ON THE CLASSIFICATION OF ALBERT ALGEBRAS

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preliminary version

1. Introduction

This is a very preliminary text. Our goal is to prove Corollary 4 at the very end of this text.

I am indebted to H. Petersson for comments on earlier versions of this text.

2. Jordan symbol algebras of degree 3

Let R be a commutative ring in which 6 is invertible. Let $\zeta \in R$ with $\zeta^2+\zeta+1=0$ and let a, b, $c \in R$. The aim of this section is to define Jordan algebras $J_\zeta(R;a,b,c)$ generalizing the first Tits construction of Albert algebras (=exceptional Jordan algebras) over fields to possibly degenerate Jordan algebras over rings. The condition $1/2 \in R$ is probably not necessary for this, but I haven't thought about details.

2.1. **First Notations.** $J(R;a) = R[X]/(X^3-a)$ is the cubic Kummer extension associated to a. The algebra $J(R;a)$ is a free R-module with basis X^i, $0 \le i \le 2$.

$A_\zeta(R;a,b)$ is the associative R-algebra with generators X, Y and relations $X^3 = a$, $Y^3 = b$ and $YX = \zeta XY$. The algebra $A_\zeta(R;a,b)$ is a free R-module with basis X^iY^j, $0 \le i,j \le 2$.

$J(R;a,b) = A_\zeta(R;a,b)^+$ is the Jordan R-algebra associated to $A_\zeta(R;a,b)$. The algebra $J(R;a,b)$ is a free R-module with basis $X^i \cdot Y^j$, $0 \le i,j \le 2$, where $x \cdot y = (xy+yx)/2$ is the Jordan product. Since $A_\zeta(R;a,b)^{\mathrm{op}} = A_{\zeta^2}(R;a,b)$, one has $J(R;a,b) = A_{\zeta^2}(R;a,b)^+$.

2.2. **The first Tits construction.** It is possible to define the Jordan algebras $J_\zeta(R;a,b,c)$ in terms of generators and relations, see proposition 1. However to carry out the details is somewhat tedious. Therefore we use an indirect way by referring to the first Tits construction over fields.

Let F be a field of characteristic different from 2, 3. Moreover let A be a central simple F-algebra of degree 3 and let $c \in F^*$, let Nrd, Trd: $A \to F$ be the reduced norm and reduced trace of A and let $\times\colon A \times A \to A$ be the symmetric bilinear product defined by $(x \times x)x = \mathrm{Nrd}(x)$. For $x \in A$, set $\overline{x} = \frac{1}{2}(\mathrm{Trd}(x) - x)$. Tits [2, 5] defined an Albert algebra

$$J(A,c) = A_0 \oplus A_1 \oplus A_2$$

Date: September 3, 2002, last revision: October 5, 2002.

as the sum of three copies of A with the product

	x_0	y_1	z_2
x'_0	$\frac{1}{2}(xx'+x'x)_0$	$(\overline{x'}y)_1$	$(z\overline{x'})_2$
y'_1	$(\overline{x}y')_1$	$c(y\times y')_2$	$(\overline{y'}z)_0$
z'_2	$(z'\overline{x})_2$	$(\overline{yz'})_0$	$\frac{1}{c}(z\times z')_1$

Now assume that F contains a primitive cube root ζ of 1 and let a, b, $c\in F^*$. We put

$$J_\zeta(F;a,b,c)=J\big(A_\zeta(a,b),c\big)$$

Note that the algebra $J(F;a,b)$ is a subalgebra of $J_\zeta(F;a,b,c)$ via $x\mapsto(x,0,0)$. Let

$$X_1=(X,0,0),\ X_2=(Y,0,0),\ X_3=(0,1,0).$$

Let $R_0=\mathbf{Z}[1/6,\zeta,a,b,c]/(\zeta^2+\zeta+1)$ and let $F_0=\mathbf{Q}(\mu_3)(a,b,c)$ be the fraction field of R_0. Let

$$J_\zeta(R_0;a,b,c)\subset J_\zeta(F_0;a,b,c)$$

be the R_0-subalgebra generated by X_1, X_2, X_3.

Now let R be a commutative ring in which 6 is invertible, let $\zeta\in R$ with $\zeta^2+\zeta+1=0$ and let a_1, a_2, $a_3\in R$. We define $J_\zeta(R,a_1,a_2,a_3)$ as the R-algebra obtained via specialization from the universal example: Let $R_0\to R$ be the homomorphism sending ζ, a, b, c to the elements ζ, a_1, a_2, resp. a_3 of R and let

$$J_\zeta(R,a_1,a_2,a_3)=J_\zeta(R_0,a,b,c)\otimes_{R_0}R$$

Let $\Gamma=(\mathbf{Z}/3)^3$ and let $\omega\colon\ \mathrm{Alt}(3,\Gamma)\to\mathbf{Z}/3$ be the isomorphism with

$$\omega\big((1,0,0)\wedge(0,1,0)\wedge(0,0,1)\big)=1$$

For $r\in\Gamma$ we denote by r_1, r_2, $r_3\in\mathbf{Z}$ the integers with $0\le r_i\le 2$ and $r=(r_1,r_2,r_3)$ mod 3.

For $r\in\Gamma$ let L_r be the R-submodule of $J_\zeta(R;a_1,a_2,a_3)$ generated by the element $(X_1^{r_1}X_2^{r_2})X_3^{r_3}$.

Proposition 1. *The R-modules L_r are free of rank 1. One has*

$$J_\zeta(R;a_1,a_2,a_3)=\bigoplus_{r\in\Gamma}L_r$$

and $L_rL_s\subset L_{r+s}$.

The R-algebra $J_\zeta(R;a_1,a_2,a_3)$ is generated by X_1, X_2, X_3.

For $r\in\Gamma$ one has

$$\big((X_1^{r_1}X_2^{r_2})X_3^{r_3}\big)^3=\frac{1}{8^n}a_1^{r_1}a_2^{r_2}a_3^{r_3}$$

where $n=2$ if the r_i are all nonzero, $n=1$ if exactly one of the r_i is equal to 0 and $n=0$ otherwise.

Let r, s, $t \in \Gamma$ be linearly independent elements (i. e., $r \wedge s \wedge t \neq 0$) and let $U \in L_r$, $V \in L_s$, $W \in L_t$. Then

$$
\begin{aligned}
U^2(UV) &= U(U^2V) = \frac{1}{4}U^3V \\
U(UV) &= -\frac{1}{2}U^2V \\
(UV)(UW) &= -\frac{1}{2}U^2(VW) \\
(UV)(U^2W) &= \frac{1}{4}U^3(VW) \\
(UV)W &= \zeta^{\omega(r\wedge s\wedge t)}(UW)V
\end{aligned}
$$

Proof. This all follows by inspection of the product in $J_\zeta(F_0; a, b, c)$. □

The significance of proposition 1 is that it provides a complete description of $J_\zeta(R; a_1, a_2, a_3)$ as commutative (but not necessarily associative) Γ-graded R-algebra with generators X_1, X_2, X_3 and certain relations. We mention one particular consequence:

Corollary 1.

$$
\begin{aligned}
J_\zeta(R; a_3, a_1, a_2) &\simeq J_\zeta(R; a_1, a_2, a_3) \\
J_\zeta(R; a_2, a_1, a_3) &\simeq J_{\zeta^2}(R; a_1, a_2, a_3)
\end{aligned}
$$

3. The $H^3(\mathbf{Z}/3)$-invariant

We summarize results from [6, 10]. See also [5].

Proposition 2. *For Albert algebras J over fields F (of characteristic different from 2, 3) there exist a unique cohomological invariant $g_3(J) \in H^3(F, \mathbf{Z}/3)$ such that: For a field F containing a primitive cube root ζ of 1 and for a_1, a_2, $a_3 \in F^*$ one has*

$$g_3\big(J_\zeta(F; a_1, a_2, a_3)\big) \otimes \zeta = (a_1) \cup (a_2) \cup (a_3)$$

in $H^3(F, \mathbf{Z}/3) \otimes \mu_3 = H^3(F, \mu_3^{\otimes 3})$ (note that $\mu_3 \otimes \mu_3 \equiv \mathbf{Z}/3$).

4. F_4-torsors

We assume that R contains a subfield k with $\zeta \in k$. The algebraic group $F_4 = \mathrm{Aut}\big(J_\zeta(k; 1, 1, 1)\big)$ (= group of k-algebra automorphisms) is a split group over k of type F_4.

Lemma 1. *Let a_1, a_2, $a_3 \in R$ be invertible.*

Then $\mathrm{Isom}\big(J_\zeta(R; 1, 1, 1), J_\zeta(R; a_1, a_2, a_3)\big)$ *is an F_4-torsor over R (in the etale topology).*

Proof. It suffices to note that after adjoining cube roots of the a_i the Jordan R-algebras $J_\zeta(R; a_1, a_2, a_3)$ and $J_\zeta(R; 1, 1, 1)$ become isomorphic. □

We will use the following corollary of the theorem of Raghunathan-Ramanathan [9] (together with a theorem of Steinberg) applied to F_4-torsors over the affine line:

Proposition 3. *Let k be perfect and let J be a algebra over $k[t]$ which is locally (in the etale topology) isomorphic to the split Albert algebra. Then $J \simeq J_0 \otimes k[t]$ for some Albert algebra J_0 over k.*

By Lemma 1, this condition is satisfied if for every irreducible polynomial $P \in k[t]$ there exists a_1, a_2, $a_3 \in (k[t]_{(P)})^*$ with $J \otimes k[t]_{(P)} \simeq J_\zeta(k[t]_{(P)}; a_1, a_2, a_3)$.

`Remark added on Sept. 10:`
Actually, we will need only the following:

Proposition 4. *Let* $\operatorname{char} k \neq 3$ *with* $\mu_3 \subset k$, *let* J *be an Albert divsion algebra of first Tits construction over* k, *and let* $J' \subset J \otimes k(t)$ *be a* $k[t]$*-subalgebra which is locally (in the etale topology) isomorphic to the split Albert algebra. Then* $J' \simeq J \otimes k[t]$.

Proof. (Sketch) One uses, I guess, Harder's method used for quadratic forms. The argument simplifies for the anisotropic case.

Extend the order J' to an order $\bar{J}$ on the projective line by taking the order $J \otimes k[1/t]$ at infinity. Let $x \in H^0\big(\mathbf{P}^1, \operatorname{Hom}\big(\mathcal{O}_{\mathbf{P}^1}(1), \bar{J}\big)\big)$ be a global section. Since J' is a vector bundle (in fact, a direct sum of line bundles $\mathcal{O}_{\mathbf{P}^1}(r)$, [1, p. 516]), one has $H^0\big(\mathbf{P}^1, \operatorname{Hom}\big(\mathcal{O}_{\mathbf{P}^1}(n), \bar{J}\big)\big) = 0$ for $n \gg 0$. Thus x is nilpotent, and since $J(t)$ is a field, it follows $x = 0$. Hence $H^0\big(\mathbf{P}^1, \operatorname{Hom}\big(\mathcal{O}_{\mathbf{P}^1}(1), \bar{J}\big)\big) = 0$. This shows that $\bar{J}$ is trivial as a vector bundle, since the vector bundle $\bar{J}$ is self dual (via the trace form). Thus $J_0 = H^0(\mathbf{P}^1, \bar{J})$ is an Albert algebra over k with $\bar{J} = J_0 \otimes \mathcal{O}_{\mathbf{P}^1}$. It suffices to note $J_0 = \bar{J}(\infty) = J$. □

5. Orders

Let $R = k[t]$ and $F = k(t)$. Let further f_1, f_2, $f_3 \in R$ be nonzero polynomials. Then

$$J_\zeta(R; f_1, f_2, f_3) \subset J_\zeta(F; f_1, f_2, f_3)$$

is an R-order (see [3, 4, 8]). Our goal in this section is to show that if $J_\zeta(F; f_1, f_2, f_3)$ is unramified on the affine line, then there exists a *regular* R-order J' with

$$J_\zeta(R; f_1, f_2, f_3) \subset J' \subset J_\zeta(F; f_1, f_2, f_3)$$

[By a regular order we understand here an order which is locally (in the etale topology) isomorphic to a split Albert algebra. For a regular order the quadratic form is nondegenerate. The converse should be also true, but we won't need this.]

By proposition 3 it follows then that if $J_\zeta(F; f_1, f_2, f_3) \simeq J_0(t)$ for some Albert algebra J_0 over k, then $J_\zeta(R; f_1, f_2, f_3)$ admits an embedding into $J_0 \otimes k[t]$ (at least if k is perfect). This gives rise to some sort of Cassels-Pfister theorem for Albert algebras.

One is tempted to take for J' a maximal R-order, as in the case of associative central simple algebras or of quadratic forms. However Knebusch [4] has given an example of a maximal order of an Albert algebra which is not regular. It is not clear to me whether such phenomena can appear in our specific applications. Anyway, we will construct J' locally by means of a sequence of elementary operations on the generators of Jordan symbol algebras.

Let $v \colon F^* \to \mathbf{Z}$ be a discrete valuation, let R be the valuation ring, let κ be the residue field and let π be a prime element of v. (F can be any field with $\operatorname{char} F \neq 2, 3$ and $\mu_3 \subset F$; we assume also $\operatorname{char} \kappa \neq 3$). Let

$$\partial_v \colon H^3(F, \mu_3) \to H^2(\kappa, \mathbf{Z}/3)$$

be the residue map.

Proposition 5. *Let f_1, f_2, $f_3 \in R$ be nonzero elements with*

$$\partial_v\big((f_1) \cup (f_2) \cup (f_3)\big) = 0$$

Then there exists an R-order

$$J_\zeta(R; f_1, f_2, f_3) \subset J' \subset J_\zeta(F; f_1, f_2, f_3)$$

such that $J' \simeq J_\zeta(R; g_1, g_2, g_3)$ with $g_i \in R^*$.

Remark 1. Actually, what we will need in the proof is that $J_\zeta(F; f_1, f_2, f_3)$ is unramified, that is, it is extended from some regular Albert algebra over R. In the application this will be obvious, since then $J_\zeta(F; f_1, f_2, f_3)$ is extended from an Albert algebra over the ground field.

We will also use the following fact: If $J_\zeta(F; f_1, f_2, f_3)$ is unramified, and if f_1 is a prime element for R and f_2, f_3 are R-units, then the reduction $J(\bar{f}_2, \bar{f}_3)$ of $J(f_2, f_3)$ is the split algebra.

Moreover, in this case, the norm of $J(f_2, f_3)$ represents a prime element.

Namely, if $J(\bar{f}_2, \bar{f}_3)$ is split, then $\bar{f}_2 = N_{\bar{L}/\kappa}(\bar{\lambda})$ where $L = R[t]/(t^3 - f_3)$ and $\lambda \in L^*$. After replacing f_2 by $f_2 N_{L/\kappa}(\lambda)^{-1}$ we may assume that $\bar{f}_2 = 1$. Then, for $e \in R$ and a prime element π one has with $S = R[s]/(s^3 - f_2) \subset J(f_2, f_3)$

$$N_{S/R}(1 + e\pi - s) = (1 + e\pi)^3 - f_2 \equiv \pi\big(3e + \frac{1 - f_2}{\pi}\big) \mod \pi^2$$

It is clear that for $e = 0$ or for $e = 1$ the right hand side is nonzero. Then $N(1 + e\pi - s)$ is a prime element.

Proof of Proposition 5. Let X_1, X_2, X_3 be the generators of $J_\zeta(R; f_1, f_2, f_3)$.

We first reduce to the case $v(f_2) = v(f_3) = 0$. Suppose that $v(f_1) \geq v(f_2) > 0$. Then $X_1 X_2^{-1}$, X_2, X_3 generate an order J' isomorphic to $J_\zeta(R; f_1 f_2^{-1}, f_2, f_3)$. Moreover $J_\zeta(R; f_1, f_2, f_3)$ is contained in J' since $X_1 = 4(X_1 X_2^{-1})X_2$, see proposition 1. Iterating this argument, possibly using the symmetry relations of corollary 1, one may indeed arrange $v(f_2) = v(f_3) = 0$.

Now write $v(f_1) = r + 3m$ with $0 \leq r \leq 2$ and $m \geq 0$. Then $X_1\pi^{-m}$, X_2, X_3 generate an order J' isomorphic to $J_\zeta(R; f_1\pi^{-3m}, f_2, f_3)$ and containing $J_\zeta(R; f_1, f_2, f_3)$. We may therefore assume $m = 0$.

Suppose $v(f_1) = 0$. Then we may (and must) take $J' = J_\zeta(R; f_1, f_2, f_3)$.

Suppose $v(f_1) = 2$. Then $X_1^2\pi^{-1}$, X_2, X_3 generate an order J' isomorphic to $J_{\zeta^2}(R; f_1^2\pi^{-3}, f_2, f_3)$. Moreover $J_\zeta(R; f_1, f_2, f_3)$ is contained in J' since $X_1 = (X_1^2\pi^{-1})^2 u$ with $u = \pi^2/f_1 \in R^*$, see proposition 1. Since $v(f_1^2\pi^{-3}) = 1$, we are reduced to the case $v(f_1) = 1$.

Finally suppose $v(f_1) = 1$. Then

$$(\bar{f}_2) \cup (\bar{f}_3) = \partial_v\big((f_1) \cup (f_2) \cup (f_3)\big) = 0$$

Hence $A_\zeta(R; f_2, f_3)$ is an Azumaya algebra over R which is split over the residue class field. It follows that the reduced norm of $A_\zeta(R; f_2, f_3)$ represents a prime element. Hence there exist an element α in the subalgebra of $J_\zeta(R; f_1, f_2, f_3)$ generated by X_2, X_3 with $v\big(N(\alpha)\big) = 1$. Let J' be the order generated by $X_1' = X_1\big(T(\alpha) - 2\alpha\big)^{-1}$ and X_2, X_3. Note that $N(X_1') = N(X_1)N(\alpha)^{-1} = f_1 N(\alpha)^{-1}$ and therefore $J' \simeq J_\zeta(R; f_1 N(\alpha)^{-1}, f_2, f_3)$. Clearly $f_1 N(\alpha)^{-1} \in R^*$. Moreover one has $X_1 = X_1'\big(T(\alpha) - 2\alpha\big)$ and therefore J' contains $J_\zeta(R; f_1, f_2, f_3)$. □

Now let again $R = k[t]$ and $F = k(t)$.

Proposition 6. *Let f_1, f_2, $f_3 \in R$ be nonzero elements with*

$$(f_1) \cup (f_2) \cup (f_3) \in H^3(k, \mu_3) \subset H^3(F, \mu_3)$$

Then there exists a regular R-order

$$J_\zeta(R; f_1, f_2, f_3) \subset J' \subset J_\zeta(F; f_1, f_2, f_3)$$

Remark 2. Actually, what we will need in the proof is that $J_\zeta(F; f_1, f_2, f_3)$ is everywhere unramified. In the application this will be obvious, since then $J_\zeta(F; f_1, f_2, f_3)$ is extended from an Albert algebra over the ground field.

Proof. Take the regular order $J_\zeta(R[(f_1 f_2 f_3)^{-1}]; f_1, f_2, f_3)$ and extend it to the affine line using proposition 5. □

Corollary 2. *Let a_1, a_2, $a_3 \in k^*$ and let f_1, f_2, $f_3 \in k[t]$ be nonzero elements with*

$$J_\zeta(k(t); f_1, f_2, f_3) \simeq J_\zeta(k(t); a_1, a_2, a_3)$$

Then there exists a homomorphism

$$J_\zeta(k[t]; f_1, f_2, f_3) \to J_\zeta(k[t]; a_1, a_2, a_3)$$

of $k[t]$-algebras. □

6. A Cassels-Pfister type theorem for Albert algebras

Let J be a Albert algebra over k of first Tits construction. We write $J(t) = J \otimes_k k(t)$ and $J[t] = J \otimes_k k[t]$. We assume $\mu_3 \subset k$.

We recall the following fact [2], [7, Corollary 3]:

Lemma 2. *Let $a \in k^*$ and suppose that $J \otimes k(\sqrt[3]{a})$ has zero divisors. Then there exist an element $x \in J$ with $T(x) = T(x^2) = 0$ and $N(x) = a$.*

Theorem 1. *Let $f \in k[t]$ and suppose that there exists $\alpha \in J(t)$ with $T(\alpha) = T(\alpha^2) = 0$ and $N(\alpha) = f$. Then there exists $\alpha \in J[t]$ with $T(\alpha) = T(\alpha^2) = 0$ and $N(\alpha) = f$.*

Proof. There exists g, $h \in k(t)^*$ such that

$$J(t) \simeq J_\zeta(k(t); f, g, h)$$

This follows from the fact that the invariant $f_3(J) \in H^3(k, \mathbf{Z}/2)$ is trivial. See the proof of [5, Proposition (40.5)].

After multiplication with cubes, we may assume that g, $h \in k[t]$. The claim follows from corollary 2. □

Corollary 3. *Let a, $b \in k^*$ and suppose that J has zero divisors over the function field of the curve*

$$r^3 + at^3 + b = 0$$

Then there exist x, $y \in J$ such that

$$\begin{aligned} T(x) &= T(x^2) = 0 \\ N(x) &= a \\ T(y) &= T(y^2) = 0 \\ N(y) &= b \\ T(xy) &= T(yx^2) = T(xy^2) = 0 \end{aligned}$$

Remark 3. Note that if we have the additional relation $T(x^2y^2) = 0$, then x, y would be the standard generators of a subalgebra $J(a,b)$ of J.

Proof. We may assume that J is a division algebra. By Lemma 2 and Theorem 1 there exists $\alpha \in J[t]$ with $T(\alpha) = T(\alpha^2) = 0$ and $N(\alpha) = at^3 + b$. Since J is a division algebra, one must have $\deg(\alpha) = 1$. Write $\alpha = xt + y$. The claimed properties are now easy to verify, by noting that $N(\alpha) = \alpha^3$, whence

$$(xt + y)^3 = at^3 + b$$

and therefore

$$x^2y + xyx + yx^2 = y^2x + yxy + xy^2 = 0$$

□

7. The main result

Let a, $b \in k^*$ and let $K = k(s,t)[u]$ with $u^3 = at^3 + b(1 - as^3)$. Then

$$b = \frac{u^3 - at^3}{1 - as^3} = N_{K \otimes k_a/K}\left(\frac{u - \alpha t}{1 - \alpha s}\right)$$

with $k_a = k[\alpha]$, $\alpha^3 = a$. In particular, the algebras $A_\zeta(a,b)$ and $J(a,b)$ and the symbol $(a) \cup (b)$ become trivial over K.

Remark 4 (We don't need this in the following). The field K is the function field of the Severi-Brauer Y variety of $A_\zeta(a,b)$. Indeed, K is the function field of the cubic surface

$$u^3 - at^3 + bv^3 - abw^3 = 0$$

which turns out to be the blow up of Y in specific subschemes $\operatorname{Spec} k_a \to Y$, $\operatorname{Spec} k_b \to Y$ (recall that the blow up of $\mathbf{P}^2$ in 6 points is a cubic surface).

Theorem 2. *Let J be a Albert algebra over k ($\operatorname{char} k \neq 2, 3$) and assume that J has zero divisors over K. Then there exists a finite field extension ℓ of k of degree prime to 3 and an embedding of $J(a,b)$ into J over ℓ.*

In particular, if ℓ contains a primitive root ζ of 1, then

$$J_\ell \simeq J_\zeta(a,b,c)$$

for some $c \in \ell^*$ (in fact, one may choose $c \in k^*$).

Proof. We may assume that k has no proper extensions of degree prime to 3. This means that every irreducible polynomial over k has degree divisible by 3 or is linear.

We may further assume that J is a Jordan field.

By Corollary 3, there exist x, $y \in J(s)$ such that

$$\begin{aligned} T(x) &= T(x^2) = 0 \\ N(x) &= a \\ T(y) &= T(y^2) = 0 \\ N(y) &= b(1 - as^3) \\ T(xy) &= T(yx^2) = T(xy^2) = 0 \end{aligned}$$

It suffices (see Remark 3) to find x_0, $y_0 \in J$ such that

$$\begin{aligned} T(x_0) &= T(x_0^2) = 0 \\ N(x_0) &= a \\ T(y_0) &= T(y_0^2) = 0 \\ N(y_0) &= bN_{k_a/k}(\lambda) \quad \text{for some } \lambda \in k_a^* \\ T(x_0y_0) &= T(y_0x_0^2) = T(x_0y_0^2) = 0 \\ T(x_0^2y_0^2) &= 0 \end{aligned}$$

Write

$$x = \frac{X}{P}, \quad y = \frac{Y}{Q}$$

where X, $Y \in J[s]$ and P, $Q \in k[s]$ with X, P resp. Y, Q relatively prime.

Then $N(X) = aP^3$. Let P' be an irreducible factor of P. Then $X \mod P'$ is nonzero and a zero divisor of J over $k[s]/(P')$. Since J is a field, P' cannot be linear. By the assumption made at the beginning of this proof, it follows that P' has degree divisible by 3. Hence $\deg P = 3n$. Since J is a field, it follows also that $\deg X = \deg P$.

Similarly one sees from $N(Y) = b(1 - as^3)Q^3$ that each factor of Q has degree divisible by 3 and that $\deg Y = 1 + \deg Q = 1 + 3m$.

Consider the polynomial $f(s) = T(X^2Y^2)$. One has $\deg f \leq d$ with $d = 2(1 + 3m + 3n)$.

If $\deg f = d$, then there exists a zero s_0 of f, by our assumption on k. In this case we take $x_0 = x(s_0)$ and $y_0 = y(s_0)$.

If $\deg f < d$, we take for x_0 and y_0 the leading coefficients of X, Y (after arranging that P, Q have leading coefficient 1). □

By Springer's theorem (and other things) we get

Corollary 4. *Let k be a field with $\operatorname{char} k \neq 2, 3$ and let J, J' be Albert algebras over k with isomorphic trace forms and with $g_3(J) = g_3(J')$. Then there exist a finite field extension F of k of degree prime to 3 and a finite field extension L of k of degree dividing 3 such that $J_F \simeq J'_F$ and $J_L \simeq J'_L$.*

Of course one would like to know $J \simeq J'$ over k.

References

[1] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.

[2] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society, Providence, R.I., 1968, American Mathematical Society Colloquium Publications, Vol. XXXIX.

[3] M. Knebusch, *Der Begriff der Ordnung einer Jordanalgebra*, Abh. Math. Sem. Univ. Hamburg **28** (1965), 168–184.

[4] ______, *Eine Klasse von Ordnungen in Jordanalgebren vom Grade* 3, Abh. Math. Sem. Univ. Hamburg **28** (1965), 185–207.

[5] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

[6] H. P. Petersson and M. L. Racine, *An elementary approach to the Serre-Rost invariant of Albert algebras*, Indag. Math. (N.S.) **7** (1996), no. 3, 343–365.

[7] ______, *Cubic subfields of exceptional simple Jordan algebras*, Proc. Amer. Math. Soc. **91** (1984), no. 1, 31–36.

[8] M. L. Racine, *The arithmetics of quadratic Jordan algebras*, American Mathematical Society, Providence, R.I., 1973, Memoirs of the American Mathematical Society, No. 136.

[9] M. S. Raghunathan and A. Ramanathan, *Principal bundles on the affine line*, Proc. Indian Acad. Sci. Math. Sci. **93** (1984), no. 2-3, 137–145.

[10] M. Rost, *A* (mod 3) *invariant for exceptional Jordan algebras*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), no. 12, 823–827.

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