On the spinor norm and $A_0(X,K_1)$ for quadrics

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I. Introduction

1) We denote by K_nF the n-th Milnor K-group of field (for convenience).

For X/F projective there is a complex

$$\bigoplus_{v\in X_{(1)}} K_{n+1}K(v) \xrightarrow{d} \bigoplus_{v\in X_{(0)}} K_nK(v) \xrightarrow{N} K_nF$$

where d is given by the tame symbol and N is given by the norm map in Milnor K-Theory. We denote $\operatorname{coker} d$ by $A_0(X,K_n)$ and by $N_X : A_0(X,K_n) \to K_nF$ the induced norm map.

2) In this note all fields have characteristic different from 2.

Let $\varphi : V \to F$ be a quadratic module. We denote by $X_\varphi \subset \mathbb{P}V$ the associated projective quadric hypersurface and we put $N_\varphi = N_{X_\varphi}$. Note that $X_{c\varphi} = X_\varphi$.

Let $C(\varphi)$, $C_0(\varphi)$ be the (even) Clifford algebra of φ and consider V as a subspace of $C(\varphi)$ in the usual way.

The special Clifford group of φ is defined as

$$S\Gamma(\varphi) = \{\alpha \in C_0(\varphi) \mid \alpha V \alpha^{-1} = V \text{ in } C(\varphi)\}.$$

One has a commutative diagram

$$\begin{array}{ccccccc} S\Gamma \longrightarrow F^* & \longrightarrow & S\Gamma(\varphi) & \longrightarrow & SO(\varphi) & \longrightarrow & 1 \\ \Vert & & \downarrow{\scriptstyle \mathrm{sn}} & & \downarrow{\scriptstyle \mathrm{sn}} & & \\ F^* & \xrightarrow{2} & F^* & \longrightarrow & F^*/(F^*)^2 & \longrightarrow & 1 \end{array}$$

where $F^* \subset S\Gamma(\varphi)$ is central and $S\Gamma(\varphi)$ acts on (V,φ) by $\alpha(v) = \alpha v \alpha^{-1}$; the spinor norm sn is given by $\mathrm{sn}(\alpha) = \alpha^t\alpha$.

If $\dim\varphi = 2$, the $S\Gamma(\varphi) = C_0(\varphi)^*$, $C_0(\varphi)$ is the quadratic extension of F defined by the discriminant of φ and sn is given by the norm for $C_0(\varphi)/F$.

If $\dim\varphi = 3$, then $S\Gamma(\varphi) = C_0(\varphi)^*$, $C_0(\varphi)$ is a quaternion algebra and sn is induced by the reduced norm for the algebra $C_0(\varphi) \mid F$.

If φ_0 is a subform of φ (i.e. $\varphi_0 = \varphi \mid V_0$ for some subspace V_0 of V), then $S\Gamma(\varphi_0) \subset S\Gamma(\varphi)$.

3) In this note we construct a natural homomorphism

$$\tilde\omega_\varphi : S\Gamma(\varphi) \longrightarrow A_0(X_\varphi, K_1)$$

such that $N_\varphi \circ \tilde{\omega}_\varphi = \mathrm{sn}$.

$\tilde{\omega}_\varphi$ is surjective (at least if F has no odd extensions). Therefore, if one investigates the injectivity of N_φ, one is let to consider the kernel of the spinor norm.

4) An element $\alpha \in S\Gamma(\varphi)$ is called plane, if $\alpha \in S\Gamma(\varphi_0)$ for a 2-dimensional subform φ_0 of φ. It is known that $S\Gamma(\varphi)$ is generated by plane elements (Dieudonné). We denote by $\overline{S\Gamma(\varphi)}$ the quotient of $S\Gamma(\varphi)$ by its commutator subgroup and elements of the form $\alpha\beta^{-1}$, where $\alpha, \beta \in S\Gamma(\varphi)$ are plane such that $\mathrm{sn}(\alpha) = \mathrm{sn}(\beta)$. Let $\overline{\mathrm{sn}} : \overline{S\Gamma(\varphi)} \to F^*$ be the homomorphism induced by sn. An element of $\overline{S\Gamma(\varphi)}$ is called plane, if it equals $\bar{\alpha}$ for some plane $\alpha \in S\Gamma(\varphi)$. It turns out that $\tilde{\omega}_\varphi$ factors through a homomorphism $\omega_\varphi : S\Gamma(\varphi) \to A_0(X_\varphi, K_1)$.

5) Since $N \circ \omega_\varphi = \overline{\mathrm{sn}}$ and ω_φ is surjective, we have

Theorem 1.
If every element of $\overline{S\Gamma(\varphi)}$ can be written as a product of two plane elements, then N_φ is injective in degree 1.

All forms φ, for which I can prove the injectivity of N_φ satisfy the the hypothesis of Theorem 1. We have

Proposition 2.
φ satisfies the hypothesis of Theorem 1 in the following cases

i) $\dim \varphi \leq 5$

ii) $\varphi = \rho \otimes (\psi \oplus \langle c \rangle)$, *where ρ is a Pfister form, ψ is a Pfister neighbor and $c \in F^*$*

iii) $\varphi = \psi \oplus c\langle 1, 1 \rangle$, *where ψ is a Pfister neighbor and $c \in F^*$.*

Recall that a Pfister neighbor is a form of type $\psi_0 \oplus b\psi_1$ where ψ_0 is a Pfister form and ψ_1 is a subform of ψ_0. Note that every Pfister neighbor (hence every Pfister form) is included in case ii).

6) The perhaps simplest type of quadratic forms not covered by i), ii) or iii) are 6-dimensional forms φ such that $C_0(\varphi)$ has (maximal) index 4 over its center. We have

Proposition 3. *There exists a field F and 6-dimensional quadratic form φ over F, such that N_φ is not injective. φ can be chosen to have discriminant* 1.

This result is based on a relation between $\mathrm{Ker} N_\varphi$ and $SK_1(C_0(\varphi))$ which is obtained by Swan's computation of $K_1(X_\varphi)$.
To give an explicit example, let $A = D(a, b) \otimes D(\bar{a}, \bar{b})$ a tensor-product of two quaternion algebras such that $|SK_1 A| > 2$. Let

$$\varphi = \langle -a, -b, ab \rangle \oplus -\langle -\bar{a}, -\bar{b}, \bar{a}\bar{b} \rangle$$

be the associated form (X_φ is the Grassmanian for submodules of A of rank 8).
By Swan we have $K_1(X) = (K_1F)^4 \oplus K_1A \oplus K_1A$. Consider

$$j : A_0(X_\varphi, K_1) = H^4(X; K_5) \longrightarrow K_1(X) \longrightarrow K_1A$$

where the last map is given by projection to one of the factors. One may check that $\mathrm{Nrd} \circ j = 2N_\varphi$. Hence $j(\ker N_\varphi) \subset SK_1A$. One can show that $SK_1(A)/j(\mathrm{Ker} N_\varphi)$ is of order at most 2; it is generated by $j(u)$ for any u such that $N_\varphi(u) = -1$.

II. The special Clifford group

Remark added in July 1996: In the following I seem to care only on anisotropic forms. The much simpler case of isotropic forms should considered in the very beginning.

Let $\varphi : V \to F$ be a quadratic module. For $\alpha \in S\Gamma(\varphi)$ let

$$\operatorname{supp}\alpha = \{v \in V;\ \alpha(v) = v\}^\perp.$$

Note that $\alpha \in S\Gamma(\varphi \mid \operatorname{supp}(\alpha))$. Clearly

$$\operatorname{supp}\ \alpha = 0 \Longleftrightarrow \alpha \in F^*$$

$$\dim\operatorname{supp}\ \alpha \leq 2 \Longleftrightarrow \alpha \text{ is plane }.$$

Since the product of any two reflections in $O(\varphi)$ is a plane rotation, we have the following consequence of the theorem of Cartan-Dieudonné.

Proposition 4.
Any element of $S\Gamma(\varphi)$ can be written as product of $\left[\frac{\dim\varphi}{2}\right]$ plane elements. □

Consequently $\dim(\operatorname{supp}\alpha) \neq 1, 3$ for $\alpha \in S\Gamma(\varphi)$. Two plane elements α, β are called to be **linked** if $\dim(\operatorname{supp}\alpha + \operatorname{supp}\beta) \leq 3$. In this case $\alpha\beta$ is again plane, because $\operatorname{supp}\alpha\beta \subset \operatorname{supp}\alpha + \operatorname{supp}\beta$ and $\dim(\operatorname{supp}\alpha\beta) = 3$ is impossible.
Note that α and β commute, if $\operatorname{supp}\alpha \perp \operatorname{supp}\beta$.

Theorem 5.
Let G be the free group on the set of all plane elements of $S\Gamma(\varphi)$. Denote by g_α the generator corresponding to α. Then

$$G \longrightarrow S\Gamma(\varphi), \quad g_\alpha \longrightarrow \alpha$$

is surjective and its kernel is the normal subgroup generated by elements of the form

R_1) $g_\alpha g_\beta g_{\alpha\beta}^{-1}$ *if α and β are linked.*

R_2) $[g_\alpha, g_\beta]$ *if $\operatorname{supp}(\alpha) \perp \operatorname{supp}(\beta)$.*

For $v_1, v_2 \in V$ such that $\varphi(v_1) = \varphi(v_2)$ we denote by $\varepsilon(v_2, v_1) \in S\Gamma(\varphi)$ the trivial element if $v_1 = v_2$, otherwise any plane element such that $\varepsilon(v_2, v_1)(v_1) = v_2$. Note that a plane $\alpha \in S\Gamma(\varphi)$ is linked with $\varepsilon(\alpha(v), v)$ for any $v \in V$.

Remark added in Jan 1998: The plane element $\varepsilon(v_2, v_1)$ is assumed to have support in the subspace generated by v_1, v_2.

Let $\hat{G}$ be the quotient of G by the relations R_1, R_2 and denote by $\hat{g}$ the image of g_α in $\hat{G}$. Theorem 5 states that $\hat{G} \to S\Gamma(\varphi)$, $\hat{g} \to \alpha$ is bijective. Surjectivity follows from Proposition 4. In the following we proof the injectivity (I don't know, whether and in how far this question has been considered in the literature).

Lemma 6.

$$\hat{g}_\alpha \hat{g}_\beta \hat{g}_\alpha^{-1} = \hat{g}_{\alpha\beta\alpha^{-1}} \quad \text{for plane} \quad \alpha, \beta \in S\Gamma(\varphi).$$

Proof. Let $W = (\operatorname{supp} \alpha)^\perp \cap \operatorname{supp} \beta$.

If $\dim W = 2$, then $\operatorname{supp} \beta \subset (\operatorname{supp} \alpha)^\perp$. Therefore $\alpha\beta\alpha^{-1} = \beta$ and the lemma follows from R_2.

If $\dim W < 2$, then there exists a nonzero $v \in W^\perp \cap \operatorname{supp} \beta$. Then α, β are linked with $\gamma = \varepsilon(\alpha(v), v)$, hence by R_2:

$$\hat{g}_\alpha \hat{g}_\beta \hat{g}_\alpha^{-1} \hat{g}_{\alpha\beta\alpha^{-1}} = \hat{g}_{\alpha'} \hat{g}_{\beta'} \hat{g}_{\alpha'}^{-1} \hat{g}_{\alpha'\beta'\alpha'^{-1}}$$

with $\alpha' = \alpha\gamma^{-1}$, $\beta' = \gamma\beta\gamma^{-1}$.

Note that γ fixes W, because $v \in W^\perp$ and α fixes W. This shows $W \subset W' = (\operatorname{supp} \alpha')^\perp \cap \operatorname{supp} \beta'$. Moreover $\alpha(v) = \gamma(v) \in W' \backslash W$, hence $\dim W' > W$ and we are left with the case $\dim W = 2$ after eventually repeating this argument. □

End of the proof of Theorem 5: Let $\alpha_1, \ldots, \alpha_N \in S\Gamma(\varphi)$ be plane such that $\alpha_1 \ldots \alpha_N = 1$. We show $\hat{g}_{\alpha_1} \ldots \hat{g}_{\alpha_N} = 1$ by induction on $\dim \varphi$.
Let $v \in V$ be any anisotropic vector, let $v_i = \alpha_i \ldots \alpha_n(v)$, $v_{N+1} = v = v_1$, let $\gamma_i = \varepsilon(v_i, v_{i+1})$ and $\beta_i = \alpha_i \gamma_i^{-1}$. α_i and γ_i are linked, because $\alpha_i(v_{i+1}) = v$, thus $\hat{g}_{\alpha_i} = \hat{g}_{\beta_i} \hat{g}_{\gamma_i}$.

Put $\delta_i = \gamma_1 \ldots \gamma_i$, $(\delta_0 = 1)$; δ_i is plane and is one choice of $\varepsilon(v, v_{i+1})$. Hence γ_i and δ_{i-1} are linked and therefore $\hat{g}_{\delta_i} = \hat{g}_{\gamma_1} \ldots \hat{g}_{\gamma_i}$.

Let $\rho_i = \delta_{i-1} \beta_i \delta_{i-1}^{-1}$. Then $g_{\delta_{i-1}} g_{\beta_i} g_{\delta_{i-1}}^{-1} = g_{\rho_i}$ by the lemma. Taking things together we find

$$g_{\alpha_1} \ldots g_{\alpha_N} = g_{\beta_1} g_{\gamma_1} g_{\beta_2} \ldots g_{\beta_N} g_{\gamma_N} = g_{\rho_1} \ldots g_{\rho_N} g_{\rho_N}.$$

Now, we are done, because $\rho_1 \ldots \rho_N$ and δ_N fix v. □

We have the following consequence of Theorem 5.

Corollary 7. Let $\bar{G}$ be the free abelian group generated by all plane elements of $S\Gamma(\varphi)$. Denote by $\bar{g}_\alpha$ the generator corresponding to α. Then

$$\bar{G} \longrightarrow \overline{S\Gamma(\varphi)}, \quad \bar{g}_\alpha \longrightarrow \bar{\alpha}$$

is surjective and its kernel is generated by elements of the form

$\bar{R}_0)$ $\quad \bar{g}_\alpha \bar{g}_\beta^{-1}$ $\quad$ if $\text{sn}(\alpha) = \text{sn}(\beta)$

$\bar{R}_1)$ $\quad \bar{g}_\alpha \bar{g}_\beta \bar{g}_{\alpha\beta}^{-1}$ $\quad$ if α and β are linked. □

The corollary is the basis of our construction of an epimorphism $\omega_\varphi : S\Gamma(\varphi) \to A_0(X, K_1)$ described in the next section.

The rest of this section is devoted to the proof of Proposition 2. Clearly Proposition 2 i) follows from Proposition 4.

Let

$$D(\varphi) = \{\varphi(v);\ v \in V \text{ anisotropic}\} \subset F^*$$

and

$$N(\varphi) = \{\text{sn}(\alpha);\ \alpha \in S\Gamma(\varphi) \text{ plane }\} \subset F^*.$$

$N(\varphi)$ consists just of all norms from quadratic extensions K/F such that φ_K is isotropic. Define

$$\textstyle\sum_\varphi : N(\varphi) \longrightarrow \overline{S\Gamma(\varphi)},\ \sum_\varphi(\text{sn}(\alpha)) = \bar{\alpha}$$

$\sum_\varphi$ is welldefined by the very definition of $\overline{S\Gamma(\varphi)}$. $\sum_\varphi$ is injective, since sn is a left inverse. $\sum_\varphi(N(\varphi))$ generates $\overline{S\Gamma(\varphi)}$ by Proposition 4.

For a subform φ_0 of φ, we denote by $i_* : \overline{S\Gamma(\varphi_0)} \to \overline{S\Gamma(\varphi)}$ the homomorphism induced by the inclusion $S\Gamma(\varphi_0) \subset S\Gamma(\varphi)$.

Lemma 8.
Let φ be a quadratic form. Then

i) $D(\varphi) \cdot D(\varphi) = N(\varphi)$.

ii) If φ represents 1, then $D(\varphi) \subset N(\varphi)$.

iii) Let $v \in V$ and $\alpha \in S\Gamma(\varphi)$ plane, such that $\varphi(v) = 1$ and $v \in \operatorname{supp} \alpha$. Then $\text{sn}(\alpha) \in D(\varphi)$.

iv) Let $v \in V$ and $\alpha \in S\Gamma(\varphi)$. Then $\alpha = \alpha_1 \ \dots \ \alpha_n$ with α_i plane and $v \in \operatorname{supp} \alpha_i$.

v) If φ represents 1, then

$$\textstyle\sum_\varphi(\varphi(v_1)\varphi(v_2)) = \sum_\varphi(\varphi(v_1)) \sum_\varphi(\varphi(v_2))$$

for any anisotropic $v_1, v_2 \in V$. (The left hand side is defined by i)).

vi) Suppose ψ represents 1 and let $\varphi = \psi \oplus \langle b \rangle$. Then

$$\overline{S\Gamma(\varphi)} = i_*(\overline{S\Gamma(\psi)}) \cdot \textstyle\sum_\varphi(D(\varphi)).$$

Proof.
It is easy to check i) - iii) for dim $\varphi = 2$ and iv) - vi) for dim $\varphi = 3$. One may however reduce to these cases by restriction to appropriate subforms. This is obvious except for iv) and vi). For iv) one has to consider only plane elements α by Proposition 4; hence one may replace φ by $\varphi \mid (\operatorname{supp} \alpha + vf)$, which is of dimension ≤ 3.

For vi) the reduction can be done as follows. Write $(V, \varphi) = (W, \psi) \oplus (F, \langle b \rangle)$ and let $v_0 = (0, 1) \in W \times F = V$. For a given $\alpha \in S\Gamma(\varphi)$ put $\beta = \varepsilon(\alpha(v_0), v_0)$. Then $\beta^{-1} \cdot \alpha$ fixes v_0 and is therefore contained in $S\Gamma(\psi)$. Hence it remains to show $\bar{\beta} \in i_*(S\Gamma(\psi)) \cdot \sum_\varphi(D(\varphi))$ for which one may restrict to $\varphi \mid V'$ with $V' = \operatorname{supp} \beta + Fv_1$, where $v_1 \in V$ is such that $\varphi(v_1) = 1$. □

Lemma 9.
Let $\psi = \langle 1 \rangle \oplus \psi'$ be a Pfister form, let $\bar{\psi}'$ be a subform of ψ' and let $\bar{\psi} = \langle 1 \rangle \oplus \bar{\psi}'$. Moreover let ζ be an arbitrary form representing 1 and let $b \in F^*$. Put $\varphi = (\psi \otimes \zeta) \oplus \langle b \rangle$ and $\hat{\varphi} = \varphi \oplus b\bar{\psi}' = (\psi \otimes \zeta) \oplus b\bar{\psi}$. Then

$$i_* : \overline{S\Gamma(\varphi)} \longrightarrow \overline{S\Gamma(\hat{\varphi})}$$

is surjective.

Proof.
Since $\hat{\varphi}$ represents 1 we know that $\sum_{\hat{\varphi}}(D(\hat{\varphi}))$ generates $\overline{S\Gamma(\hat{\varphi})}$ by Lemma 8 i). Note that

$$D(\hat{\varphi}) \subset D(\varphi) \cdot D(\bar{\psi}) \subset D(\varphi) \cdot D(\varphi) = N(\varphi)$$

since ψ is multiplicative and ζ represents 1. Hence

$$\textstyle\sum_{\hat{\varphi}}(D(\hat{\varphi})) \subset \sum_{\hat{\varphi}}(N(\varphi)) \subset i_*(\overline{S\Gamma(\varphi)})$$

by Lemma 8 v). □

Lemma 10.
Let φ be a Pfister neighbor, i.e. $\varphi = \psi \oplus b\bar{\psi}$ where ψ is a Pfister form, $\bar{\psi}$ is a subform of ψ and $b \in F^*$. Then $\overline{\mathrm{sn}} : \overline{S\Gamma(\varphi)} \to F^*$ is injective and has image $N(\varphi) = D(\psi \otimes \langle\langle -b \rangle\rangle)$.

Proof.
Let $\rho = \psi \otimes \ll -b \gg$. Then $D(\rho)$ is a group as for every Pfister form and therefore $D(\rho) = N(\rho)$ by Lemma 8 i). Since φ_K is isotropic if and only if ρ_K is isotropic, we have $N(\varphi) = D(\rho)$. Therefore $N(\varphi)$ is a group, hence $\mathrm{Im}\ \mathrm{sn} = N(\varphi) = D(\rho)$.

Injectivity of $\overline{\mathrm{sn}}$: If φ is a Pfister form (i.e. $\bar{\psi} = \psi$), then $\sum_\varphi$ is a homomorphism in view of Lemma 8 v) and is a left inverse to $\overline{\mathrm{sn}}$.

In the general case one may apply Lemma 9 with $\zeta = \langle 1 \rangle$ to reduce to case $\bar{\psi} = \langle 1 \rangle$. In this case we find by Lemma 8 vi) and the above remarks for Pfister forms:

$$\overline{S\Gamma(\varphi)} = i_*(\overline{S\Gamma(\psi)}) \cdot \textstyle\sum_\varphi(D(\varphi)) = \sum_\varphi(N(\psi)) \cdot \sum_\varphi(D(\varphi)).$$

Hence every element in $\overline{S\Gamma(\varphi)}$ can be written as product of two plane elements and we are done. □

Proof of Proposition 2 ii).
By Lemma 9 we may replace $\varphi = \rho \otimes (\psi \oplus \langle c \rangle)$ by $\tilde{\varphi} = (\rho \otimes \psi) \oplus \langle c \rangle$. Since $\rho \otimes \psi$ is itself a Pfister neighbor, we may assume $\varphi = \psi \oplus \langle c \rangle$. By Lemma 8 vi) and Lemma 10 we have

$$\overline{S\Gamma}(\varphi) = \textstyle\sum_\varphi(N(\psi)) \cdot \sum_\varphi(D(\varphi)). \qquad \square$$

Proof of Proposition 2 iii).
Write $(V, \varphi) = (W, \psi) + (F \times F, \langle c, c \rangle)$ and let $v_0 = (0, 0, 1)$, $v_1 = (0, 1, 0) \in V = W \times F \times F$. For given $\alpha \in S\Gamma(\varphi)$ let $\beta = \varepsilon(\alpha(v_0), v_0)$ and $\gamma = \varepsilon(\beta^{-1}\alpha(v_1), v_1)$. Then $\delta = \gamma^{-1}\beta^{-1}\alpha$ fixes v_0 and v_1, hence $\delta \in S\Gamma(\psi)$ and $\bar{\delta} \in \sum_\varphi(N(\psi))$ by Lemma 10. By Lemma 8 iii) we have $\mathrm{sn}(\beta)$, $\mathrm{sn}(\gamma) \in D(c\varphi)$ and therefore $\bar{\beta} \cdot \bar{\gamma} \in \sum_\varphi(D(c\varphi)) \cdot \sum_\varphi(D(c\varphi)) = \sum_\varphi(N(\varphi))$. Hence $\bar{\alpha} = \bar{\beta} \cdot \bar{\gamma} \cdot \bar{\delta}$ is in $\sum_\varphi(N(\psi)) \cdot \sum_\varphi(N(\varphi))$. □

III. The map $\overline{S\Gamma(\varphi)} \longrightarrow A_0(X_\varphi, K_1)$

We will use the following

Theorem 11.

i) $A_0(X_\varphi, K_0) \hookrightarrow K_0 F$ *for arbitrary* φ

ii) $A_0(X_\varphi, K_n) \hookrightarrow K_n F$ *for isotropic* φ

iii) $A_0(X_\varphi, K_1) \hookrightarrow K_1 F$ *for* $\dim \varphi = 3$.

i) is proved in [Merkuriev, Suslin; On the norm homomorphism in degree 3].

ii) follows from i) by the norm principle.

iii) is one of the main points in the proof of Hilbert Satz 90 for K_2 for quadratic extensions. It stands at the heart of our construction. It shows that for a 3-dimensional form φ the groups $\overline{S\Gamma(\varphi)} = S\Gamma(\varphi)/[S\Gamma(\varphi), S\Gamma(\varphi)]$ and $A_0(X_\varphi, K_1)$ are naturally isomorphic, because $\overline{\mathrm{sn}}$ and N_φ are injective and have the same image in $F^* = K_1 F$.

Theorem 12.
For quadratic forms φ *over* F *there exists unique homomorphisms*

$$\omega_\varphi : \overline{S\Gamma(\varphi)} \longrightarrow A_0(X_\varphi, K_1)$$

such that

i) If $\dim\ \varphi = 3$, *then* $\omega_\varphi = N_\varphi^{-1} \circ \overline{\mathrm{sn}}$.

ii) If φ_0 *is a subform of* φ, *then*

$$\begin{array}{ccc} \overline{S\Gamma(\varphi_0)} & \xrightarrow{\omega_{\varphi_0}} & A_0(X_{\varphi_0}, K_1) \\ \Big\downarrow{\scriptstyle i_*} & & \Big\downarrow{\scriptstyle i_*} \\ \overline{S\Gamma(\varphi)} & \xrightarrow{\omega_\varphi} & A_0(X_\varphi, K_1) \end{array}$$

is commutative.

Proof.
Let $\bar{G}$ be as in Corollary 7 and define

$$\hat{\omega}_\varphi : \bar{G} \longrightarrow A_0(X_\varphi, K_1)$$

as follows. For $\alpha \in S\Gamma(\varphi)$ plane choose a subform φ_0 of φ of dimension 2 such that $\alpha \in S\Gamma(\varphi_0)$. ($\varphi_0$ is unique if $\alpha \notin F^*$). Then $\alpha \in S\Gamma(\varphi_0) = K_1 F(X_{\varphi_0}) = A_0(X_{\varphi_0}, K_1)$ and we define $\hat{\omega}_\varphi(\bar{g}_\alpha)$ to be the image of α under $A(X_{\varphi_0}, K_1) \to A(X_\varphi, K_1)$. This definition does not depend on the choice of φ_0 because of Theorem 11 i). In order to prove Theorem 12, it suffices to show that $\hat{\omega}_\varphi$ vanishes on the relations $\bar{R}_0$), $\bar{R}_1$) in Corollary 7. This is clear for $\bar{R}_1$) because of Theorem 11 iii) and follows for $\bar{R}_0$) from

Proposition 12.
Let $v_1, v_2 \in X_{(0)}$ *be two points of degree 2, let* $F_i = K(v_i)$ *and let* $\alpha_i \in F_i^*$ *such that* $N_{F_1|F}(\alpha_1) = N_{F_2|F}(\alpha_2)$. *Then* $[\{\alpha_1\}, v_1] - [\{\alpha_2\}, v_2]$ *is in the image of*

$$\bigoplus_{v \in X_{(1)}} K_2K(v) \xrightarrow{d} \bigoplus_{v \in X_{(0)}} K_1K(v).$$

Proof.
Let $F_0 = F_1 \otimes_F F_2$. We have

$$\alpha_i = \{\beta\} + \{N_{F_0|F_i}(\gamma)\}$$

for some $\beta \in F^*$ and $\gamma \in F_0^*$ (take $\beta = (\operatorname{tr}\alpha_1 + \operatorname{tr}\alpha_2)^{-1}$, $\gamma = \alpha_1 + \alpha_2$ in the generic case). Hence

$$[\{\alpha_1\}, v_1] - [\{\alpha_2\}, v_2] = \{\beta\} \cdot (v_1 - v_2) + \operatorname{cor}_{F_1|F}(u_1 - u_2)$$

where $u_1, u_2 \in \bigoplus_{v \in (X_{F_1})_{(0)}} K_1K(v)$ are given by

$$u_1 = [N_{F_0|F_1}\{\gamma\}, \tilde{v}_1],$$

with $\tilde{v}_1$ a rational point of X_{F_1} and

$$u_2 = [\gamma, \tilde{V}_2]$$

with $\tilde{v}_2$ the point over v_2.

Now $\{\beta\} \cdot (v_1 - v_2) \in \operatorname{Im} d$, because $A_0(X_\varphi, K_0) \hookrightarrow K_1F$.

Furthermore, over F_1 we have $N_\varphi(u_1 - u_2) = 0$, hence $u_1 - u_2 \in \operatorname{Im} d$, because X_{F_1} has a rational point. □

Proposition 13.
If F *has no extension of odd degree, then* $\omega_\varphi : \overline{S\Gamma(\varphi)} \to A_0(X_\varphi, K_2)$ *is surjective.*

It follows from Knebusch's norm principle that $\operatorname{Im}(N_\varphi \circ \omega_\varphi) = \operatorname{Im} N_\varphi$. One may use the proof of Knebusch's norm principle to show that ω_φ is surjective in general. Since this is a bit tedious I omit a proof here.

Proof of Theorem 1.
Since $A_0(X_\varphi, K_1) \hookrightarrow K_1F$ for isotropic φ, we have $2\operatorname{Ker} N_\varphi = 0$ by a transfer argument. Hence we may assume that F has no odd extension, again using transfers. But then by Proposition 13 and the very definition of $\overline{S\Gamma(\varphi)}$:

$$\operatorname{Ker} N_\varphi = \omega_\varphi(\operatorname{Ker} \overline{\mathrm{sn}}) = 0.$$ □

For a quadratic form put $D_1(\varphi) = \operatorname{Im} \mathrm{sn} = \operatorname{Im} N_\varphi$. For the proof of Proposition 13 we need the following lemma which can be deduced also from the arguments in [Merkuriev, Suslin; On the norm homomorphism in degree 3].

Lemma 14.
Let $\dim\varphi = 4$ and let H/F be a quadratic extension. Then for every $u \in D_1(\varphi_H)$ there exists (over F) two 3-dimensional subforms φ', φ'' of φ such that $u \in D_1(\varphi'_H)\cdot D_1(\varphi''_H)$.

Proof. (sketch)
Write $\varphi = \langle -a, -b, ab, c\rangle$. It is easy to check that

$$D_1(\varphi) = \mathrm{Nrd}\,(D(a,b) \otimes F(\sqrt{c})) \cap F^* \subset F(\sqrt{c})^*.$$

Using this for φ_H one finds

$$u = \mathrm{Nrd}\,(d)\cdot(1 + c\,\mathrm{Nrd}\,(d'))$$

for some $d, d' \in D(a,b) \otimes_F H$ with $d' + \overline{d'} = 0$. Now put $\varphi' = \langle -a, -b, ab\rangle$, then $\mathrm{Nrd}\,(d) \in D_1(\varphi'_H)$. It is not hard to find $\bar{a}, \bar{b} \in F^*$ such that $D(a,b) \simeq D(\bar{a},\bar{b})$ and $1 + c\mathrm{Nrd}\,(d') \in \mathrm{Nrd}\,(D(\bar{a}c, \bar{b}c) \otimes_F H)$. Since $\bar{\varphi} = \langle -\bar{a}c, -\bar{b}c, a\bar{b}, c\rangle$ has the same even Clifford algebra as φ, we know that $\bar{\varphi}$ is similar to φ by quadratic form theory. Hence $\langle -\bar{a}c, -\bar{b}c, \bar{a}\bar{b}\rangle$ is similar to a subform φ'' of φ. Now we are done because $1 + c\,\mathrm{Nrd}\,(d') \in D_1(\varphi''_H)$.

Consequence.
Let H/F be a quadratic extension. Then

$$\mathrm{cor}_{H/F}(\mathrm{Im}\,\omega_{\varphi_H}) \subset \mathrm{Im}\,\omega_{\varphi}.$$

Proof. We may assume $\dim\varphi = 4$.
For $\alpha \in S\Gamma(\varphi_H)$ plane there exists by Lemma 14 subforms φ', φ'' of φ of dimension 3 and $\alpha' \in S\Gamma(\varphi'_H)$, $\alpha'' \in S\Gamma(\varphi''_H)$ such that $\mathrm{sn}(\alpha) = \mathrm{sn}(\alpha')\,\mathrm{sn}(\alpha'')$. We know that $\overline{\mathrm{sn}}$ is injective, hence

$$\mathrm{cor}_{H/F}(\omega_{\varphi_H}(\alpha)) = \mathrm{cor}_{H/F}(\omega_{\varphi_H}(\overline{\alpha'}) + \omega_{\varphi_H}(\overline{\alpha''})).$$

Now $\mathrm{cor}_{H/F}(\omega_{\varphi_H}(\overline{\alpha'}))$ is in the image of $A_0(X_{\varphi'}, K_1) \to A_0(X_\varphi, K_1)$. But we know that $\omega_{\varphi'}$ is surjective, hence $\mathrm{cor}_{H/F}(\omega_{\varphi_H}(\overline{\alpha'})) \in \mathrm{Im}\,\omega_\varphi$. Similarly $\mathrm{cor}_{H/F}(\omega_{\varphi_H}(\overline{\alpha''})) \in \mathrm{Im}\,\omega_\varphi$. □

Proof of Proposition 13.
By the consequence we have the norm principle for $\mathrm{Im}\,(\omega_\varphi)$ if F has no odd extensions. Since $A_0(X_\varphi, K_1)$ is generated by corestrictions from splitting fields K of φ and since $A_0(X_{\varphi_K}, K_1) = \mathrm{Im}\,\omega_{\varphi_K}$ we conclude $A_0(X_\varphi, K_n) = \mathrm{Im}\,\omega_\varphi$. □