mathbb{Z}} \mathbb{Z}$$

where v is the map defined on p. 9. And consider the map $\hat{\alpha}_{ijk}\colon \overline{X}_Q \longrightarrow S^1$ in terms of the double angles. Then one has

$$\hat{\alpha}_{ijk} \equiv \left(\mathrm{id}_{S^1} \otimes \alpha_{ijk}\right) \circ v$$

using the canonical isomorphism $S^1 \longrightarrow S^1 \otimes_{\mathbb{Z}} \mathbb{Z}$, $\quad s \longmapsto s \otimes 1$.

Proof. Let $[x] \in \overline{X}_Q$ and let i, j, k be pairwise distinct elements of Q. One finds

$$\begin{aligned}\big(\big(\mathrm{id}_{S^1} \otimes \alpha_{ijk}\big) \circ v\big)([x]) &= \big(\mathrm{id}_{S^1} \otimes \alpha_{ijk}\big)\left(\big[(\hat{u}_s(x))_{s\in R}\big]\right) \\ &= \big(\mathrm{id}_{S^1} \otimes \alpha_{ijk}\big)\left(\sum_{s\in R} \hat{u}_s(x) \otimes [s]\right) \\ &= \hat{u}_{jk}(x) \otimes 1 - \hat{u}_{ij}(x) \otimes 1 \\ &= \frac{\hat{u}_{jk}(x)}{\hat{u}_{ij}(x)} \otimes 1 \\ &= \hat{\alpha}_{ijk}([x])\end{aligned}$$

where we use the canonical isomorphism $S^1 \cong S^1 \otimes_{\mathbb{Z}} \mathbb{Z}$ in the last step. $\square$

Corollary 1.21. *$M_{\hat{\alpha}}$ has 24 elements.*

Proof. One has $|M_\alpha| = 24$. Due to Lemma 1.20, it is sufficient to prove that $\hat{\alpha}_{djk} \not\equiv \hat{\alpha}_{hmn}$ for all $d, j, k, h, m, n \in Q$ where d, j, k are pairwise distinct, as well as h, m, n are pairwise distinct. Assume $\hat{\alpha}_{djk} \equiv \hat{\alpha}_{hmn}$, then one has

$$\hat{\alpha}_{djk}([x]) = \hat{\alpha}_{hmn}([x]) \qquad \forall\ [x] \in \overline{X}_Q \tag{7}$$

Thus this equation holds for $[x] \in \overline{X}_Q$ with points $x_d = 0$, $x_j = 1$, $x_k = i$ and $x_l = 2+2i$, where i denotes the imaginary unit. Easy calculations yield that

$$\hat{\alpha}_{hmn} \equiv \hat{\alpha}_{jdl} \ \text{ or } \ \hat{\alpha}_{hmn} \equiv \hat{\alpha}_{ldk} \ \text{ or } \ \hat{\alpha}_{hmn} \equiv \hat{\alpha}_{jkd} \ \text{ or } \ \hat{\alpha}_{hmn} \equiv \hat{\alpha}_{djk}$$

Now consider the Q-labeled quadrangle y with points $y_d = 0$, $y_j = 1$, $y_k = 2+2i$ and $y_l = i$. Equation (7) also holds for y. But one easily observes that $\hat{\alpha}_{hmn}([y])$ does not coincide with either $\hat{\alpha}_{jdl}([y])$ or $\hat{\alpha}_{ldk}([y])$ or $\hat{\alpha}_{jkd}([y])$. This completes the proof. $\square$

Not only are the double angles $\hat{\alpha}_{ijk}$ and the maps α_{ijk} closely related, but so are the cross ratios $\hat{\gamma}_{ijkl}$ and the maps γ_{ijkl}.

Lemma 1.22. *For pairwise distinct elements $i, j, k, l \in Q$ consider the composition of maps*

$$\big(\mathrm{id}_{S^1} \otimes \gamma_{ijkl}\big) \circ v \colon \overline{X}_Q \longrightarrow S^1 \otimes_{\mathbb{Z}} \mathbb{Z}$$

where v is the map defined on p. 9. And consider the map $\hat{\gamma}_{ijkl} \colon \overline{X}_Q \longrightarrow S^1$ in terms of the cross ratios. Then one has

$$\hat{\gamma}_{ijkl} \equiv \big(\mathrm{id}_{S^1} \otimes \gamma_{ijkl}\big) \circ v$$

using the canonical isomorphism $S^1 \cong S^1 \otimes_{\mathbb{Z}} \mathbb{Z}$.

Proof. This follows from Lemma 1.20, using the fact that one has $\hat{\gamma}_{ijkl} = \hat{\alpha}_{ijk} \cdot \hat{\alpha}_{kli}$ and $\gamma_{ijkl} = \alpha_{ijk} + \alpha_{kli}$ by definition. $\square$

Corollary 1.23. *$M_{\hat{\gamma}}$ has 6 elements. The considerations above for M_γ yield $M_{\hat{\gamma}} = \{\, \hat{\gamma}_{0ijk} \mid$ pairwise distinct $i, j, k \in Q \setminus \{0\} \,\}$, where $0 \in Q$.*

Proof. Due to Lemma 1.22, one has $|M_{\hat{\gamma}}| \leqslant |M_\gamma| = 6$. Assume that $\hat{\gamma}_{0jkl} \equiv \hat{\gamma}_{0hmn}$ for pairwise distinct $0, j, k, l \in Q$ and pairwise distinct $0, h, m, n \in Q$. Then

$$\hat{\gamma}_{0jkl}([x]) = \hat{\gamma}_{0hmn}([x]) \qquad \forall\ [x] \in \overline{X}_Q$$

This equation especially holds for $x \in X_Q$ with points $x_0 = 0$, $x_j = 1$, $x_k = i$ and $x_l = 2 + 2i$. Easy calculations show that

$$\hat{\gamma}_{0hmn} \equiv \hat{\gamma}_{0ljk} \text{ or } \hat{\gamma}_{0hmn} \equiv \hat{\gamma}_{0jkl}$$

But for the Q-labeled quadrangle y with points $y_0 = 0$, $y_j = 1$, $y_k = 2 + 2i$ and $y_l = i$ one has $\hat{\gamma}_{0jkl}([y]) \neq \hat{\gamma}_{0ljk}([y])$. Thus $\hat{\gamma}_{0hmn} \equiv \hat{\gamma}_{0jkl}$. This shows that $|M_{\hat{\gamma}}| = 6$. □

1.3. **The Pedal Triangle.** We will now consider the so-called *pedal triangle* construction, which will later on yield an action on a subset of the set of quadrangles up to affine automorphisms.

In [1], one finds the following exercise (cf. p. 16, ex. 12).

Given a triangle ABC and a point P in its plane (but not on a side nor on the circumcircle), let $A_1B_1C_1$ be the derived triangle formed by the feet of the perpendiculars from P to the sides BC, CA, AB. Let $A_2B_2C_2$ be derived analogously from $A_1B_1C_1$ (using the same P), and $A_3B_3C_3$ from $A_2B_2C_2$. Then $A_3B_3C_3$ is directly similar to ABC. $\big($Hint: $\angle PBA = \angle PA_1C_1 = \angle PC_2B_2 = \angle PB_3A_3\big)$

During this section, let $D = \{1, 2, 3\}$ and $Q = \{0, 1, 2, 3\}$.

Definition 1.24. Given a D-labeled triangle x and an additional point x_0. Then the *pedal triangle of x relative to point x_0* is defined as

$$\phi_{x_0}(x) := x' = (x'_1, x'_2, x'_3)$$

where x'_i, $1 \leqslant i \leqslant 3$, is the foot of the perpendicular from x_0 to the side x_kx_j with $i \notin \{k, j\}$. See Figure 6.

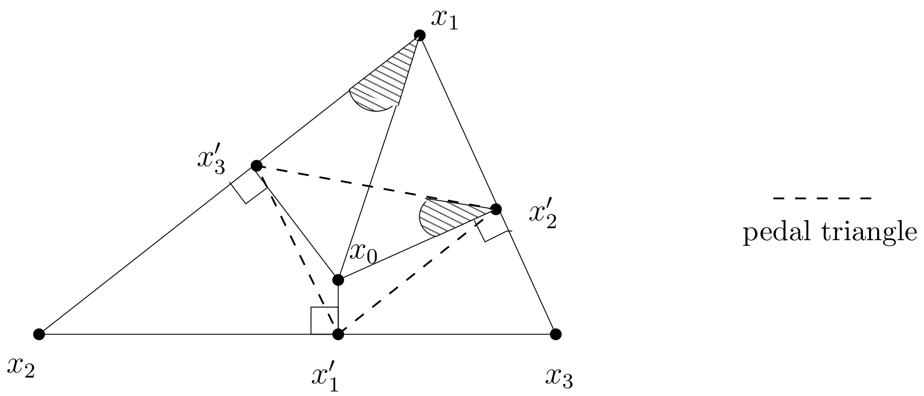

FIGURE 6. The pedal triangle construction

We summarise some properties of this construction. In the next three lemmas, let x be a D-labeled triangle and let x_0 be an additional point.

Lemma 1.25. *The points of the pedal triangle of x relative to x_0 are collinear if and only if x_0 lies on the circumcircle of x.*

Proof. Cf. [2], Th. 2.51 □

Lemma 1.26. *If x_0 does not lie on the circumcircle of x, then it does not lie on the circumcircle of the pedal triangle of x relative to x_0.*

Proof. Suppose the point x_0 does not lie on the circumcircle of x, but x_0 lies on the circumcircle of the pedal triangle $\phi_{x_0}(x)$. For $k \in D$ denote by g_k the line through x'_k and x_0 (cf. Figure 7). We construct the triangle x (with points x_1, x_2, x_3) from x_0 and $\phi_{x_0}(x)$. From the definition of the pedal triangle we deduce that, for each $i \in D$, x_i is the intersection of the perpendiculars to g_k, $k \in D \setminus \{i\}$, through the point x'_k. Hence the side x_3x_0 is the hypothenuse of the right triangle with points x_0, x'_1, x_3 as well as of the triangle with points x_0, x'_2, x_3. Due to Thales' Theorem the four points x_0, x'_1, x'_2, x_3 lie on a circle with diameter x_3x_0. It is easy to see that this is the circumcircle of the pedal triangle. Since x'_3 lies on this circle, we deduce from Thales' Theorem that the triangle with points x_0, x_3, x'_3 is a right triangle with the right angle at x'_3. But x_3 is the intersection of the 3 perpendiculars to g_1, g_2 and g_3 through the points x'_1, x'_2, x'_3 respectively. Hence $x_1 = x_2 = x_3$, which is a contradiction, because x is a D-labeled triangle. □

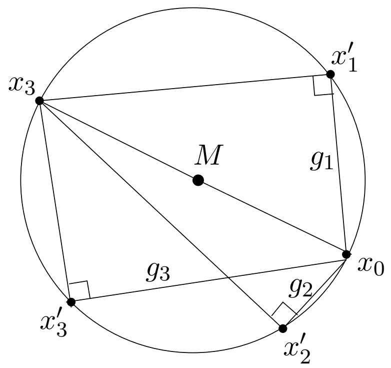

FIGURE 7

Lemma 1.27. *If x_1, x_2, x_3 are not collinear, then the point x_0 is not collinear with any two points of x'.*

Proof. It is sufficient to show that x_1, x_2, x_3 are collinear, if x_0, x'_1, x'_2 are collinear. Since the sides x_3x_2 and $x_0x'_1$ are orthogonal as well as the sides x_3x_1 and $x_0x'_2$, one deduces from the collinearity of the points x_0, x'_1, x'_2 that x_1, x_2, x_3 are collinear. □

We can iterate the construction and consider the pedal triangle of the pedal triangle, again relative to x_0. We call it the second pedal triangle of x with respect to x_0 and write $\phi^2_{x_0}(x)$. In general, we denote by $\phi^n_{x_0}(x)$ the n-th pedal triangle of x with respect to the point x_0.

Remark 1.28. If x_0 lies on a side of x, then x_0 itself is a vertex of $\phi_{x_0}(x)$, whence the second pedal triangle $\phi^2_{x_0}(x)$ has only two points.

The following lemma gives a proof of the exercise above. (Part of this proof can be found in [2], Th. 1.92.)

Lemma 1.29. *Let $y = (x_0, x_1, x_2, x_3)$ be a Q-labeled quadrangle such that x_0 does not lie on the circumcircle of the triangle $x = (x_1, x_2, x_3)$. Consider the third pedal triangle $\phi^3_{x_0}(x) = (x_1''', x_2''', x_3''')$ relative to x_0. Then the quadrangles y and $(x_0, x_1''', x_2''', x_3''')$ are similar under a similarity with fixed point x_0.*

Proof. We begin by showing that the triangles x and $\phi^3_{x_0}(x)$ are similar. Consider Figure 6. Setting $\beta := \angle x_0x_1x_2$ and $\delta := \angle x_0x_2'x_3'$, we firstly prove the claim $\beta = \delta$. Since the triangles (x_3', x_0, x_1) and (x_2', x_0, x_1) are both right triangles with the same hypothenuse x_0x_1, we deduce from Thales' Theorem that the four points x_0, x_1, x_2', x_3' all lie on a circle, namely the circumcircle of the above triangles. Hence the angles β and δ belong to the same chord x_0x_3'. Due to the Inscribed Angle Theorem, β and δ are equal. It follows that

$$\angle x_0x_1x_2 = \angle x_0x_2'x_3' = \angle x_0x_3''x_1'' = \angle x_0x_1'''x_2''' \tag{8}$$

Thus one in general has

$$\angle x_0x_jx_k = \angle x_0x_j'''x_k''' \tag{9}$$

for pairwise distinct $i,\ j,\ k,\quad 1 \leqslant i,j,k \leqslant 3$. One furthermore has

$$\angle x_ix_0x_j = \angle x_i'''x_0x_j'''$$

for $1 \leqslant i,j \leqslant 3,\ i \neq j$. This can be easily deduced from equation (9). Namely, one has

$$\begin{aligned}\angle x_ix_0x_j &= \pi - \angle x_0x_ix_j - \angle x_0x_jx_i \\ &= \pi - \angle x_0x_i'''x_j''' - \angle x_0x_j'''x_i''' \\ &= \angle x_i'''x_0x_j'''\end{aligned}$$

for all $1 \leqslant i,j \leqslant 3\ i \neq j$. Thus all corresponding angles are the same. $\square$

Definition 1.30. Let $Y_Q \subset X_Q$ be the set of Q-labeled quadrangles y such that for each $i \in Q$ holds

- y_i does not lie on the circumcircle of the triangle with points y_j, y_k, y_l, where $i \notin \{j,k,l\}$

Define $\overline{Y}_Q := Y_Q/\operatorname{Aff}(1,\mathbb{C})$.

Corollary 1.31. *For each $i \in Q$, there is the pedal triangle construction relative to i*

$$\begin{aligned}\phi_i : \overline{Y}_Q &\longrightarrow \overline{Y}_Q \\ [y] &\longmapsto [(y_i, y_j', y_k', y_l')]\end{aligned}$$

where (y_j', y_k', y_l') is the pedal triangle relative to y_i of the triangle consisting of the three points y_j, y_k, y_l. One has $\phi_i^3 \equiv \mathrm{id}_{\overline{Y}_Q}$.

Proof. Without loss of generality, we may assume $i = 0$. We show that the map ϕ_0 is well defined. Let $y \in Y_Q$. According to Lemmas 1.25, 1.26, 1.27 and Remark 1.28, one has $\phi_0^n(y) \in Y_Q$ for all $n \geqslant 1$.

For $g \in \mathrm{Aff}(1, \mathbb{C})$ all corresponding angles of the quadrangles y and $g(y)$ coincide. Due to equation (8), one in general has

$$\begin{aligned}\angle x_0 x_l' x_j' &= \angle x_0 x_k x_l \\ &= \angle g(x_0) g(x_k) g(x_l) \\ &= \angle g(x_0) \left(g(x_l)\right)' \left(g(x_j)\right)'\end{aligned}$$

where $j, k, l \in Q \setminus \{0\}$ are pairwise distinct. Hence one also has

$$\angle x_j' x_0 x_k' = \angle \left(g(x_j)\right)' g(x_0) \left(g(x_k)\right)'$$

for all $j, k \in Q \setminus \{0\}$, $j \neq k$. This shows that all corresponding angles of the quadrangles $\phi_0(y)$ and $\phi_0\big(g(y)\big)$ coincide. Hence all partial triangles are similar. By Remark 1.16 the quadrangles $\phi_0(y)$ and $\phi_0\big(g(y)\big)$ are similar. This shows that ϕ_0 is well defined. The other assertion has already been proven in Lemma 1.29. □

We wish to understand what happens with the double angles and the cross ratios of a quadrangle in $\overline{Y}_Q$ when applying the pedal triangle construction. Therefore we define

$$\hat{M} := M_{\hat{\alpha}} \cup M_{\hat{\gamma}}$$

Then $\hat{M}$ contains 30 elements according to Corollaries 1.21 and 1.23. For $x \in X_Q$ define the set $\hat{M}(x)$ as the union of the two sets $M_{\hat{\alpha}}(x)$ and $M_{\hat{\gamma}}(x)$, where

$$M_{\hat{\alpha}}(x) := \left\{ \left(\hat{\alpha}_{ijk}(x)\right)_{ijk} \;\middle|\; i, j, k \in Q \text{ pairwise distinct} \right\}$$

and

$$M_{\hat{\gamma}}(x) := \left\{ \left(\hat{\gamma}_{0ijk}(x)\right)_{0ijk} \;\middle|\; i, j, k \in Q \setminus \{0\} \text{ pairwise distinct} \right\}$$

where 0 is an element of Q.

Lemma 1.32. *Let $y = (x_0, x_1, x_2, x_3) \in \overline{Y}_Q$ and consider the pedal triangle construction ϕ_0. Then one finds*

$$\hat{M}(y) = \hat{M}(\phi_0(y)),$$

i.e., the pedal triangle construction relative to x_0 leaves the set $\hat{M}$ invariant.

Proof. Set $y' := \phi_0(y)$. For ijk a permutation of 123 one has

(1) $\hat{\alpha}_{0ij}(y') = \hat{\alpha}_{0jk}(y)$ after equation 1.29(8)

(2) $\hat{\alpha}_{ij0}(y') = (\hat{\alpha}_{0ji}(y'))^{-1} \overset{(1)}{=} (\hat{\alpha}_{0ik}(y))^{-1} = \hat{\alpha}_{ki0}(y)$

(3) $\hat{\alpha}_{ijk}(y') = \hat{\alpha}_{0jk}(y') \cdot \hat{\alpha}_{ij0}(y') \overset{(1)}{=} \hat{\alpha}_{0ki}(y) \cdot \hat{\alpha}_{ki0}(y) = \hat{\alpha}_{k0i}(y)$

(4) $\hat{\alpha}_{i0j}(y') = \hat{\alpha}_{0ij}(y') \cdot \hat{\alpha}_{ij0}(y') \overset{(1)}{=} \hat{\alpha}_{0jk}(y) \cdot \hat{\alpha}_{ki0}(y) = \hat{\gamma}_{0jki}(y)$

(5) $\hat{\gamma}_{0ijk}(y') = \hat{\alpha}_{0ij}(y') \cdot \hat{\alpha}_{jk0}(y') \overset{(1)}{=} \hat{\alpha}_{0jk}(y) \cdot \hat{\alpha}_{ij0}(y) = \hat{\alpha}_{ijk}(y)$

where we use Lemma 1.18 in (2) to (5). Comparing the left hand sides of the equations above with the right hand sides yields $\hat{M}(y) = \hat{M}(y')$. □

Remark 1.33. Furthermore we deduce from Lemma 1.32(4) that, in general, one has $M_{\hat{\alpha}}(y') \neq M_{\hat{\alpha}}(y)$. This means that the pedal triangle construction does not leave invariant the set of double angles of a given triangle. But when we identify the group $\mathrm{Aut}\left(\hat{M}\right)$ of bijections of $\hat{M}$ with the symmetric group S_{30}, then Lemma 1.32 shows the existence of an element in S_{30} that corresponds to the pedal triangle construction.

2. An action of $S_6 \times \{\pm 1\}$ on $\mathbb{Z}^R/\Delta(\mathbb{Z})$

Let $G = \mathrm{Aut}(Q)$ be the group of bijections of Q. Hence $G \cong S_4 := \mathrm{Aut}(\{0,1,2,3\})$.

Definition 2.1. (1) An ordering $(i_0 i_1 i_2 i_3)$ of Q is an enumeration i_0, i_1, i_2, i_3 of the elements of Q. In other words, an ordering is a bijection $\{0,1,2,3\} \longrightarrow Q$.
(2) A cyclic ordering of Q is an ordering of Q defined up to a cyclic permutation. Cyclic orderings are denoted by symbols $[i_0 i_1 i_2 i_3]$. These symbols are subject to the relations

$$[i_0 i_1 i_2 i_3] = [i_1 i_2 i_3 i_0] = [i_2 i_3 i_0 i_1] = [i_3 i_0 i_1 i_2]$$

One can "normalise" cyclic orderings by choosing an element $i \in Q$ and by writing cyclic orderings starting with i, namely $[ijkl]$ where $i, j, k, l \in Q$ are pairwise distinct. Then this representation is uniquely determined by the cyclic ordering.
(3) An orientation of Q is an ordering of Q well defined up to an even permutation. Orientations are denoted by symbols

$$\langle i_0 i_1 i_2 i_3 \rangle$$

They are subject to the relations

$$\langle i_0 i_1 i_2 i_3 \rangle = \left\langle i_{\sigma(0)} i_{\sigma(1)} i_{\sigma(2)} i_{\sigma(3)} \right\rangle$$

for any element σ of the alternating group A_4 .

Denote by $\mathcal{C} = \mathcal{C}_Q$ the set of cyclic orderings of Q, and by $\mathcal{O} = \mathcal{O}_Q$ the set of orientations of Q. Then one has $|\mathcal{C}_Q| = 6$ and $|\mathcal{O}_Q| = 2$. Consider the following diagonal maps

$$\begin{aligned} \Delta_{\mathbb{Z}}^{R}\colon \mathbb{Z} &\longrightarrow \mathbb{Z}^R, \\ 1 &\longmapsto \sum_{r \in R} 1 \cdot r \end{aligned} \qquad \begin{aligned} \Delta_{\mathbb{Z}}^{\mathcal{C}}\colon \mathbb{Z} &\longrightarrow \mathbb{Z}^{\mathcal{C}} \\ 1 &\longmapsto \sum_{c \in \mathcal{C}} 1 \cdot c \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Delta_{\mathbb{Z}}^{\mathcal{O}}\colon \mathbb{Z} &\longrightarrow \mathbb{Z}^{\mathcal{O}} \\ 1 &\longmapsto \sum_{q \in \mathcal{O}} 1 \cdot q \end{aligned}$$

Obviously, these maps are G-module homomorphisms induced by the canonical action of G on R, $\mathcal{C}$ and $\mathcal{O}$. They induce a natural action of G on the cokernels of $\Delta_{\mathbb{Z}}^{R}$, $\Delta_{\mathbb{Z}}^{\mathcal{C}}$ and of $\Delta_{\mathbb{Z}}^{\mathcal{O}}$. We will denote all diagonal maps by Δ, if the context prevents confusion.

Set

$$\mathbb{Z}(Q) := \mathbb{Z}^{\mathcal{O}}/\Delta(\mathbb{Z})$$

$\mathbb{Z}(Q)$ is the free abelian group of rank 1 with generator $[\sigma_1] = -[\sigma_2]$, where σ_1, σ_2 are the orientations of Q. The G-action on $\mathbb{Z}^{\mathcal{O}}$ yields the following

G-action on $\mathbb{Z}(Q)$ via the map

$$\mathrm{sgn}\colon G \longrightarrow \{\pm 1\}, \quad \mathrm{sgn}(g) = \begin{cases} 1 & \text{if g is an even permutation} \\ -1 & \text{else} \end{cases}$$

where we use $G \cong S_4$. For $[\sigma_1] \in \mathbb{Z}(Q)$ and $g \in G$, one has

$$g \cdot [\sigma_1] = \mathrm{sgn}(g)[\sigma_1] \tag{10}$$

Lemma 2.2. *There exists a unique homomorphism of G-modules*

$$\widetilde{\Psi}\colon \mathbb{Z}^R \longrightarrow \Big(\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})\Big) \otimes_{\mathbb{Z}} \mathbb{Z}(Q)$$

with

$$\widetilde{\Psi}(\{i,j\}) = \big[\,[iljk] + [iklj] + [ijlk]\,\big] \otimes [\langle ijkl\rangle] \tag{11}$$

for any ordering $(ijkl)$ of Q, where $[\,\cdot\,]$, $[\,\cdot\,]$ denotes the residue class in $\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})$ and $\mathbb{Z}(Q)$, respectively.

Proof. First, we show that $\widetilde{\Psi}$ is well-defined. Denote by $\widetilde{\widetilde{\Psi}}\big((ijkl)\big)$ the right hand side of (11). We show that $\widetilde{\widetilde{\Psi}}\big((ijkl)\big)$ only depends on the subset $\{i,j\}$ of Q. For the ordering $(jikl)$ of Q, one finds

$$\widetilde{\widetilde{\Psi}}\big((jikl)\big) = \big[\,[jlik] + [jkli] + [jilk]\,\big] \otimes [\langle jikl\rangle] \tag{12}$$

We use $[\langle jikl\rangle] = -[\langle ijkl\rangle]$ and normalise the cyclic orderings with $i \in Q$. Subtracting (12) from (11), we get

$$\begin{aligned}
&\widetilde{\widetilde{\Psi}}\big((ijkl)\big) - \widetilde{\widetilde{\Psi}}\big((jikl)\big) \\
&= \big[\,[iljk] + [iklj] + [ijlk]\,\big] \otimes [\langle ijkl\rangle] \\
&\quad - \big[\,[ikjl] + [ijkl] + [ilkj]\,\big] \otimes \big(-[\langle ijkl\rangle]\big) \\
&= \big[\,[iljk] + [iklj] + [ijlk] + [ikjl] + [ijkl] + [ilkj]\,\big] \otimes [\langle ijkl\rangle] \\
&= \big[\,\Delta(1)\,\big] \otimes [\langle ijkl\rangle] \\
&= 0
\end{aligned}$$

Hence, interchanging i and j does not change (11). Now consider the ordering $(ijlk)$.

$$\widetilde{\widetilde{\Psi}}\big((ijlk)\big) = \big[\,[ikjl] + [ilkj] + [ijkl]\,\big] \otimes [\langle ijlk\rangle] \tag{13}$$

One has

$$\begin{aligned}
&\widetilde{\widetilde{\Psi}}\big((ijkl)\big) - \widetilde{\widetilde{\Psi}}\big((ijlk)\big) \\
&= \big[\,[iljk] + [iklj] + [ijlk]\,\big] \otimes [\langle ijkl\rangle] \\
&\quad - \big[\,[ikjl] + [ilkj] + [ijkl]\,\big] \otimes \big(-[\langle ijkl\rangle]\big) \\
&= \big[\,[iljk] + [iklj] + [ijlk] + [ikjl] + [ilkj] + [ijkl]\,\big] \otimes [\langle ijkl\rangle] \\
&= \big[\,\Delta(1)\,\big] \otimes [\langle ijkl\rangle] \\
&= 0
\end{aligned}$$

Thus interchanging k and l does not change (11). From these two observations follows immediately that for the ordering $(jilk)$ one also has

$$\widetilde{\widetilde{\widetilde{\Psi}}}\big((jilk)\big) = \widetilde{\widetilde{\widetilde{\Psi}}}\big((ijkl)\big)$$

This shows that $\widetilde{\widetilde{\widetilde{\Psi}}}\big((ijkl)\big)$ depends only on the subset $\{i,j\}$ of Q. Hence $\widetilde{\Psi}$ is well defined. Furthermore, it is easy to see that $\widetilde{\Psi}$ is G-equivariant. □

Remark 2.3. It will be useful to calculate $\widetilde{\Psi}(t)$ for an arbitrary element $t \in \mathbb{Z}^R$. Let $t = \sum_{r\in R} t_r \cdot r$, where $t_r \in \mathbb{Z}$ for all $r \in R$. Then one has

$$\begin{aligned}
&\widetilde{\Psi}\left(\sum_{r\in R} t_r \cdot r\right)\\
&= \Big[\, t_{\{i,j\}} \cdot \Big([iljk] + [iklj] + [ijlk]\Big) + t_{\{k,l\}} \cdot \Big([ikjl] + [ijlk] + [iklj]\Big)\\
&\quad - t_{\{i,l\}} \cdot \Big([ijlk] + [ikjl] + [iljk]\Big) - t_{\{i,k\}} \cdot \Big([ilkj] + [ijlk] + [iklj]\Big)\\
&\quad - t_{\{j,k\}} \cdot \Big([iklj] + [iljk] + [ikjl]\Big) - t_{\{j,l\}} \cdot \Big([ijkl] + [ijlk] + [iklj]\Big)\,\Big]\\
&\quad \otimes [\langle ijkl\rangle]\\
&= \Big[- t_{\{j,l\}} \cdot [ijkl] + \Big(t_{\{i,j\}} - t_{\{i,l\}} - t_{\{j,k\}}\Big) \cdot [iljk] + \Big(t_{\{i,j\}} + t_{\{k,l\}} - t_{\{i,k\}}\\
&\quad - t_{\{j,k\}} - t_{\{j,l\}}\Big) \cdot [iklj] + \Big(t_{\{i,j\}} + t_{\{k,l\}} - t_{\{i,l\}} - t_{\{i,k\}} - t_{\{j,l\}}\Big) \cdot [ijlk]\\
&\quad + \Big(t_{\{k,l\}} - t_{\{i,l\}} - t_{\{j,k\}}\Big) \cdot [ikjl] - t_{\{i,k\}} \cdot [ilkj]\,\Big] \otimes [\langle ijkl\rangle]
\end{aligned}$$

Lemma 2.4. *One has* $\widetilde{\Psi} \circ \Delta^R_{\mathbb{Z}} = 0$.

Proof. Using Remark 2.3, one finds

$$\begin{aligned}
&\widetilde{\Psi} \circ \Delta^R_{\mathbb{Z}}(1)\\
&= \widetilde{\Psi}\left(\textstyle\sum_{r\in R} 1 \cdot r\right)\\
&= \big[- [ijkl] - [iljk] - [iklj] - [ijlk] - [ikjl] - [ilkj]\big] \otimes [\langle ijkl\rangle]\\
&= \big[\Delta^{\mathcal{C}}_{\mathbb{Z}}(-1)\,\big] \otimes [\langle ijkl\rangle]\\
&= 0
\end{aligned}$$

□

Corollary 2.5. $\widetilde{\Psi}$ *induces a homomorphism of* G*-modules*

$$\Psi\colon \mathbb{Z}^R/\Delta(\mathbb{Z}) \longrightarrow \big(\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})\big) \otimes_{\mathbb{Z}} \mathbb{Z}(Q)$$

Proof. This follows immediately from Lemma 2.4 by the first group isomorphism theorem. □

Consider the augmentation map

$$\begin{aligned}
\epsilon\colon \mathbb{Z}^{\mathcal{C}} &\longrightarrow \mathbb{Z}, \quad \text{given by}\\
c &\longmapsto 1 \qquad \text{for all } c \in \mathcal{C}
\end{aligned}$$

Now consider the composition of ϵ with the projection pr onto $\mathbb{Z}/3\mathbb{Z}$

$$\begin{aligned} \mathrm{pr}\circ\epsilon\colon \mathbb{Z}^{\mathcal{C}} &\longrightarrow \mathbb{Z}/3\mathbb{Z} \\ c &\longmapsto 1+3\mathbb{Z} \qquad \text{for all } c\in\mathcal{C} \end{aligned}$$

One has $\Delta(\mathbb{Z})\subset \operatorname{Ker}(\mathrm{pr}\circ\epsilon)$, because

$$\epsilon\circ\Delta_{\mathbb{Z}}^{\mathcal{C}}(1) = 6 \equiv 0 \bmod 3$$

Thus $\mathrm{pr}\circ\epsilon$ factors through the projection $\mathbb{Z}^{\mathcal{C}} \longrightarrow \mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})$, which yields the map

$$\widetilde{\mathrm{pr}\circ\epsilon}\colon \mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z}) \longrightarrow \mathbb{Z}/3\mathbb{Z}$$

We consider the tensor product of group homomorphisms

$$\varepsilon = \widetilde{\mathrm{pr}\circ\epsilon}\otimes \mathrm{id}_{\mathbb{Z}(\mathrm{Q})}$$

and set

$$\mathbb{Z}/3(Q) := (\mathbb{Z}/3\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}(Q)$$

Then one has

$$\begin{aligned} \varepsilon\colon \mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}(Q) &\longrightarrow \mathbb{Z}/3(Q) \\ \left[\sum_{c\in\mathcal{C}} a_c\cdot c\right]\otimes t\cdot[\sigma_1] &\longmapsto \left(\sum_{c\in\mathcal{C}} a_c + 3\mathbb{Z}\right)\otimes t\cdot[\sigma_1] \end{aligned}$$

where $a_c\in\mathbb{Z}$ for all $c\in\mathcal{C}$, $t\in\mathbb{Z}$ and σ_1 is an orientation of Q.
$\mathbb{Z}/3(Q)$ is a G-module via the G-action on $\mathbb{Z}(Q)$ and via the trivial action on $\mathbb{Z}/3\mathbb{Z}$.

Theorem 2.6. *The sequence*

$$0\longrightarrow \mathbb{Z}^R/\Delta(\mathbb{Z}) \xrightarrow{\Psi} \big(\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})\big)\otimes_{\mathbb{Z}}\mathbb{Z}(Q) \xrightarrow{\varepsilon} \mathbb{Z}/3(Q)\longrightarrow 0$$

is an exact sequence of G-modules.

Proof. We already know that the involved maps are homomorphisms of G-modules. We begin by showing that ε is surjective. It is sufficient to show that $z = (1+3\mathbb{Z})\otimes[\sigma_1]$ lies in $\operatorname{Im}(\varepsilon)$. One easily observes that, for an arbitrary $c\in\mathcal{C}$, the element $[c]\otimes[\sigma_1]$ lies in $\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}(Q)$ and

$$\varepsilon\left([c]\otimes[\sigma_1]\right) = (1+3\mathbb{Z})\otimes[\sigma_1] = z$$

Now we shall prove the injectivity of Ψ by showing $\operatorname{Ker}(\widetilde{\Psi}) = \Delta(\mathbb{Z})$. Due to Lemma 2.4, one has $\Delta(\mathbb{Z})\subset\operatorname{Ker}(\widetilde{\Psi})$, whence it remains to show the other inclusion. Let $\sum_{r\in R} t_r\cdot r \in \operatorname{Ker}(\widetilde{\Psi})$. Using Remark 2.3, one finds

$$\begin{aligned} 0 &= \widetilde{\Psi}\left(\sum_{r\in R} t_r\cdot r\right) \\ &= \Big[-t_{\{j,l\}}\cdot[ijkl] + \Big(t_{\{i,j\}} - t_{\{i,l\}} - t_{\{j,k\}}\Big)\cdot[iljk] + \Big(t_{\{i,j\}} + t_{\{k,l\}} - t_{\{i,k\}} \\ &\quad - t_{\{j,k\}} - t_{\{j,l\}}\Big)\cdot[iklj] + \Big(t_{\{i,j\}} + t_{\{k,l\}} - t_{\{i,l\}} - t_{\{i,k\}} - t_{\{j,l\}}\Big)\cdot[ijlk] \\ &\quad + \Big(t_{\{k,l\}} - t_{\{i,l\}} - t_{\{j,k\}}\Big)\cdot[ikjl] - t_{\{i,k\}}\cdot[ilkj]\Big]\otimes[\langle ijkl\rangle] \end{aligned}$$

Since this identity holds in $\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}(Q)$, it follows that

$$\begin{aligned} -t_{\{j,l\}} &= t_{\{i,j\}} - t_{\{i,l\}} - t_{\{j,k\}} = t_{\{i,j\}} + t_{\{k,l\}} - t_{\{i,k\}} - t_{\{j,k\}} - t_{\{j,l\}} \\ &= t_{\{i,j\}} + t_{\{k,l\}} - t_{\{i,l\}} - t_{\{i,k\}} - t_{\{j,l\}} = t_{\{k,l\}} - t_{\{i,l\}} - t_{\{j,k\}} = -t_{\{i,k\}} \end{aligned}$$

Now easy calculations show $t_r = t_{r'}$ for all $r, r' \in R$. Therefore $\sum_{r\in R} t_r \cdot r$ is an element of $\Delta(\mathbb{Z})$. It remains to show that the sequence is exact at $\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}(Q)$, i.e., $\mathrm{Im}(\Psi) = \mathrm{Ker}(\varepsilon)$. Clearly, $\mathrm{Im}(\Psi) \subset \mathrm{Ker}(\varepsilon)$, because for all $r \in R$ holds

$$\varepsilon \circ \Psi([r]) = (3 + 3\mathbb{Z}) \otimes (\pm\langle ijkl\rangle) = 0$$

In order to show $\mathrm{Ker}(\varepsilon) \subset \mathrm{Im}(\Psi)$, let $x \in \mathrm{Ker}(\varepsilon)$. Then

$$x = \sum_{j\in J} \left(\left[\sum_{c\in\mathcal{C}} a_{j_c} \cdot c \right] \otimes t_j[\sigma_1] \right)$$

where J is some finite set, $\sigma_1 = \langle ijkl\rangle$, $a_{j_c} \in \mathbb{Z}$ for all $j \in J$ and all $c \in \mathcal{C}$, and $t_j \in \mathbb{Z}$ for all $j \in J$. One has

$$x = \sum_{c\in\mathcal{C}} \left[\left(\sum_{j\in J} t_j \cdot a_{j_c} \right) \cdot c \right] \otimes [\sigma_1]$$

Since $x \in \mathrm{Ker}(\varepsilon)$, we find

$$0 = \varepsilon(x) = \left(\sum_{c\in\mathcal{C}} \sum_{j\in J} t_j a_{j_c} + 3\mathbb{Z} \right) \otimes [\sigma_1]$$

Thus

$$\sum_{c\in\mathcal{C}} \sum_{j\in j} t_j a_{j_c} \equiv 0 \mod 3$$

whence there exists an element $h \in \mathbb{Z}$ with

$$\sum_{c\in\mathcal{C}} \sum_{j\in j} t_j a_{j_c} = 3h \tag{14}$$

Set $b_c := \sum_{j\in J} t_j a_{j_c}$. We want to find an element $y = \left[\sum_{r\in R} y_r \cdot r\right] \in \mathbb{Z}^R/\Delta(\mathbb{Z})$ such that $\Psi(y) = x$. Solving the resulting system of linear equations for the y_r, $r \in R$, yields

$$\begin{aligned} y_{\{i,j\}} &= \frac{1}{3} \sum_{c\in\mathcal{C}\setminus\mathcal{C}'} b_c - \frac{2}{3} \sum_{c\in\mathcal{C}'} b_c, & y_{\{k,l\}} &= \frac{1}{3} \sum_{c\in\mathcal{C}\setminus\mathcal{C}''} b_c - \frac{2}{3} \sum_{c\in\mathcal{C}''} b_c, \\ y_{\{i,l\}} &= -\frac{1}{3} \sum_{c\in\mathcal{C}\setminus\{[iklj]\}} b_c + \frac{2}{3}\, b_{[iklj]}, & y_{\{j,k\}} &= -\frac{1}{3} \sum_{c\in\mathcal{C}\setminus\{[ijlk]\}} b_c + \frac{2}{3}\, b_{[ijlk]}, \\ y_{\{i,k\}} &= -b_{[ilkj]} \ \in \mathbb{Z} \qquad \text{and} & y_{\{j,l\}} &= -b_{[ijkl]} \ \in \mathbb{Z} \end{aligned}$$

where $\mathcal{C}' := \{[ikjl], [ilkj], [ijkl]\}$ and $\mathcal{C}'' := \{[iljk], [ilkj], [ijkl]\}$ are subsets of $\mathcal{C}$. It remains to check that $y_r \in \mathbb{Z}$ for all $r \in R$. This is easily obtained by using equation (14). One has

$$y_{\{i,j\}} = \frac{1}{3}\sum_{c\in\mathcal{C}} b_c - \sum_{c\in\mathcal{C}'} b_c = h - \sum_{c\in\mathcal{C}'} b_c \quad \in \mathbb{Z}$$

$$y_{\{k,l\}} = \frac{1}{3}\sum_{c\in\mathcal{C}} b_c - \sum_{c\in\mathcal{C}''} b_c = h - \sum_{c\in\mathcal{C}''} b_c \quad \in \mathbb{Z}$$

$$y_{\{i,l\}} = -\frac{1}{3}\sum_{c\in\mathcal{C}} b_c + b_{[iklj]} = -h + b_{[iklj]} \quad \in \mathbb{Z}$$

$$y_{\{j,k\}} = -\frac{1}{3}\sum_{c\in\mathcal{C}} b_c + b_{[ijlk]} \quad \in \mathbb{Z}$$

Consequently, $y = \left[\sum_{r\in R} y_r \cdot r\right]$ lies in $\mathbb{Z}^R/\Delta(\mathbb{Z})$ with $\Psi(y) = x$, which completes the proof. □

We want to use the previous result to establish an action of $S_6 \times \{\pm 1\}$ on $\mathbb{Z}^R/\Delta(\mathbb{Z})$.

Define $\overline{G} := \mathrm{Aut}(\mathcal{C})$, then $\overline{G} \cong S_6$. Furthermore, $\overline{G}$ acts on $\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})$ (by acting canonically on $\mathcal{C}$), and we let it act trivially on $\mathbb{Z}(Q)$ and on $\mathbb{Z}/3(Q)$. Thus we have an action of $\overline{G}$ on $\left(\mathbb{Z}^{\mathcal{C}}/\Delta(\mathbb{Z})\right) \otimes_{\mathbb{Z}} \mathbb{Z}(Q)$. Now one easily observes that the map ε is $\overline{G}$-equivariant, which yields the fact that $\mathrm{Ker}(\varepsilon)$ is $\overline{G}$-invariant. But due to Theorem 2.6, $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(\Psi) \cong \mathbb{Z}^R/\Delta(\mathbb{Z})$. We hence get an action of $\overline{G}$ on $\mathbb{Z}^R/\Delta(\mathbb{Z})$.

Lemma 2.7. *Consider the group homomorphism*

$$\begin{aligned} g\colon G &\longrightarrow \overline{G} \\ \tau &\longmapsto g_\tau \end{aligned}$$

where g_τ is induced by the natural action of G on $\mathcal{C}$, namely

$$g_\tau([ijkl]) = [\tau(i)\tau(j)\tau(k)\tau(l)]$$

for $[ijkl] \in \mathcal{C}$. Then g is injective.

Proof. Let $Q = \{0,1,2,3\}$ and let $\tau \in \mathrm{Ker}(g)$. Then $g_\tau\colon \mathcal{C} \longrightarrow \mathcal{C}$ is the identity map. One has

$$[0123] = g_\tau([0123]) = [\tau(0)\tau(1)\tau(2)\tau(3)]$$

Thus $\tau(i) = i + k \bmod 4$ for some $k \in \mathbb{N}$. One finds

$$\begin{aligned} [0132] = g_\tau([0132]) &= [\tau(0)\tau(1)\tau(3)\tau(2)] \\ &= \left[\,\overline{k}\ \ \overline{1+k}\ \ \overline{3+k}\ \ \overline{2+k}\,\right] \end{aligned}$$

where $\overline{}$ denotes the residue class mod 4.

If $k \equiv 1 \bmod 4$, then

$$[0132] = g_\tau([0132]) = [1203] = [0312]$$

which is a contradiction. Similarly, if $k \equiv 2 \bmod 4$, one has

$$[0132] = g_\tau([0132]) = [2310] = [0231]$$

This is a contradiction as well. Finally, for $k \equiv 3 \bmod 4$ follows

$$[0132] = g_\tau([0132]) = [3021] = [0213]$$

which is again a contradiction. Therefore one has $k \equiv 0 \bmod 4$, which shows that $\tau = \mathrm{id}_G$. This completes the proof. $\square$

Define $\widehat{G} := \overline{G} \times \mu_2$, where $\mu_2 = \{\pm 1\}$. Consider the group homomorphism

$$\begin{aligned} \delta\colon G &\longrightarrow \widehat{G} \\ \tau &\longmapsto \big(g(\tau), \operatorname{sgn}(\tau)\big) \end{aligned}$$

where $g\colon G \longrightarrow \overline{G}$ is the map defined in Lemma 2.7.

In section 1.2 we embedded the set $\overline{X}_Q$ of quadrangles up to affine automorphisms into $V = S^1 \otimes_{\mathbb{Z}} \left(\mathbb{Z}^R/\Delta(\mathbb{Z})\right) = \left(S^1\right)^R/\Delta\left(S^1\right)$. Now we receive an action of $\widehat{G}$ on V via the action on $\mathbb{Z}^R/\Delta(\mathbb{Z})$. Namely, for $(\eta, \xi) \in \widehat{G}$ and a decomposable tensor $z \otimes \left[\sum_{r\in R} t_r \cdot r\right] \in V$, one has

$$(\eta, \xi) \cdot \left(z \otimes \left[\sum_{r\in R} t_r \cdot r\right]\right) = \xi \cdot z \otimes \sum_{r\in R} \left(\Psi^{-1}\left(\eta \cdot \Psi\left([t_r \cdot r]\right)\right)\right) \tag{15}$$

The next lemma shows that the group homomorphism $\delta\colon G \longrightarrow \widehat{G}$ is compatible with the injective map $v\colon \overline{X}_Q \longrightarrow V$ (cf. p. 9).

Lemma 2.8. *The map v is δ-equivariant, i.e., one has $v\left(\tau \cdot [x]\right) = \delta(\tau) \cdot v\left([x]\right)$ for all $\tau \in G$ and all $[x] \in \overline{X}_Q$.*

Proof. The map

$$\begin{aligned} v\colon \overline{X}_Q &\longrightarrow V \cong \left(S^1\right)^R/\Delta\left(S^1\right) \\ [x] &\longmapsto \left[\left(\hat{u}_r(x)\right)_{r\in R}\right] \end{aligned}$$

obviously is a G-map. Furthermore, the identity map on V is δ-equivariant. For if $z \otimes \left(\sum_{r\in R} t_r \cdot [r]\right) \in V$ and $\tau \in G$, then one finds the following identity due to the G-equivariance of Ψ .

$$\begin{aligned} \delta(\tau) \cdot \left(z \otimes \left(\sum_{r\in R} t_r \cdot [r]\right)\right) &= z \otimes \left(\sum_{r\in R} t_r \cdot \Psi^{-1}\Big(\big(g_\tau, \operatorname{sgn}(\tau)\big) \cdot \Psi([r])\Big)\right) \\ &= z \otimes \left(\sum_{r\in R} t_r \cdot \Psi^{-1}\Psi(\tau \cdot [r])\right) \\ &= z \otimes \left(\sum_{r\in R} t_r \cdot (\tau \cdot [r])\right) \\ &= \tau \cdot \left(z \otimes \left(\sum_{r\in R} t_r \cdot [r]\right)\right) \end{aligned}$$

Therefore one concludes

$$v(\tau \cdot [x]) = \tau \cdot v([x]) = \delta(\tau) \cdot v([x])$$

for all $[x] \in \overline{X}_Q$ and all $\tau \in G$. $\square$

Lemma 2.9. *The group homomorphism*

$$\begin{aligned}\delta\colon G &\longrightarrow \widehat{G}\\ \tau &\longmapsto \big(g(\tau), \operatorname{sgn}(\tau)\big)\end{aligned}$$

is injective. Hence $\operatorname{Im}(\delta) \cong S_4$.

Proof. This follows directly from Lemma 2.7. □

3. Interpretation of the S_6-symmetry

There is the canonical action of $\operatorname{Aut}(Q) \cong S_4$ on the set $\overline{X}_Q$ of Q-labeled quadrangles up to affine automorphisms, i.e., the permutations of the points of a quadrangle. Besides this action, we established another action on the subset $\overline{Y}_Q$ of $\overline{X}_Q$ induced by the pedal triangle construction (cp. section 1.3). On the other hand, we found an action of $S_6 \times \mu_2$ on the $\mathbb{Z}$-module $V = \big(S^1\big)^R/\Delta\big(S^1\big)$ (cf. p. 23). V contains the set of $\overline{X}_Q$ via the injective map $v\colon \overline{X}_Q \longrightarrow V$ (cf. Lemma 1.17). This section will show that the two operations – permutations of the points of a quadrangle and the pedal triangle construction – generate a group isomorphic to S_6.

Theorem 3.1. *Let $Q = \{0, i, j, k\}$ and consider the pedal triangle construction $\phi_0\colon \overline{Y}_Q \longrightarrow \overline{Y}_Q$ (cf. Corollary 1.31). Then there exists an element $\widehat{\phi}_0 \in \widehat{G} = \operatorname{Aut}(\mathcal{C}) \times \mu_2$ with*

$$v\big(\phi_0([x])\big) = \widehat{\phi}_0\big(v([x])\big) \qquad \forall\ [x] \in \overline{Y}_Q$$

This element is $\widehat{\phi}_0 = (\tau, 1)$, where

$$\tau = \begin{pmatrix}[i0jk] & [ik0j] & [ij0k] & [ikj0] & [i0kj] & [ijk0]\\ [ijk0] & [i0kj] & [i0jk] & [ik0j] & [ikj0] & [ij0k]\end{pmatrix}$$

in the usual permutation notation.

Proof. It will be useful to consider the operation of $\widehat{\phi}_0$ on $\mathbb{Z}^R/\Delta(\mathbb{Z})$. Therefore we calculate $\widehat{\phi}_0 \cdot [s]$ for all $s \in R$, using Remark (2.3). One has

$$\begin{aligned}\widehat{\phi}_0 \cdot [\{i,j\}] &= \Psi^{-1}\Big(\tau \cdot \Psi\big([\{i,j\}]\big)\Big)\\ &= \Psi^{-1}\Big(\tau \cdot \Big(\big[[i0jk] + [ik0j] + [ij0k]\big] \otimes [\langle ijk0\rangle]\Big)\Big)\\ &= \Psi^{-1}\Big(\big[-[ik0j] - [ij0k] - [ikj0]\big] \otimes [\langle ijk0\rangle]\Big)\\ &= \big[-\{k,0\}\big]\end{aligned}$$

$$\begin{aligned}\widehat{\phi}_0 \cdot [\{k,0\}] &= \Psi^{-1}\Big(\tau \cdot \Psi\big([\{k,0\}]\big)\Big)\\ &= \Psi^{-1}\Big(\tau \cdot \Big(\big[[ikj0] + [ij0k] + [ik0j]\big] \otimes [\langle ijk0\rangle]\Big)\Big)\\ &= \Psi^{-1}\Big(\big[[ik0j] + [i0jk] + [i0kj]\big] \otimes [\langle ijk0\rangle]\Big)\\ &= \big[-\{k,0\} - \{i,k\} - \{j,k\}\big]\end{aligned}$$

$$\begin{aligned}
\widehat{\phi}_0 \cdot [\{i,0\}] &= \Psi^{-1}\Big(\tau \cdot \Psi\big([\{i,0\}]\big)\Big) \\
&= \Psi^{-1}\Big(\tau \cdot \Big(\big[-[ij0k]-[ikj0]-[i0jk]\big] \otimes [\langle ijk0\rangle]\Big)\Big) \\
&= \Psi^{-1}\Big(\big[-[i0jk]-[ik0j]-[ijk0]\big] \otimes [\langle ijk0\rangle]\Big) \\
&= \big[-\{i,j\}-\{i,0\}-\{i,k\}\big]
\end{aligned}$$

$$\begin{aligned}
\widehat{\phi}_0 \cdot [\{i,k\}] &= \Psi^{-1}\Big(\tau \cdot \Psi\big([\{i,k\}]\big)\Big) \\
&= \Psi^{-1}\Big(\tau \cdot \Big(\big[-[i0kj]-[ij0k]-[ik0j]\big] \otimes [\langle ijk0\rangle]\Big)\Big) \\
&= \Psi^{-1}\Big(\big[[ijk0]+[ij0k]+[ik0j]\big] \otimes [\langle ijk0\rangle]\Big) \\
&= \big[-\{j,0\}\big] \\
&= \sum_{s\in R\setminus\{\{j,0\}\}} [s]
\end{aligned}$$

$$\begin{aligned}
\widehat{\phi}_0 \cdot [\{j,k\}] &= \Psi^{-1}\Big(\tau \cdot \Psi\big([\{j,k\}]\big)\Big) \\
&= \Psi^{-1}\Big(\tau \cdot \Big(\big[-[ik0j]-[i0jk]-[ikj0]\big] \otimes [\langle ijk0\rangle]\Big)\Big) \\
&= \Psi^{-1}\Big(\big[[i0jk]+[ij0k]+[ikj0]\big] \otimes [\langle ijk0\rangle]\Big) \\
&= \big[-\{i,0\}\big]
\end{aligned}$$

$$\begin{aligned}
\widehat{\phi}_0 \cdot [\{j,0\}] &= \Psi^{-1}\Big(\tau \cdot \Psi\big([\{j,0\}]\big)\Big) \\
&= \Psi^{-1}\Big(\tau \cdot \Big(\big[-[ijk0]-[ij0k]-[ik0j]\big] \otimes [\langle ijk0\rangle]\Big)\Big) \\
&= \Psi^{-1}\Big(\big[-[ij0k]-[i0jk]-[i0kj]\big] \otimes [\langle ijk0\rangle]\Big) \\
&= \big[\{i,k\}+\{k,0\}+\{i,0\}\big]
\end{aligned}$$

For $[x] \in \overline{Y}_Q$ and τ defined as in the claim, one finds

$$\begin{aligned}
v\left(\phi_0([x])\right) &= \big[\left(\hat{u}_s\left(\phi_0(x)\right)\right)_{s\in R}\big] \\
&= \sum_{s\in R} \hat{u}_s\left(\phi_0(x)\right) \otimes [s] \\
&= \sum_{s\in R\setminus\{\{0,j\}\}} \hat{u}_s\left(\phi_0(x)\right) \otimes [s] \;+\; \hat{u}_{0j}\left(\phi_0(x)\right) \otimes \left[\sum_{s\in R\setminus\{\{0,j\}\}} -s\right] \\
&= \sum_{s\in R\setminus\{\{0,j\}\}} \frac{\hat{u}_s\left(\phi_0(x)\right)}{\hat{u}_{0j}\left(\phi_0(x)\right)} \otimes [s]
\end{aligned}$$

We know by equation 1.32(1) that certain relations hold between the double angles of the triangle x and the double angles of its pedal triangle $\phi_0(x)$. Consider $s = \{i,j\}$. By equation 1.32(1), one has

$$\hat{\alpha}_{0ji}\big(\phi_0(x)\big) = \hat{\alpha}_{0ik}(x)$$

Hence one has

$$\frac{\hat{u}_{ij}\left(\phi_0(x)\right)}{\hat{u}_{0j}\left(\phi_0(x)\right)} = \frac{\hat{u}_{ik}(x)}{\hat{u}_{0i}(x)}$$

Similarly, we express

$$\frac{\hat{u}_{s}\left(\phi_0(x)\right)}{\hat{u}_{0j}\left(\phi_0(x)\right)}$$

for all $s \in R \setminus \{\{j,0\}\}$ in terms of the algebraic sides of the triangle x. Inserting our calculations into the right-hand term of the equation above yields

$$\begin{aligned}
v\big(\phi_0([x])\big) =& \frac{\hat{u}_{ik}(x)}{\hat{u}_{i0}(x)} \otimes \big[\{i,j\}\big] \;+\; \frac{\hat{u}_{ik}(x)\hat{u}_{j0}(x)}{\hat{u}_{ij}(x)\hat{u}_{k0}(x)} \otimes \big[\{k,0\}\big] \\
&+ \frac{\hat{u}_{ik}(x)\hat{u}_{j0}(x)}{\hat{u}_{i0}(x)\hat{u}_{jk}(x)} \otimes \big[\{i,0\}\big] \;+\; \frac{\hat{u}_{ik}(x)\hat{u}_{j0}(x)}{\hat{u}_{k0}(x)\hat{u}_{i0}(x)} \otimes \big[\{i,k\}\big] \\
&+ \frac{\hat{u}_{ik}(x)}{\hat{u}_{k0}(x)} \otimes \big[\{j,k\}\big] \\
=& \;\hat{u}_{ij}(x) \otimes \big[-\{k,0\}\big] \;+\; \hat{u}_{k0}(x) \otimes \big[-\{k,0\} - \{i,k\} - \{j,k\}\big] \\
&+ \hat{u}_{i0}(x) \otimes \big[-\{i,j\} - \{i,0\} - \{i,k\}\big] \\
&+ \hat{u}_{ik}(x) \otimes \big[\{i,j\} + \{k,0\} + \{i,0\} + \{i,k\} + \{j,k\}\big] \\
&+ \hat{u}_{jk}(x) \otimes \big[-\{i,0\}\big] + \hat{u}_{j0}(x) \otimes \big[\{k,0\} + \{i,0\} + \{i,k\}\big]
\end{aligned}$$

Finally, we make use of our calculations for $\widehat{\phi}_0 \cdot [s]$, $s \in R$, and get

$$\begin{aligned}
v\big(\phi_0([x])\big) &= \sum_{s\in R} \hat{u}_s(x) \otimes \left(\widehat{\phi}_0 \cdot [s]\right) \\
&= \widehat{\phi}_0 \cdot \left(\sum_{s\in R} \hat{u}_s(x) \otimes [s]\right) \\
&= \widehat{\phi}_0 \cdot \big[(\hat{u}_s(x))_{s\in R}\big] \\
&= \widehat{\phi}_0 \cdot v([x])
\end{aligned}$$

□

Remark 3.2. Let $f\colon \{1,\dots,6\} \longrightarrow \mathcal{C}$ be the bijection with $f(1) = [i0jk]$, $f(2) = [ik0j]$, $f(3) = [ij0k]$, $f(4) = [ikj0]$, $f(5) = [i0kj]$ and $f(6) = [ijk0]$. Under this bijection, one has

$$\widehat{\phi}_0 = \big((163)(254), 1\big)$$

written as a product of disjoint cycles.

Due to Lemma 2.9, the action on V induced by permuting the vertices of a quadrangle is an action of $S_4 \subset \widehat{G}$. Furthermore, we proved the existence of an element $\widehat{\phi}_0 \in \widehat{G}$, whose operation on V corresponds to the pedal triangle construction on the subset $\overline{Y}_Q$ of $\overline{X}_Q$.

Define

$$H = \{\, (g, \operatorname{sgn}(g)) \mid g \in S_6 \,\}$$

Then H is a subgroup of $S_6 \times \mu_2$ that is obviously isomorphic to S_6.

Theorem 3.3. *We identify $\widehat{G}$ with $S_6 \times \mu_2$ via the bijection f from Corollary 3.2. Then the set* $\operatorname{Im}(\delta)$ *and the element $\widehat{\phi}_0$ generate the group $H \cong S_6$.*

Proof. Let $Q = \{i, j, k, 0\}$. The transpositions (ij), (jk) and $(k0)$ generate $\operatorname{Aut}(Q)$, whence their images under δ generate $\operatorname{Im}(\delta)$. At first, we calculate $\omega_1 := \delta((ij))$, $\omega_2 := \delta((jk))$ and $\omega_3 := \delta((k0))$. Afterwards we show that ω_1, ω_2, ω_3 and $\widehat{\phi}_0$ generate the subgroup H.

For the transposition ω_1 holds

$$\omega_1 = \big(\, g_{(ij)}, \operatorname{sgn}((ij))\,\big) = \big(g_{(ij)}, -1\big) \quad \in \operatorname{Aut}(\mathcal{C}) \times \mu_2$$

where $g_{(ij)}(c) = (ij) \cdot c$ for $c \in \mathcal{C}$. Hence

$$\begin{aligned} g_{(ij)}\big([i0jk]\big) &= [j0ik] = [ikj0] \\ g_{(ij)}\big([ik0j]\big) &= [jk0i] = [ijk0] \\ g_{(ij)}\big([ij0k]\big) &= [ji0k] = [i0kj] \end{aligned}$$

This shows that, under the bijection f, the automorphism $g_{(ij)}$ corresponds to the element $(14)(26)(35)$ in S_6. Thus $\omega_1 = \big((14)(26)(35), -1\big)$. Analogously, one finds $\omega_2 = \big((15)(23)(46), -1\big)$ and $\omega_3 = \big((14)(25)(36), -1\big)$. By Corollary 3.2, we have $\widehat{\phi}_0 = \big(\,(163)(254), 1\,\big)$. Denote by H' the subgroup of $S_6 \times \mu_2$ generated by these four elements. Since they are clearly contained in H, one has $H' \subset H$. Now one easily checks that the following holds.

$$\begin{aligned} \omega_4 &:= \big((26), -1\big) = \Big(\omega_3 \widehat{\phi}_0\Big)^2 \omega_1 \quad \in H' \\ \omega_5 &:= \big((35), -1\big) = \omega_3\omega_1\omega_4\omega_3\omega_1 \quad \in H' \\ \omega_6 &:= \big((14), -1\big) = \omega_3\omega_4\omega_3\omega_1\omega_4 \quad \in H' \\ \omega_7 &:= \big((45), -1\big) = (\omega_3\omega_2\omega_4)^2\,\omega_5 \quad \in H' \\ \omega_8 &:= \big((24), -1\big) = \omega_7\widehat{\phi}_0\omega_2\widehat{\phi}_0\omega_1 \quad \in H' \\ \omega_9 &:= \big((25), -1\big) = \omega_8\omega_7\omega_8 \quad \in H' \\ \omega_{10} &:= \big((12), -1\big) = \omega_6\omega_8\omega_6 \quad \in H' \\ \omega_{11} &:= \big((34), -1\big) = \omega_5\omega_7\omega_5 \quad \in H' \\ \omega_{12} &:= \big((23), -1\big) = \omega_{11}\omega_8\omega_{11} \quad \in H' \\ \omega_{13} &:= \big((56), -1\big) = \omega_4\omega_9\omega_4 \quad \in H' \end{aligned}$$

Thus all elements $\big((r, r+1), -1\big)$, $1 \leqslant r \leqslant 5$, are contained in H', which proves $H' = H$. $\square$

Corollary 3.4. *The operations on $\overline{Y}_Q$ given by*

(1) *permutations of the points of a Q-labeled quadrangle x*

(2) *the derived triangle construction relative to x_0*

generate a group isomorphic to S_6.

References

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