NOTES ON INVARIANTS FOR QUADRATIC FORMS

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All invariants take value in a cycle module M.

1. Isometry invariants

The Stiefel-Whitney classes w_i....

2. Similiarity invariants

$\mathbf{H}$ denotes a hyperbolic plane, $\mathbf{H}_k$ denotes a hyperbolic space of dimension $2k$.

Lemma 1. *Let q be a quadratic form of dimension $i+2k-1$, $k \geq 0$. Then*

$$v_i(q) = w_i(q - \mathbf{H}_k)$$

is a similarity invariant.

Proof. Note that $v_i(cq) - v_i(q)$ can be expressed in the $w_j(q)$, $j \leq i-1$. Therefore it suffices to check $v_i(cq) - v_i(q) = 0$ for q with anisotropic dimension less that i, that is, one reduces to $k = 0$. But then $v_i(q) = 0$. □

Lemma 2. *Any similarity invariant α for n-dimensional forms can be uniquely written as*

$$\alpha = \alpha_0 + \sum_{i=1}^{[\frac{n}{2}]} v_{n+1-2i}(q)\alpha_i$$

with $2\alpha_i = 0$ for $i > 0$.

Proof. By induction on n. For $(n-2)$-dimensional forms q' define

$$\alpha'(q') = \alpha(q' \perp \mathbf{H}).$$

Then α' is a similarity invariant and therefore

$$\alpha'(q') = \alpha_0 + \sum_{i=1}^{[\frac{n-2}{2}]} v_{n+1-2i}(q')\alpha_i.$$

for some α_i with $2\alpha_i = 0$ for $i > 0$. After replacing α by

$$\alpha - \alpha_0 - \sum_{i=1}^{[\frac{n-2}{2}]} v_{n+1-2i}\alpha_i.$$

we may assume that α vanishes for isotropic q.

Write

$$\alpha = \beta_0 + \sum_{i=1}^{n} w_i\beta_i.$$

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Then

$$\alpha(cq) - \alpha(q) = \{c\}(w_{n-1}(q)\beta_n + \sum_{i=0}^{n-2} w_i(q)\gamma_i)$$

for some γ_i. It follows that $\beta_n = 0$. After replacing α by $\alpha - v_{n-1}\beta_{n-1}$ we may assume $\beta_{n-1} = 0$.

But then it suffices to test α on forms $\langle t_1, \dots, t_{n-2}, 1, -1\rangle$. It follows that $\alpha = 0$ since α vanishes on isotropic q. □

Lemma 3. *Let q be a quadratic form of dimension i and of determinant $-(-1)^i$. Then $w_i(q) = 0$.*

Proof. Let $q = \langle t_1, \dots, t_{i-1}, -(-1)^i t_1 \cdots t_{i-1}\rangle$ and let $x_j = \{t_j\}$. Then

$$\begin{aligned} w_i(q) &= x_1 \cdots x_{i-1}\big((i-1)\{-1\} + x_1 + \cdots + x_{i-1})\big) \\ &= x_1 \cdots x_{i-1}\big((i-1)\{-1\} + \{-1\} + \cdots + \{-1\}\big) = 0 \end{aligned}$$

□

Lemma 4. *Let q be a quadratic form of dimension $2i + 2k$ and of determinant $-(-1)^k$, $k \geq 0$. Then*

$$\eta_i(q) = w_{2i}(q - \mathbf{H}_k)$$

is a similarity invariant.

Proof. Note that $\eta_i(cq) - \eta_i(q)$ can be expressed in the $w_j(q)$, $j \leq 2i - 1$. Therefore it suffices to check $\eta_i(cq) - \eta_i(q) = 0$ for q with anisotropic dimension less that $2i$, that is, one reduces to $k = 0$. But then $\eta_i(q) = 0$ by the previous lemma. □

The last lemma can be extended to forms of arbitrary fixed determinant δ:

Lemma 5. *Let q be a quadratic form of dimension $2i + 2k$ and of determinant δ, $k \geq 0$. Then*

$$\eta_i(q) = w_{2i}(q - \mathbf{H}_k) \mod \{-(-1)^k\delta\}M(F)$$

is a similarity invariant. □

Recall that $\operatorname{disc}(q) = (-1)^n \det(q)$

Note that we also obtained also invariants for $2n$-dimensional forms of fixed discriminant δ:

$$\overline{\eta}_i(q) = j(\eta_i(q)) \in H^{2i}(F, \mu_4^{\otimes i}(\delta))$$

where j denotes the injective (modulo Milnor's conjecture) map

$$K_{2i}F/\big(2K_{2i}F + \{-(-1)^i\delta\}K_{2i-1}F\big) \to H^{2i}(F, \mu_4^{\otimes i}(\delta)).$$

Theorem 6. *Every similarity invariant for $2n$-dimensional forms of fixed discriminant δ can be uniquely written as*

$$\alpha = \alpha_0 + \sum_{i=1}^{n-1} \eta_i\alpha_i$$

with $2\alpha_i = 0$ and $\{-(-1)^i\delta\}\alpha_i = 0$ for $i > 0$.

Proof. Not yet provided. □

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