Euler's "Art of Reckoning"[1]

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Abstract:

The Art of Reckoning has always been part of human culture, but to my knowledge there have been only two eminent mathematicians who wrote a book on the subject: Leonardo of Pisa in 1202 and Leonhard Euler in 1738 (for schools in St. Petersburg). While Fibonacci assumes some prior familiarity with it, Euler develops it from its beginning in a timelessly clear fashion. In his preface he expresses his conviction that learning is without value as long as it does not lead to understanding. I think that even today his book can serve as a guide for anyone who wishes to teach arithmetic to children. In my talk I will pay particular attention to the kind of problems used in Liping Ma's survey.

During the last four years, I have been involved in the math–education of future elementary school teachers, and for much of that time, I was searching for a book in which elementary arithmetic would be developed from scratch in a manner understandable to T. C. MITS[3]. When — by chance — I came across EULER'S '*Einleitung zur Rechenkunst*', I immediately realized that I found it. EULER'S book contains all the arithmetic that — in my opinion — every elementary school teacher should have '*learned*'[4]. I think that Liping Ma's findings, published in her book *'Knowing and Teaching Elementary Mathematics': Teachers' Understanding of Fundamental Mathematics in China and the United States*', confirm that view.

To illustrate my conviction I shall juxtapose the first two questions from Liping Ma's book with corresponding parts of EULER'S '*Einleitung zur Rechenkunst*'.

To a German audience I could directly present EULER'S words. They sound a bit old fashioned and long–winded to modern ears, but they would force the reader to read slowly. Thereby the reader would discover that every word is chosen carefully and would also find that an adaption of EULER'S language to modern German would not be easy.

An adequate translation into English is even less so. My friend KLAUS HOECHSMANN took on the demanding job of translating those parts of EULER'S book which I want to use in this talk. I think that he did it in a way which preserved EULER'S intentions, and I thank him for his invaluable help.

I am indebted to him in another respect. When I started my job as mathematics educator for primary school teachers four years ago, I rarely used a

[1] LEONHARD EULER, *Einleitung zur Rechenkunst*, OPERA OMNIA, SER. III, VOL. 2

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[3] T. C. MITS := **T**he **C**elebrated **M**an **I**n **T**he **S**treet. LILLIAN R. LIEBER and HUGH GRAY LIEBER, The education of T. C. MITS.

[4] "*Please forget what you have learned in school; you havent't learned it.*" EDMUND LANDAU in the 'Preface for the Student' in his '*Grundlagen der Analysis*'.

pocket–calculator. At that time — long before I came to know EULER'S 'Art of Reckoning' — he showed me how to extend the range of a pocket–calculator 'beyond its natural bounds'. This will be used in example at the end of my talk.

Subtraction With Regrouping: Approaches To Teaching A Topic

Scenario

Let's spend some time thinking about one particular topic that you may work with when you teach subtraction with regrouping. Look at these questions

$$\begin{array}{r} 52 \\ -25 \\ \hline \end{array} \qquad \begin{array}{r} 91 \\ -79 \\ \hline \end{array} \qquad \text{etc.}$$

How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

Reading Euler

Rule

But if the pieces of a certain value are more abundant in the second number than in the first one — so that subtraction cannot succeed in the manner described — then one piece of the next higher value must be removed from the first number and changed into 10 pieces of the lower value, thus making subtraction possible. For the next step in the subtraction, however, it must be remembered that the first number has been diminished by one.

Reason

Just as in an addition, when there are more than 9 pieces of a certain value, and ten of them are removed and changed into one piece of the next higher value, so — but in reverse — it happens in subtraction: when not enough pieces of one value are available in the first number to allow subtracting the amount required by the second number, one piece of the next higher value is taken off, yielding 10 of the previous type, and those ten are put in with the rest. For,

- if from a number one Hundred, say, is taken away
- but 10 Tens are added in again,
- the size of number does not change.

Consequently a change of this sort may surely be used to facilitate subtraction.

Multidigit Number Multiplication: Dealing With Students' Mistakes

Scenario

Some sixth–grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate

$$\begin{array}{r} 123 \\ \times\ 645 \\ \hline \end{array}$$

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{r} 123 \\ \times\ 645 \\ \hline 615 \\ 492 \\ 738 \\ \hline 1845 \end{array} \qquad \text{instead of this:} \qquad \begin{array}{rrrrr} & & 1 & 2 & 3 \\ \times & & 6 & 4 & 5 \\ \hline & & 6 & 1 & 5 \\ & 4 & 9 & 2 & \\ 7 & 3 & 8 & & \\ \hline 7 & 9 & 3 & 3 & 5 \end{array}$$

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this?

Reading Euler

Rule

- *If a number, however large it may be, is to be multiplied by 10, one only needs to append a zero to the right of it.*
- *If a number is to be multiplied by 100, one has to append two zeroes.*
- *To multiply by 1000 one appends three zeroes,*
- *. . . and so on, always as many zeroes as follow the 1 in the multiplier.*

Explanation

If a number is to be multiplied by 10 then every part of that number must be multiplied by 10.

- Multiplying the Units by 10 yields as many Tens as there were Units.
- The Tens, however, are changed into Hundreds,
- the Hundreds into Thousands,
- . . . and so on.

Since, by appending a 0 to the right of a given number, every value is changed into the next higher one – which is ten times greater — it makes the entire number ten times greater. Consequently 10 times 5783 is as much as 57830.

Likewise, if two zeroes are appended to the right of a number the Units become Hundreds, the Tens become Thousands, the Hundreds become Tenthousands and so on; every value is changed into another, which is 100 times greater.

- Hence by appending two zeroes, the entire number is multiplied by 100. Thus, when 328 is multiplied by 100, one obtains 32 800.
- In a similar way one sees that, if three zeroes are appended to a number it becomes 1000 times bigger, and so forth. If, for example, 5430 is multiplied by 1000000, the product will be 5430000000.
- In this way, one sees how to multiply any number by a number written as a 1 followed by a string of zeroes.

And this is the basis of all rules for multiplication by large composite numbers, as will be further explained below.

Rule

If the multiplier – that is, the number by which a given number is to be multiplied – is a simple number followed by a string of zeros, such as 60, 300, 4000, 70000 etc., then one finds the product by multiplying first with the simple number and then appending to the right hand as many zeros as are in the multiplier.

Explanation

Fundamental Rule

If the multiplier for a given number is itself the result of multiplying two numbers, one obtains the true product by first multiplying the given number by one of these two numbers and then multiplying this product by the other number.

Explanation of the Fundamental Rule

When I am to multiply 47 with 6 — because 6 is as much as 2 times 3 — I find the product by multiplying first with 2, which makes 94, and then multiplying those 94 with 3, which gives 282; and that is the number which results when 47 is multiplied by 6. Because 6 is 2 times 3, the product to be determined — namely 6 times 47 — is the same as 2 times 3 times 47 or 3 times 2 times 47. To find what 3 times 2 times 47 is, one first finds 2 times 47, which is 94. Therefore 3 times 2 times 47 is 3 times 94, and consequently 3 times 94 is the same as 6 times 47. This shows the reason for the fundamental rule which is equally valid in all possible cases.

By the fundamental rule, many multiplications can be performed even if the multiplier is not a simple number[5]. So, for example if 127 is to be multiplied by 63, then since 63 is the same as 7 times 9, one may first multiply the

[5] The fundamental rule is one of the ingredients for daily life calculations outside the standard schemes. It is known that Gauss was inclined to this type of reckoning. See: C. F. Gauss, *Werke* X_2, P. Maennchen, '*Gauss als Zahlenrechner*' and C. F. Gauss *Disquisitiones Arithmeticae*, Part Six.

number 127 by 7, getting 889. Thereafter multiplication of 889 by 9 gives 8001, and this is as many as 63 times 127. Since 8001 is 9 times 889, and since 889 is 7 times 127, it follows that 8001 is 9 times 7 times 127. But 9 times 7 ist 63, and therefore 8001 is 63 times 127. This example clarifies truth of the fundamental rule even further.

To arrive at the proposed rule itself, however, it is to be noted that a simple number with a string of trailing zeroes results from multiplying that simple number by a 1 with as many zeros attached. Therefore, to multiply by such a number, one can first multiply by the simple number, and the resulting product by a 1 with as many zeroes attached. It has already been shown that this can be done by simply appending that many zeroes.

So, if the multiplier is a single digit with a string of trailing zeroes,

- one multiplies first with the simple number
- and then appends to the product as many zeroes as the multiplier has to the right of its first digit.

If, for instance, 543 is to be multiplied by 700 one first multiplies 543 by 7, which is 3801, and with two appended zeroes gives 380100 — and that is the desired product, namely 700 times 543. Since multiplication by a simple number is carried out from right to left, one may immediatly write down on the right as many zeros as there are in the multiplier and then carry out the multiplication by the simple number. In this manner the operation in the preceeding example is done as follows:

543	number
700	multiplier
380100	product

Rule

If the multiplier is a number with many digits, the given number must be multiplied by every part of the multiplier, and all these products must be added, since the resulting sum will be the desired product.

Reason

We have already shown that if the given number is composed of several parts, then every part must be multiplied by the multiplier and all these particular products be put together, as their sum will give the desired product.

- But since the two factors may be interchanged, this also applies to the multiplier.

Therefore if the latter has several digits, the given number must be multiplied by each part, and all these individual products must be added: because their sum gives the desired product.

The parts of a multidigit number, however, are the different value groups, such as Units, Tens, Hundreds, etc., each containing no more than 9 pieces.

- Therefore the given number must be multiplied first by as many Units as there are in the multiplier,
- then with as many Tens
- and similarly with as many Hundreds, and so on as there are in the multiplier,
- and all the products obtained in this way must be put into a sum.

It has already been shown how to multiply by a simple number with a string of trailing zeroes, and the various parts of the multiplier are numbers of precisely this type — whence multiplication with such a compound or multidigit multiplier will cause no difficulties.

If for example one has to multiply 4738 with 358 then the parts of the multiplicator are firstly 8, then 50 and thirdly 300.

- Because of this one multiplies the number 4738 first with 8, which gives 87904.
- Then one multiplies the number 4738 with 50, getting 236900.
- Thirdly one multiplies the given number with 300, and this yields 1421400.

These three products have to be added as follows:

$$\begin{array}{r} 37904 \\ 236900 \\ 1421400 \\ \hline 1696204 \end{array}$$

and the sum is the demanded product.

In order to make the whole operation more comfortable one writes the resulting products under each other in such a manner that Units come under Units and all the like value types are placed under each other, so that addition can be carried out immediately as follows:

$$\begin{array}{rl} 4738 & \quad\text{number} \\ 358 & \quad\text{multiplier} \\ \hline 37904 & \\ 236900 & \\ 1421400 & \\ \hline 1696204 & \quad\text{product} \end{array}$$

- One first writes the multiplier under the number to be multiplied, and traces a line under them.

- After this one multiplies the number by as many units as there are in the multiplier, namely 8, and writes the product under the line as been taught in the multiplication with simple numbers.
- Then one multiplies with the Tens of the multiplier, i.e., by 50, writing — according to the rule given above — into the place of the Units a zero, and then simply multiplying by 5.
- Thirdly one multiplies with the Hundreds, in this case by 300, writing into the first two places from the right two zeroes, and then simply multiplying by 3.
- If one writes the digits properly under each other, then, in the resulting products, the different value types are placed under each other, making addition much easier.
- After one has found all the particular products, a line is drawn under them and they are added together, thus yielding the required product.

One may — for the sake of brevity — leave out the zeroes which had to be written on the right for the higher value types, because they contribute nothing to the addition, as follows:

$$\begin{array}{r} 4738 \\ 358 \\ \hline 37904 \\ 23690 \\ 14214 \\ \hline 1696204 \end{array}$$

Here it may be noted that the product for each digit of the multiplier starts — from the right — exactly where this digit is placed.

- So, the product with the first digit from the right of the multiplier starts from the first place.
- The product with the second digit starts in the second place,
- that with the third in the third place,
- and so on.

An Application

In the preface to his book EULER says:

> *Since learning the art of reckoning without some basis in reason is neither sufficient for treating all possible cases nor apt to sharpen the mind — as should be our special intent — so we have striven, in the present guide, to expound and explain the reasons for all rules and operations in such a way that even persons who are not yet skilled in thourough discussion can see and understand them; nonetheless, the rules and shortcuts appropriate to*

calculation were described in detail and extensively clarified by examples.

By this device, we hope that young people, besides acquiring an adequate proficiency in calculation, will always be aware of the true reason behind every operation, and in this way gradually become accustomed to thorough reflection. For, when they thus not only grasp the rules, but also clearly see their basis and origin, they will in some measure be enabled to invent new rules of their own, and, by means of these, solve problems for which the ordinary rules are insufficient.

A Babylonian clay tablet from the Seleucid period contains the following sequence of numbers:

$$1 \qquad 2 \qquad 4 \qquad 8 \qquad 16 \qquad 32 \qquad 1{:}4 \qquad 2{:}8 \qquad 4{:}16 \qquad 8{:}32.$$

The tablet contains also the remark that the sum of these numbers is 17:3.

Evidently the first six numbers are terms of the doubling sequence. Appending the word '*seconds*' to these numbers suggests to continue after '*32 seconds*' with '*1 minute and 4 seconds*', '*2 minutes and 8 seconds*' and so on. This shows that the numbers on the tablet consist of subsequent terms of the doubling sequence. Hence the sum of these numbers must be 17:3, as remarked.

The sum of the first 64 terms of the doubling sequence — well known as the number of wheat–grains on the chess–board — appears in Al Birunis *Chronology* ($\sim$ 1000 A.D.) in decimal and in Babylonian writing:

$$18446\_74407\_37095\_51615$$

$$30{:}30{:}27{:}9{:}5{:}3{:}50{:}40{:}31{:}0{:}15.$$

Question: Do these two 'strings of digits' represent the same number?

Al Biruni used the pattern of the chess–board to structure his computation:

1. Onto the 8 fields of first line of the board he put the numbers

   $$1 \qquad 2=2^1 \qquad 4=2^2 \qquad 8=2^3 \qquad 16=2^4 \qquad 32=2^5 \qquad 64=2^6 \qquad 128=2^7$$

   with sum $255 = 2^8-1$ (of course not in this notation, which was introduced more than six centuries later by Viète and Descartes).
2. Then he proceeded by squaring $2^8 = 256$ and arrived at $(2^8)^2 = 2^{16} = 65536$, which is one more than the sum of the numbers placed onto the 16 fields of the first two lines,
3. then — again by squaring – he arrived at $(2^{16})^2 = 2^{32} = 42949\_67296$, which is one more than the sum of the numbers placed onto the 32 fields of the first four lines,

4. and finally — again by squaring — he arrived at

$$(2^{32})^2 = 2^{64} = 18446\_74407\_37095\_51616,$$

   and that is one more than the sum of all the numbers placed onto the 64 fields of the chess–board.

At that time — of course — the calculation had to be carried out by hand.

Today one would prefer the use a pocket–calculator for a calculation of this type. If the calculator displays 10 digits, the first three steps above are executed simply by pushing buttons. But squaring 2^{32} leads to 1.8446744074e+19 and that poses a problem for the determination of the precise number of wheat–grains on the board: we all know that subtracting 1 from a number decreases that number. But pushing buttons $\boxed{1}$ and $\boxed{-}$ does not change the display of the calculator at all. So the modern tool for calculation seems to be useless to determine the precise value of the sum of all the numbers placed on the chess–board!

One more time:

Reading Euler

> The characters used to describe numbers are arbitrary: every manner to express the numbers has its particular rules for the arithmetic operations. These rules have to be derived from the properties of the pieces of different value.

Therefore — since the pocket–calculator displays the product of two *five–digit* numbers correctly — it seems natural to change the system of numeration in a way which takes this into account. So we now use '*Five digit numbers*' or '*Hands of digits*' as the units. Then EULER'S remarks for the multiplication of multidigit numbers may be carried over to the present situation and the process of squaring takes the following form:

$$\begin{array}{r}
42949_67296 \\
\underline{42949_67296} \\
45287_51616 \\
28902_95904 \\
28902_95904 \\
\underline{18446_16601} \\
18446_74407_37095_51616
\end{array}$$

Subtracting one from the result supplies the desired sum.

FIBONACCI in his *Liber Abbaci* went one step further and asked for the sum of all the numbers of the doubling–sequence placed onto two adjacent chessboards. This number is obtained — of course — by squaring

$$18446\_74407\_37095\_51616$$

and subtracting 1 from the result. The *Liber Abbaci* contains the following result:

$$3402\_82366\_92093\_84634\_83374\_60743\_17682\_11455$$

Exercise:
One of the 8's in FIBONACCI'S result is wrong. Which one is it and by what does it have to be replaced?

Final Remark

I think that even today there is no better introduction to elementary arithmetic than EULER'S '*Einleitung zur Rechenkunst*'.

Therefore the recommendation which LAPLACE gave to his students,

"Lisez Euler, c'est notre maître à tous"[6]

as well as the words of GAUSS,

"Das Studium der Werke Euler's bleibt die beste Schule in den verschiedenen Gebieten der Mathematik und kann durch nichts ersetzt werden"[7]

are as relevant today as they were two–hundred years ago. Therefore:

Lisez Euler — Read Euler!

[6] "Read Euler. He is the master of us all."

[7] "Studying Euler's works remains the best school for the different areas of mathematics and can be replaced by nothing else."