NOTES ON STRICT BICOMMUTATIVE HOPF ALGEBRAS

MARKUS ROST

Contents

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Meanwhile (2021-2022) I have understood things better. For instance, the word "strict" is entirely unnecessary: Definitions 1 and 2 just yield bi-cocommutatve resp. cocommutatve Hopf algebras in the usual sense.

I hope to replace this text at some time.

Preface

This is an outline. The two sections are introductory and can be read in any order.

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Introduction

Let V be a locally free R-module V over a ring R and consider the exterior algebra $H = \Lambda^\bullet V$. The multiplication μ, comultiplication Δ and antipode S of H together with some standard rules establishes it as a Hopf algebra (commutative and cocommutative in the graded sense).

It seems that many general considerations on the exterior algebra are formal consequences of its structure as a Hopf algebra, together with its grading.

The examples we have in mind are general treatments of the Caley-Hamilton theorem and of Schur functors. We hope to consider these things elsewhere.

To formalize properties of $\Lambda^\bullet V$ and some proofs, an obvious idea is to work in the framework of "bicommutative graded Hopf algebras in the category of R-modules" or "commutative group objects in the category of cocommutative graded R-coalgebras" (or the other way round). However for our purposes we didn't find that very enlightening. First, for the tensor product of graded R-modules there are two involutions, the signed one (which is part of the Hopf algebra structure) and the unsigned one (which is not part of the Hopf algebra structure and which we don't use). Second, we are considering a variety of combinations of S, μ, Δ and the signed involution τ. Two work effectively with such combinations, it is convenient to describe them with integral matrices.

So far we came up with a very general definition (Definition 1 in Section 1).

This definition facilitates in some basic cases to establish an object as a Hopf algebra. Examples are the exterior algebra $\Lambda^\bullet V$ and the rings H of commutative affine algebraic groups $G = \operatorname{Spec} H$. In both examples it becomes a 1-liner to identify them as Hopf algebras as in Definition 1. (For non-commutative G see Definition 2.)

Proposition 1 in Section 1 and Proposition 2 in Section 2 are first attempts to relate Definition 1 with standard definitions of Hopf algebras in terms of the basic morphisms S, μ, Δ. It turns out that there are some subtleties regarding the transpose τ.

In some future version we hope to make the Propositions more precise (a better scenario would be if someone could tell me references on these things...).

As for the example $\Lambda^\bullet V$: This text doesn't cover the grading. We are not sure yet about how to formalize the grading in a way that fits the applications we have in mind.

1. Strict bicommutative Hopf algebras

The material of this text grew out from computations with the exterior algebra $H = \Lambda^\bullet V$. So we begin with this example.

We found it appropriate not to start out with S, μ, Δ but in a slightly different way. Since one has

$$H^{\otimes n} = \Lambda^\bullet(V \otimes_{\mathbf{Z}} \mathbf{Z}^n)$$

there is a natural action of integral matrices on the tensor powers of H (by acting on the $\mathbf{Z}^n$). We denote this action by

$$M(n, m, \mathbf{Z}) \times H^{\otimes m} \to H^{\otimes n}$$
$$(A, x) \mapsto [A](x)$$

This way the basic morphisms S, μ, Δ have the descriptions

$$S = [-1], \quad \mu = [1, 1], \quad \Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To mention a further example, consider

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : H^{\otimes 2} \to H^{\otimes 2}$$

In concrete terms this is the morphism

$$(\mu \otimes 1) \circ (1 \otimes \Delta) \colon (\Lambda^\bullet V)^{\otimes 2} \to (\Lambda^\bullet V)^{\otimes 2}$$
$$x \otimes y \mapsto \sum_i x y_i \otimes y_i' \qquad \big(\sum_i y_i \otimes y_i' = \Delta(y)\big)$$

Using the matrix notation the sometimes tiring computations in terms of S, μ, Δ can be written in a more compact form.

Another and much simpler example is that of the Hopf algebra H of a commutative affine algebraic group $G = \operatorname{Spec} H$. One has

$$\operatorname{Spec} H^{\otimes n} = G^n = G \otimes \mathbf{Z}^n$$

and the resulting action of integral matrices on the family $H^{\otimes n}$ is on group level just the collection of homomorphisms

$$[A]^* \colon G^n \to G^m$$
$$(g_1, \ldots, g_n) \mapsto \big(\ldots, \prod_i g_i^{a_{ij}}, \ldots\big)$$

with $A = (a_{ij}) \in M(n, m, \mathbf{Z})$.

The natural way to set up such operations of integral matrices

$$M(n, m) = M(n, m, \mathbf{Z}) = \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}^m, \mathbf{Z}^n)$$

seems to be to consider certain monoidal functors (for monoidal categories see [1]).

Let $\mathcal{Z}$ be the category with objects $\mathbf{Z}^n$ $(n \geq 0)$ and with morphisms

$$\operatorname{Hom}_{\mathcal{Z}}(\mathbf{Z}^m, \mathbf{Z}^n) = \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}^m, \mathbf{Z}^n)$$

the $\mathbf{Z}$-linear homomorphisms (or integral $n \times m$-matrices). We consider $\mathcal{Z}$ as a monoidal category with respect to the coproduct

$$\mathbf{Z}^{n_1} \oplus \mathbf{Z}^{n_2} = \mathbf{Z}^{n_1 + n_2}$$

Thus $\mathcal{Z}$ is the full monoidal subcategory of the category of abelian groups generated by the object $\mathbf{Z}$.

Definition 1. A strict bicommutative Hopf algebra is a monoidal functor

$$F\colon (\mathcal{Z},\oplus)\to(\mathcal{C},\square)$$

of monoidal categories.

Maybe there is a better name for such a functor, but in the end it is a Hopf algebra in some restricted form. One just doesn't mention the axioms for a Hopf algebra in the definition explicitly, rather one derives them from the axioms of a group.

The word bicommutative stands for commutative and cocommutative. The object $H=F(\mathbf{Z})$ is called a (strict) bicommutative Hopf algebra (in $\mathcal{C}$) as well.

Let

$$\begin{aligned} S&=(-1)\in M(1,1)\\ \mu&=(1,1)\in M(1,2)\\ \Delta&=\begin{pmatrix}1\\1\end{pmatrix}\in M(2,1)\\ \tau&=\begin{pmatrix}0&1\\1&0\end{pmatrix}\in M(2,2)\end{aligned}$$

I believe that something like the following is correct:

Proposition 1. *Let H be an object in a monoidal category $(\mathcal{C},\square)$. Let further*

$$S_H\in\operatorname{Hom}(H,H),\quad \mu_H\in\operatorname{Hom}(H\square H,H),$$
$$\Delta_H\in\operatorname{Hom}(H,H\square H),\quad \tau_H\in\operatorname{Hom}(H\square H,H\square H)$$

be morphisms in $\mathcal{C}$.

There is a monoidal functor

$$\widetilde{H}\colon(\mathcal{Z},\oplus)\to(\mathcal{C},\square)$$

with

$$\widetilde{H}(\mathbf{Z})=H,\quad \widetilde{H}(S)=S_H,\quad \widetilde{H}(\mu)=\mu_H,\quad \widetilde{H}(\Delta)=\Delta_H,\quad \widetilde{H}(\tau)=\tau_H$$

if and only if the following conditions hold:

(1) *The monoidal subcategory $\mathcal{H}$ of $(\mathcal{C},\square)$ generated by H and $S_H,\mu_H,\Delta_H,\tau_H$ is a symmetric monoidal category with respect to τ_H.*
(2) *(H,S_H,μ_H,Δ_H) is a commutative and cocommutative Hopf algebra in the symmetric monoidal category $\mathcal{H}$.*

Condition (1) is somewhat unprecise. The objects of $\mathcal{H}$ are $H^{\square n}$ (only up to canonical isomorphisms if $(\mathcal{C},\square)$ is not strict). We understand that the isomorphisms

$$H^{\square n}\,\square\,H^{\square m}\simeq H^{\square m}\,\square\,H^{\square n}$$

of the symmetric monoidal category $\mathcal{H}$ are build in the obvious way from τ_H.

Condition (2) refers to a standard (and obvious) definition of Hopf algebras in symmetric monoidal categories (reference?).

It is easy to see that S, μ, Δ, τ generate all morphisms in $\mathcal{Z}$ with respect to compositions and direct products (details are omitted for now). It follows that the functor $\widetilde{H}$ in Proposition 1 is uniquely determined.

Interestingly, τ can be expressed in terms of S, μ, Δ, see relation (R) in Section 2. However I didn't find a useful way to eliminate τ from Proposition 1.

To summarize, we understand Proposition 1 as a justification to call a functor as in Definition 1 a "bicommutative Hopf algebra".

For the applications to $H = \Lambda^\bullet V$ we have in mind, Proposition 1 is not necessary at all: one just works with the functor

$$\begin{aligned} \mathcal{Z} &\to \text{graded } R\text{-algebras} \\ X &\mapsto \Lambda^\bullet(V \otimes_{\mathbf{Z}} X) \end{aligned}$$

Probably one may generalize these considerations to commutative but not necessarily cocommutative Hopf algebras. One replaces the category $(\mathcal{Z}, \oplus)$ by the full monoidal subcategory $(\mathcal{F}, *)$ of the category of groups generated by the object $\mathbf{Z}$. Here the monoidal operation $*$ is again the coproduct (the free product of groups). Thus the objects of $\mathcal{F}$ are the free groups F_n on n letters $x_1, x_2, \dots$ and an element

$$A \in \operatorname{Hom}(F_m, F_n)$$

is a collection

$$A = (w_1, \dots, w_m)$$

of words w_j in the $x_1, \dots, x_n$.

For a group G the functor

$$\begin{aligned} \mathcal{F}^{\text{op}} &\to \text{Sets} \\ F_n &\mapsto \operatorname{Hom}(F_n, G) = G^n \end{aligned}$$

yields maps

$$\operatorname{Hom}(F_m, F_n) \to \operatorname{Maps}(G^n, G^m)$$

It associates to

$$A = (w_1, \dots, w_m) \in \operatorname{Hom}(F_m, F_n)$$

the map

$$\begin{aligned} [A]^* \colon G^n &\to G^m \\ (g_1, \dots, g_n) &\mapsto \big(w_1(g_i), \dots, w_m(g_i)\big) \end{aligned}$$

For an affine algebraic group $G = \operatorname{Spec} H$, this yields maps

$$\operatorname{Hom}(F_m, F_n) \to \operatorname{Hom}(H^{\otimes m}, H^{\otimes n})$$

A (tentative) definition would be

Definition 2. A strict commutative Hopf algebra is a monoidal functor

$$F \colon (\mathcal{F}, *) \to (\mathcal{C}, \square)$$

of monoidal categories.

I have no applications in mind for the non-cocommutative commutative case. However concerning our discussions this generalization of the bicommutative case seems to be obvious. I haven't looked at the case of general Hopf algebras (not necessarily commutative or cocommutative).

By the way, I don't think τ can be expressed in terms of S, μ, Δ in the category $\mathcal{F}$ (with obvious definitions of S, μ, Δ, τ in $\mathcal{F}$).

2. Generators and relations for integral matrices

This section consists of another line of introduction to the material.

For the family of groups of integral matrices

$$M(n,m) = M(n,m,\mathbf{Z}) = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^m, \mathbf{Z}^n)$$

consider the following two kinds of operations: The product or composition of matrices

$$\begin{aligned} M(n,m) \times M(m,k) &\to M(n,k) \\ (A,B) &\mapsto AB \end{aligned}$$

and the direct sum:

$$\begin{aligned} M(n_1,m_1) \times M(n_2,m_2) &\to M(n_1+n_2, m_1+m_2) \\ (A_1,A_2) &\mapsto A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \end{aligned}$$

One may ask: Which set of matrices will generate all matrices using these operations? And what would be the relations among such generators?

As for a set of generators, there is an obvious choice:

$$\begin{aligned} S &= (-1) \in M(1,1) \\ \mu &= (1,1) \in M(1,2) \\ \Delta &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in M(2,1) \\ \tau &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M(2,2) \end{aligned}$$

It is easy to see that S, μ, Δ, τ yield all matrices (details are omitted for now).

For instance one gets the basic elementary 2×2 matrices as follows

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = (\mu \oplus 1)(1 \oplus \Delta)$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (1 \oplus \mu)(\Delta \oplus 1)$$

Their inverses can be obtained like this

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = (\mu \oplus 1)(1 \oplus S \oplus 1)(1 \oplus \Delta)$$

Moreover, the transposition τ yields the transpositions

$$(1_s \oplus \tau \oplus 1_t) \in \mathrm{GL}(s+t+2, \mathbf{Z})$$

and therefore by compositions all permutation matrices of any size.

Now, what are the relations among our generators S, μ, Δ, τ?

Interestingly, the transposition τ can be expressed in terms of the other generators S, μ, Δ because of

$$(R) \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This relation is well-known: The group $\mathrm{SL}(2,\mathbf{Z})$ is generated by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and (R) is essentially the standard expression of $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ in terms of these generators (as for instance in [3]).

Proposition 2 (preliminary version)**.** *The answer for a set of generating relations among the S,μ,Δ,τ is (probably) essentially this:*

(1) *The relations which ensure that the transposition τ together with direct product operations yields indeed the permutation groups. By the Coxeter type presentation for the permutation groups this amounts to the relations*

$$1=\tau^2$$
$$1=\sigma^3$$

with

$$\sigma=(\tau\oplus 1)(1\oplus\tau)$$

Moreover the compatibilities of S,μ,Δ with τ:

$$(1\oplus S)\tau=\tau(S\oplus 1)$$
$$(1\oplus\Delta)\tau=\sigma(\Delta\oplus 1)$$
$$\tau(\mu\oplus 1)=(1\oplus\mu)\sigma$$

(2) *The axioms of a commutative and cocommutative Hopf algebra with S the antipode, μ the product, Δ the coproduct and τ the transpose. These include associativity*

$$\mu(1\oplus\mu)=\mu(1\oplus\mu)$$
$$(1\oplus\Delta)\Delta=(1\oplus\Delta)\Delta$$

commutativity

$$\mu=\mu\tau$$
$$\Delta=\tau\Delta$$

and of course the major Hopf algebra axioms

$$0=\mu(1\oplus S)\Delta$$
$$\Delta\mu=(\mu\oplus\mu)(1\oplus\tau\oplus 1)(\Delta\oplus\Delta)$$

(3) *Clearly the set up should include the "monoidal" relations*

$$(f\oplus 1)(1\oplus g)=(f\oplus g)=(1\oplus g)(f\oplus 1)$$

... □

To make this precise, one should consider the category $\mathcal{Z}$ in Section 1.

Just for an illustration: The bialgebra axiom (the last relation of (2) in Proposition 2) reads in matrix form as

$$\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1&1\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0\\1&0\\0&1\\0&1\end{pmatrix}$$

Informally speaking, the mentioned relations permit to move in a composition the Δ to the right and the μ to the left. In the middle there remains a combination of τ and S. The latter is a permutation matrix with entries ± 1, that is an element of the normalizer of the diagonal matrices in $\mathrm{GL}_n(\mathbf{Z})$.

By the way, to think about the relations, I found it useful to use diagrammatic calculus as for many tensor categories.

It is obvious to think of the classical presentations of $\mathrm{GL}(n, \mathbf{Z})$ or $\mathrm{SL}(n, \mathbf{Z})$ in terms of generators and relations. My standard reference here is [2]. A quick reference for $\mathrm{SL}(n, \mathbf{Z})$ is Wikipedia: “Special linear group”, section “Generators and relations”. The generators

$$T_{ij} = e_{ij}(1) \qquad (i \neq j)$$

have the relations

$$(T_{12}T_{21}^{-1}T_{12})^4 = 1 \tag{1}$$

$$[T_{ij}, T_{k\ell}] = 1 \qquad (i, j, k, \ell \text{ pairwise distinct}) \tag{2}$$

$$[T_{ij}, T_{ik}] = 1 \qquad (j \neq k) \tag{3}$$

$$[T_{ji}, T_{ki}] = 1 \qquad (j \neq k) \tag{4}$$

$$[T_{ij}, T_{jk}] = T_{ik} \qquad (i \neq k) \tag{5}$$

Relation (1) considered for $\mathrm{GL}(2, \mathbf{Z})$ is essentially the relation (R) together with relations for S, τ, namely $S^2 = 1$, $\tau^2 = 1$ and a commutation relation.

Relation (2) is in our context encoded in the word “monoidal”.

Relation (3) is related with the associativity of the product μ, relation (4) with the associativity of the coproduct Δ.

If I am not mistaken, relation (5) is useful for the application $H = \Lambda^\bullet V$.

References

[1] S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[2] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966.

[3] J. Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J., 1971, Annals of Mathematics Studies, No. 72.

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

Email address: `rost at math.uni-bielefeld.de`

URL: `www.math.uni-bielefeld.de/~rost`