ON THE BASIC CORRESPONDENCE OF A SPLITTING VARIETY

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Preface

This text should be considered as private notes, made public only to have a reference for discussions on the subject.

The technical ad hoc computations of Section 8 are obsolete; the computation of $c(X)$ can be done in a much simpler way.

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1. No Introduction

Looking for an introduction? See Remark 2.5.

2. Definitions and a brief introduction

Let p be a prime and let k be a field with $\operatorname{char} k \neq p$.

Moreover let $n \geq 2$. We are going to discuss some part of the proof of the Bloch-Kato conjecture in weight n in [11].

For simplicity we will assume throughout $\operatorname{char} k = 0$ and $\mu_p \subset k$. Moreover for $p = 2$ we will often assume that -1 is a square.

Let B be a smooth variety over k and let $X \to B$ be a smooth morphism. We denote by $\mathcal{X}(X/B)$ the simplicial scheme

$$\mathcal{X}(X/B): \qquad X \leftleftarrows X \times_B X \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, X \times_B X \times_B X \cdots$$

For the following we refer to [11, Proof of Lemma 6.5].

Lemma 2.1. *Assume the Bloch-Kato conjecture holds in weights $\leq n-1$. Then for any smooth simplicial scheme $\mathcal{X}$ the sequence*

$$0 \to H^{n,n-1}(\mathcal{X}, \mathbf{Z}/p) \to H^n_{\text{et}}(\mathcal{X}, \mu_p^{\otimes(n-1)}) \to H^0\big(\mathcal{X}, \underline{H}^n_{\text{et}}(\mu_p^{\otimes(n-1)})\big)$$

is exact. Here the first map is given by changing from Nisnevich to etale topology and the second map is induced from the associated spectral sequence. □

Corollary 2.2. *Assume the Bloch-Kato conjecture holds in weights $\leq n-1$. Let $X \to B$ be as above and surjective. Then there is a natural exact sequence*

$$H^{n,n-1}\big(\mathcal{X}(X/B), \mathbf{Z}/p\big) \subset H^n_{\text{et}}(B, \mu_p^{\otimes(n-1)}) \to \coprod_\alpha H^n_{\text{et}}\big(k(X_\alpha), \mu_p^{\otimes(n-1)}\big)$$

where the X_α are the components of X.

Proof. Since $X \to B$ has sections on an etale covering of B, it follows that

$$\mathcal{X}(X/B) \to \mathcal{X}(B/B)$$

induces an isomorphism of the etale topological types. Hence $\mathcal{X}(X/B)$ and B have the same etale cohomology.

Moreover,

$$H^0(\mathcal{X}, F) \subset H^0(\mathcal{X}_0, F) = H^0(X, F) \subset \coprod_\alpha H^0\big(k(X_\alpha), F\big)$$

as in [11, Proof of Lemma 6.5].

Now Lemma 2.1 yields the stated exact sequence. □

The Milnor operations

$$Q_i\colon H^{r,s}(-,\mathbf{Z}/p) \to H^{r+2p^i-1,s+p^i-1}(-,\mathbf{Z}/p)$$

are defined by $Q_0 =$ Bockstein and $Q_{i+1} = [P^{p^i}, Q_i]$ where

$$P^h\colon H^{r,s}(-,\mathbf{Z}/p) \to H^{r+2h(p-1),s+h(p-1)}(-,\mathbf{Z}/p)$$

are the Steenrod operations in motivic cohomology (we assume that -1 is a square if $p = 2$). Let further

$$\tilde{Q}_0\colon H^{r,s}(-,\mathbf{Z}/p) \to H^{r+1,s}(-,\mathbf{Z})$$

be the integral valued Bockstein.

Let

$$\delta \in H^{n,n-1}\big(\mathcal{X}(X/B), \mathbf{Z}/p\big)$$

be an element. As in [11, (5.2)], we consider the element

$$\mu = \mu(\delta) = \hat{Q}(\delta) \in H^{2b+1,b}\big(\mathcal{X}(X/B), \mathbf{Z}\big)$$

where

$$\hat{Q} = \tilde{Q}_0 \circ Q_1 \circ \cdots \circ Q_{n-2}$$

and

$$b = \frac{p^{n-1}-1}{p-1}$$

Let

$$\Delta_{X/B}\colon \mathcal{X}(X/X) \to \mathcal{X}(X/B)$$

be the diagonal map. We consider the induced maps

$$\Delta^*_{X/B}\colon H^{r,s}\big(\mathcal{X}(X/B)\big) \to H^{r,s}\big(\mathcal{X}(X/X)\big) = H^{r,s}(X)$$

and their kernels

$$H^{r,s}\big(\mathcal{X}(X/B)\big)^{[1]} = \ker \Delta^*_{X/B} \subset H^{r,s}\big(\mathcal{X}(X/B)\big)$$

The natural filtration on the simplicial scheme $\mathcal{X}(X/B)$ yields then maps

$$H^{r,s}\big(\mathcal{X}(X/B)\big)^{[1]} \xrightarrow{\text{proj}} \mathcal{H}^1(X/B, H^{r-1,s})$$

where $\mathcal{H}^1(X/B, F)$ is an abbreviation for the homology of the complex

$$F(X) \xrightarrow{\pi_0^*-\pi_1^*} F(X\times_B X) \xrightarrow{\pi_0^*-\pi_1^*+\pi_2^*} F(X\times_B X\times_B X)$$

Note that $H^{2b+1,b}(Y,\mathbf{Z}) = 0$ and $H^{2b,b}(Y,\mathbf{Z}) = \mathrm{CH}^b(Y)$ (the classical Chow group) for any smooth variety Y. Therefore we get a map

$$H^{2b+1,b}\big(\mathcal{X}(X/B), \mathbf{Z}\big) \xrightarrow{\text{proj}} \mathcal{H}^1(X/B, \mathrm{CH}^b)$$

We put

$$\rho = \rho(\delta) = \mathrm{proj}\big(\mu(\delta)\big) \in \mathcal{H}^1(X/B, \mathrm{CH}^b)$$

Remark 2.3. We are particularly interested in the following situation. Let

$$u \in H^n_{\mathrm{et}}(k, \mu_p^{\otimes(n-1)})$$

be a symbol and let X be a smooth splitting variety of u, i. e., $u_{k(X)} = 0$. By Corollary 2.2 with $B = \operatorname{Spec} k$ we can identify u with an element

$$\delta \in H^{n,n-1}\big(\mathcal{X}(X/k), \mathbf{Z}/p\big)$$

as above and by the preceding discussion we get eventually an element

$$\rho \in \mathcal{H}^1(X/k, \mathrm{CH}^b)$$

We call ρ the *basic correspondence* of the splitting variety X of the symbol u.

Example 2.4. Let $n = 2$ and take for X the Severi-Brauer variety associated with u. Then $b = 1$ and $\dim X = p - 1$.

There is a standard class

$$\tilde{\rho} \in \mathrm{CH}^1(X^2) = \mathrm{Pic}(X^2)$$

It is represented by the line bundle

$$\mathrm{Hom}_A(\pi_0^* I, \pi_1^* I)$$

on X. Here A is the central simple algebra of degree p associated with u and I is the tautological sheaf on $X = \mathrm{SB}(A)$ (the geometric points of X are the right ideals of A of rank p). It is easy to see that $\tilde{\rho}$ is in the kernel of

$$\mathrm{Pic}(X^2) \xrightarrow{\pi_0^* - \pi_1^* + \pi_2^*} \mathrm{Pic}(X^3)$$

One finds that ρ is represented by $\tilde{\rho}$ (up to sign).

Remark 2.5. Let us return to the general case with $B = \operatorname{Spec} k$ and X proper and irreducible. Consider a representative

$$\tilde{\rho} \in \mathrm{CH}^b(X^2)$$

of ρ. Let

$$\theta = \tilde{\rho}^{p-1} \in \mathrm{CH}^d(X^2)$$

with $d = b(p-1) = p^{n-1} - 1$. Assume further that $\dim X = d$ and let

$$c(X) = c(X, u) = (\pi_0)_*(\theta) \in \mathrm{CH}^0(X) = \mathbf{Z}$$

be the push forward of θ under the first projection (the other projection will also do the job).

I am pretty sure that (modulo p and up to sign) the number $c(X)$ coincides with the number "c" appearing in [11, Lemma 5.15]. The non-triviality of "c" is crucial in [11]. Voevodsky uses [11, Theorem 3.8] in order to show that $s_d(X)/p \neq 0 \mod p$ implies $c \neq 0 \mod p$. The proof of [11, Theorem 3.8] relies on [11, Lemma 2.2, Lemma 2.3]. However proofs of [11, Lemma 2.2, Lemma 2.3] are unfortunately not available.

The aim of this text is to find a smooth proper splitting variety X of u of dimension d such that

$$c(X) \neq 0 \mod p$$

This will be based on our construction of splitting varieties "with good characteristic number".

I haven't actually worked out a proof that $c(X)$ agrees essentially with Voevodsky's "c", because that is not really necessary anymore.

Instead I plan to prove in a sequel[1] that if $c(X) \neq 0 \mod p$, then X splits off a certain motive M (cf. Proposition 5.9, Section 7) which is very likely isomorphic to Voevodsky's motive M_{p-1} constructed from the class μ in [11]. It will also turn out that $c(X)$ mod p does not depend on the choice of $\tilde{\rho}$ provided that u is nonzero (cf. Corollary 5.8). In the forthcoming text we will use only classical methods for correspondences.

Given that motive M, one may use it to prove Bloch-Kato in a similar way as the motive of a Pfister quadric was used for the Milnor conjecture in [9]. One shows first that one has an "exact quadrangle" (with $\mathcal{X} = \mathcal{X}(X/k)$)

$$M(\mathcal{X})\{pb\} \to M\{b\} \xrightarrow{\rho^{p-2}} M \to M(\mathcal{X})$$

which means more precisely that there are exact triangles

$$M(\mathcal{X})\{d\} \to M \to D \to M(\mathcal{X})\{d\}[1]$$
$$C \to M \to M(\mathcal{X}) \to C[1]$$

and an isomorphism

$$D\{b\} \simeq C$$

(The object D is very likely isomorphic to Voevodsky's motive M_{p-2} in [11].) One may then prove [11, Lemma 6.13] in the same way as in [11], but now using the quadrangle in place of Voevodsky's motives M_i. This reduces Bloch-Kato to the injectivity of

$$H^{2d+1,d+1}(M, \mathbf{Z}_{(p)}) \to H^{1,1}(k, \mathbf{Z}_{(p)})$$

This follows from my results on the injectivity of the norm map on zero cycles with K_1-coefficients.

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[1] This is meanwhile included in this text.

Remark 2.6. By the way, for $n = 1$ there is a well known analogous "quadrangle": A (nonzero) element $u \in H^1_{\text{et}}(k, \mathbf{Z}/p)$ is a cyclic field extension of k of degree p. Let

$$G = \mathbf{Z}/p = \langle\, t \mid t^p \,\rangle$$

be its Galois group. Then there is the exact sequence of G-modules

$$0 \to \mathbf{Z} \to \mathbf{Z}[G] \to \mathbf{Z}[G] \to \mathbf{Z} \to 0$$

where the middle map is multiplication with $1 - t$.

The G-module $\mathbf{Z}$ has infinite cohomological dimension, while the G-module $\mathbf{Z}[G]$ has cohomological dimension 0. This reflects the general situation: The motive $M(\mathcal{X})$ has unbounded nontrivial motivic cohomology, while M is a direct summand of a finite-dimensional variety.

See also [11, Proof of Lemma 3.7] and [11, Example 5.6].

3. Expressing $c(X)$

In [6, Section 6] we considered a specific form (R, L, γ) of degree p which we will also use in this text. See also [7].

Given elements $a_1, \ldots, a_{n-1} \in k^\times$, let

$$v = (a_1) \cdots (a_{n-1}) \in H^{n-1}_{\text{et}}(k, \mu_p^{\otimes(n-1)})$$

We have constructed in [6]:

(1) A smooth proper cellular variety R with $\dim R = p^{n-1} - p$.
(2) Forms (R, L, γ), (R, L'_i, γ'_i), $i = 1, \ldots, n-2$, of degree p on line bundles L, L'_i over R.
(3) Let $R_0 \subset R$ be the singular locus of the forms γ, γ'_i and let $B = R \setminus R_0$. Moreover let (γ), $(\gamma'_i) \in H^1_{\text{et}}(B, \mu_p)$ be the classes of the corresponding forms.

 Then for any $x \in R_0$ one has

$$v_{\kappa(x)} = 0$$

 in $H^{n-1}_{\text{et}}\big(\kappa(x), \mu_p^{\otimes(n-1)}\big)$.

 Moreover, one has

$$v_{k(B)} = \big((\gamma)(\gamma'_1) \cdots (\gamma'_{n-2})\big)_{k(B)}$$

 in $H^{n-1}_{\text{et}}\big(k(B), \mu_p^{\otimes(n-1)}\big)$.

Given such data, let

$$A = \bigoplus_{i=0}^{p-1} L^{\otimes i}$$

be the (commutative) $\mathcal{O}_R$-algebra of rank p associated to the form (R, L, γ), let

$$N_A\colon A \to \mathcal{O}_R$$

be its norm form, and let $\mathbf{A}(A)$ be the associated scheme (a bundle of affine spaces over R).

Now let $a_n \in k^\times$ and let

$$u = v \cup (a_n) = (a_1) \cdots (a_n) \in H^n_{\mathrm{et}}(k, \mu_p^{\otimes n})$$

Furthermore let

$$\hat{X} \subset \mathbf{P}(A \oplus \mathcal{O}_R)$$

be the subscheme defined by the equation

$$N_A(x) = a_n t^p \qquad \text{for } [x, t] \in \mathbf{P}(A \oplus \mathcal{O}_R)$$

Let further $A' = A|B$ be the restriction to B. Note that A' is a separable $\mathcal{O}_B$-algebra of rank p. Let

$$U \subset \mathbf{A}(A')$$

be the subscheme defined by the equation

$$N_A(x) = a_n \text{ for } x \in \mathbf{A}(A')$$

Then $\pi\colon U \to B$ is a torsor over the norm-1 torus of the algebra A'/B.

Let $L = k[z]/(z^p - a_n)$.

Lemma 3.1. *Suppose that $u \neq 0$. There exists a smooth completion X of U such that:*

For $x \in X \setminus U$ one has

$$v_{\kappa(x)} = 0$$

Moreover, for any closed point $x \in (X \setminus U) \times \operatorname{Spec} L$ one has

$$[\kappa(x) : L] \in p\mathbf{Z}$$

Proof. For X we may take any resolution of singularities of $\hat{X}$. It suffices to check the claimed properties for $X = \hat{X}$.

Let $x \in \hat{X} \setminus U$. If x lies over R_0, then $v_{\kappa(x)} = 0$ by (3). If x lies over B, then $x = [y, 0]$ with $[y] \in \mathbf{P}(A')$ and $N_A(y) = 0$. But then $A_{\pi(x)}$ has zero divisors and is therefore split. Thus $\big(\gamma\big(\pi(x)\big)\big) = 0$ and $v_{\kappa(x)} = 0$ by (3).

For the second claim it suffices to have

$$v_L \neq 0 \in H^{n-1}_{\mathrm{et}}(L, \mu_p^{\otimes(n-1)})$$

This follows from

$$N_{L/k}\big(v_L \cup (z)\big) = v \cup (a_n) = u \neq 0$$

(Recall that $p \neq 2$. If $p = 2$ this argument works also if -1 is a square.) $\square$

Our aim is to express $c(X) \bmod p$ in terms of the Chern classes of the line bundles L, L_i'.

Let $\zeta \in k$ be a primitive p-th root of 1. For

$$\alpha \in H^i_{\mathrm{et}}(-,\mu_p^{\otimes j}) = H^i_{\mathrm{et}}(-,\mu_p^{\otimes j-1}) \otimes \mu_p$$

we write α/ζ for the element of $H^i_{\mathrm{et}}(-,\mu_p^{\otimes j-1})$ with $\alpha = \alpha/\zeta \otimes \zeta$.

For the element

$$u/\zeta = (a_1)\cdots(a_n)/\zeta \in H^n_{\mathrm{et}}(k,\mu_p^{\otimes(n-1)})$$

we have $u/\zeta = \delta$ with

$$\delta \in H^{n,n-1}\big(\mathcal{X}(X/k),\mathbf{Z}/p\big)$$

as above. (The notation δ is introduced for clarity.) Let also

$$\rho \in \mathcal{H}^1(X/k,\mathrm{CH}^b)$$

be as above and let

$$\rho' \in \mathcal{H}^1(U/k,\mathrm{CH}^b)$$

be its image.

We will use the following morphisms:

$$b_0\colon\ \operatorname{Spec} k \to B$$

is some rational point. Moreover let

$$j\colon\ B \times \operatorname{Spec} L \to U \subset \mathbf{A}(A')$$
$$j = z\cdot 1_A$$

where $z \in L$, $z^p = a_n$. In particular we have the L-point

$$h = j \circ (b_0 \times \mathrm{id}_L)\colon\ \operatorname{Spec} L \to U$$

of U.

The morphism

$$\mathrm{id}_U \times h\colon U \times \operatorname{Spec} L \to U \times U$$

and its induced map

$$(\mathrm{id}_U \times h)^*\colon\ \mathrm{CH}^b(U\times U) \to \mathrm{CH}^b(U \times \operatorname{Spec} L)$$

yields a map

$$\bar h\colon \mathcal{H}^1(U/k,\mathrm{CH}^b) \to \mathrm{CH}^b(U\times \operatorname{Spec} L)/\operatorname{res}_{L/k}\big(\mathrm{CH}^b(U)\big)$$

Let

$$\rho'' = \bar h(\rho')$$

and let

$$\tilde\rho'' \in \mathrm{CH}^b(U \times \operatorname{Spec} L)$$

be a representative of ρ''. Put

$$c(\tilde{\rho}'') = \deg\bigl(\tilde{\rho}''^{p-1}\bigr) \in \mathbf{Z}/p$$

Note here that the degree map over L

$$\deg\colon\ \mathrm{CH}_0(U \times \operatorname{Spec} L) \to \mathbf{Z}/p$$

is well defined by Lemma 3.1.

Note that $c(X)$ is given by $c(\tilde{\rho}'')$. Hence our aim is to conclude $c(\tilde{\rho}'') \neq 0$. (Say, for some representative $\tilde{\rho}''$ of ρ''. One can however show that $c(\tilde{\rho}'')$ depends only on ρ'').

Consider the morphism

$$f\colon U \times \operatorname{Spec} L \to U \times U \times U$$
$$f(u,\lambda) = \bigl(u, j(b_0,\lambda), j(\pi(u),\lambda)\bigr)$$

Let

$$\tilde{\rho}' \in \mathrm{CH}^b(U \times U)$$

be a representative of ρ'.

Then

$$(\pi_2 \circ f)^*(\tilde{\rho}') \in \mathrm{CH}^b(U \times \operatorname{Spec} L)$$

is obviously a representative of ρ''.

Let us consider

$$(\pi_0 \circ f)^*(\tilde{\rho}') \in \mathrm{CH}^b(U \times \operatorname{Spec} L)$$

The morphism

$$\pi_0 \circ f\colon U \times \operatorname{Spec} L \to U \times U$$
$$(u,\lambda) \mapsto \bigl(j(b_0,\lambda), j(\pi(u),\lambda)\bigr)$$

factors through $U \times \operatorname{Spec} L \to B \times \operatorname{Spec} L$. Since B is an open subvariety of a cellular variety, the restriction map

$$\mathrm{res}_{L/k}\colon\ \mathrm{CH}^*(B) \to \mathrm{CH}^*(B \times \operatorname{Spec} L)$$

is surjective. Thus

$$(\pi_0 \circ f)^*(\tilde{\rho}') \in \mathrm{res}_{L/k}\bigl(\mathrm{CH}^b(U)\bigr)$$

(The actual computation of $c(X)$ later on will provide another argument avoiding the reference to the cellularity of R).

Finally let us consider

$$(\pi_1 \circ f)^*(\tilde{\rho}') \in \mathrm{CH}^b(U \times \operatorname{Spec} L)$$

The morphism $\pi_1 \circ f$ factors through

$$\hat{f}\colon U \times \operatorname{Spec} L \to U \times_B U \subset U \times U$$
$$(u,\lambda) \mapsto \bigl(u, j(\pi(u),\lambda)\bigr)$$

On the other hand we have

$$(\pi_0^* - \pi_1^* + \pi_2^*)(\tilde\rho') = 0$$

We conclude: Let

$$\delta_B \in H^{n,n-1}\big(\mathcal{X}(U/B), \mathbf{Z}/p\big)$$

be the image of δ, let

$$\rho_B = \operatorname{proj}\big(\hat Q(\delta_B)\big) \in \mathcal{H}^1(U/B, \mathrm{CH}^b)$$

and let

$$\tilde\rho_B \in \mathrm{CH}^b(U \times_B U)$$

be a representative of ρ_B. Then

$$\hat f^*(\tilde\rho_B) \in \mathrm{CH}^b(U \times \operatorname{Spec} L)$$

is a representative of ρ''.

It follows that all we need to compute $c(X)$ is the image δ_B of δ.

We have in $H^n_{\mathrm{et}}(B, \mu_p^{\otimes(n-1)})$:

$$(u/\zeta)_B = (a_1)\cdots(a_n)/\zeta = u_1 \cup (a_n) \pm u_2 \cup (\gamma_1')\cdots(\gamma_{n-2}')$$

where

$$u_1 = \big((a_1)\cdots(a_{n-1}) - (\gamma_1)(\gamma_1')\cdots(\gamma_{n-2}')\big)/\zeta \in H^{n-1}_{\mathrm{et}}(B, \mu_p^{\otimes(n-2)})$$
$$u_2 = (\gamma)/\zeta \cup (a_n) \in H^2_{\mathrm{et}}(B, \mu_p)$$

Lemma 3.2. *One has*

$$\pi^*(u_2) = 0$$

in $H^2_{\mathrm{et}}(U, \mu_p)$.

Proof. Let $q\colon \tilde B = \operatorname{Spec} A \to B$ be the etale covering of degree p corresponding to $(\gamma)/\zeta \in H^1_{\mathrm{et}}(B, \mathbf{Z}/p)$. Then $q^*(\gamma) = 0$ and

$$\big((\gamma) \cup (a_n)\big)_U = (\gamma)_U \cup (N_A(x)) = q_*\big(q^*(\gamma)_U \cup (x)\big) = 0$$

□

It follows that u_2 vanishes over $k(U)$. By Corollary 2.2, u_2 lies in the subgroup

$$H^{2,1}\big(\mathcal{X}(U/B), \mathbf{Z}/p\big) \subset H^2_{\mathrm{et}}(B, \mu_p)$$

If we map this to

$$H^{2,1}\big(\mathcal{X}(U/U), \mathbf{Z}/p\big) \subset H^2_{\mathrm{et}}(U, \mu_p)$$

it follows by Lemma 3.2 that $u_2 = \delta_2$ with

$$\delta_2 \in H^{2,1}\big(\mathcal{X}(U/B), \mathbf{Z}/p\big)^{[1]}$$

Let

$$\rho_2 = \operatorname{proj}(\delta_2) \in \mathcal{H}^1\big(U/B, H^{1,1}(-, \mathbf{Z}/p)\big)$$

The element u_1 vanishes in the generic point of B. Hence $u_1 = \delta_1$ with

$$\delta_1 \in H^{n-1,n-2}\big(\mathcal{X}(B/B), \mathbf{Z}/p\big) \subset H^{n-1}_{\text{et}}(B, \mu_p^{\otimes(n-2)})$$

Let

$$\delta_B \in H^{n,n-1}\big(\mathcal{X}(U/B), \mathbf{Z}/p\big) \subset H^{n}_{\text{et}}(B, \mu_p^{\otimes(n-1)})$$

be the element corresponding to $(u/\zeta)_B$.

Then

$$\delta_B = (\delta_1)_{\mathcal{X}(U/B)} \cup (a_n) \pm \delta_2 \cup (\gamma'_1) \cdots (\gamma'_{n-2})$$

in $H^{n,n-1}\big(\mathcal{X}(U/B), \mathbf{Z}/p\big)$, with (a_n), (γ'_i) considered as elements of $H^{1,1}(-, \mathbf{Z}/p)$.

We have

$$\hat{Q}(\delta_1) \in H^{2b,b-1}\big(\mathcal{X}(B/B), \mathbf{Z}\big) = 0$$

and therefore

$$\hat{Q}\big((\delta_1)_{\mathcal{X}(U/B)} \cup (a_n)\big) = 0$$

Hence (up to sign)

$$\begin{aligned}
\rho_B &= \operatorname{proj}\big(\hat{Q}(\delta_B)\big) \\
&= \operatorname{proj}\big(\hat{Q}\big(\delta_2 \cup (\gamma'_1) \cdots (\gamma'_{n-2})\big)\big) \\
&= \hat{Q}\big(\operatorname{proj}\big(\delta_2 \cup (\gamma'_1) \cdots (\gamma'_{n-2})\big)\big) \\
&= \hat{Q}\big(\rho_2 \cup (\gamma'_1) \cdots (\gamma'_{n-2})\big)
\end{aligned}$$

Let

$$\tilde{\rho}_2 \in H^{1,1}(U \times_B U, \mathbf{Z}/p)$$

be a representative of ρ_2 and let

$$\omega = \hat{f}^*(\tilde{\rho}_2) \in H^{1,1}(U \times \operatorname{Spec} L, \mathbf{Z}/p)$$

It follows that

$$\hat{Q}\big(\omega \cup (\gamma'_1) \cdots (\gamma'_{n-2})\big) \in \mathrm{CH}^b(U \times \operatorname{Spec} L)$$

is a representative of ρ''. Hence all we have to show is that

$$\deg\Big(\big(\hat{Q}\big(\omega \cup (\gamma'_1) \cdots (\gamma'_{n-2})\big)\big)^{p-1}\Big) \neq 0$$

Note that $U \times \operatorname{Spec} L = T \times \operatorname{Spec} L$, where T is the norm-1 torus of the algebra A'/B. One has

$$T = R_{A'/B}(\mathbf{G}_\mathrm{m})/\mathbf{G}_\mathrm{m}$$

and the norm map

$$N\colon R_{A'/B}(\mathbf{G}_\mathrm{m}) \to \mathbf{G}_\mathrm{m}$$

yields a canonical class

$$\alpha \in H^{1,1}(T, \mathbf{Z}/p) = H^1_{\text{et}}(T, \mu_p)$$

Lemma 3.3. *One has (up to sign)*

$$\omega = \operatorname{res}_{L/k}(\alpha) \mod \operatorname{res}_{L/k}\big(H^{1,1}(U,\mathbf{Z}/p)\big)$$

Proof. Well, what else can it be?

Seriously: This is a not so difficult exercise, but I haven't found a nice way yet to write things down. Note that this is a very general statement for the torus T and T-torsor U associated to a smooth variety B and classes $(A'/B) \in H^1_{\mathrm{et}}(B,\mathbf{Z}/p)$ and $(a_n) \in H^1_{\mathrm{et}}(k,\mu_p)$. □

Corollary 3.4. *In particular we have*

$$c(X) = \deg\Big(\big(\hat{Q}\big(\alpha \cup (\gamma'_1)\cdots(\gamma'_{n-2})\big)\big)^{p-1}\Big) \mod p$$

Here $\alpha \in H^{1,1}(T,\mathbf{Z}/p)$, $(\gamma'_i) \in H^{1,1}(B,\mathbf{Z}/p)$, $\hat{Q}$ *operates as map*

$$\hat{Q}\colon H^{n-1,n-1}\big(T,\mathbf{Z}/p\big) \to H^{2b,b}\big(T,\mathbf{Z}\big) = \mathrm{CH}^b(T)$$

and

$$\deg\colon \mathrm{CH}_0(T) \to \mathbf{Z}/p$$

is the degree map (well defined because $v \neq 0$*).* □

Hence eventually the proof of $c(X) \neq 0$ resorts to an explicit computation for very concrete classes. The complexity of this computation is similar to the one of $\delta(L) \neq 0 \bmod p$ in [6]. I have no doubts[2] that indeed $c(X) \neq 0$ and have checked this in some cases, for instance for $n = 3$.

Remark 3.5. For $n = 3$, the variety T is very close to $\mathrm{SL}_1(A)$ where A is a central simple algebra of degree p. In fact, there should be a morphism $T \to \mathrm{SL}_1(A)$ of degree prime to p. I believe that my computations will show that for a certain standard element $u \in \mathrm{CH}^{p+1}(\mathrm{SL}_1(A))$ one has $\deg(u^{p-1}) \neq 0 \bmod p$. (The degree mod p is well defined, if A is non-division.)

4. A computation for the Q_i

For a start, let me formulate a general computation for $\hat{Q}$ (with the index n shifted by 1). In the following X can be any smooth variety. Let

$$Q_{[n]} = Q_{n-1} \circ \cdots \circ Q_1 \circ Q_0$$

Note that for

$$\alpha \in H^{1,1}(X,\mathbf{Z}/p) = H^1_{\mathrm{et}}(X,\mu_p)$$

[2]The computations for all n have been meanwhile included in this text. See Section 8.

the element

$$\tilde{Q}_0(\alpha) \in H^{2,1}(X, \mathbf{Z}) = \mathrm{Pic}(X) = \mathrm{CH}^1(X)$$

is just the first Chern class of the line bundle associated to α under

$$H^1_{\mathrm{et}}(X, \mu_p) \to H^1_{\mathrm{et}}(X, \mathbf{G}_{\mathrm{m}})$$

We will use standard properties of the Steenrod operations, see [8]. Let $\alpha \in H^{1,1}(X, \mathbf{Z}/p)$ and $\beta \in H^{2,1}(X, \mathbf{Z}/p)$. One has $Q_i^2 = 0$, the Q_i anti-commute and are derivations in the $\mathbf{Z}/2$-graded sense (with α odd and β even). Moreover one has

$$\begin{aligned} &P^h(\alpha) = 0 \quad (h \geq 1) \\ &P^1(\beta) = \beta^p \\ &P^h(\beta) = 0 \quad (h > 1) \end{aligned}$$

and the total Steenrod operation

$$P^\bullet = P^0 + P^1 + P^2 + \cdots$$

is multiplicative.

Lemma 4.1. *Let* $\alpha_i \in H^{1,1}(X, \mathbf{Z}/p)$ $(i = 1, \ldots, n)$ *and let* $\beta_i = Q_0(\alpha_i)$. *Then*

$$Q_{[n]}(\alpha_1 \cdots \alpha_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} (\beta_{\sigma(i)})^{p^{i-1}} \mod p$$

in $\mathrm{CH}^b(X) \otimes \mathbf{Z}/p$ *with*

$$b = \frac{p^n - 1}{p - 1}$$

Proof. This is an exercise using the standard properties of the Steenrod operations mentioned above. □

5. Generalities for special correspondences

Let

$$b = \frac{p^n - 1}{p - 1}$$

Let X be a smooth proper irreducible variety of dimension

$$d = b(p-1) = p^n - 1$$

Let

$$\rho \in \mathrm{CH}^b(X^2)$$

and let

$$c(\rho) = (\pi_0)_*(\rho^{p-1}) \in \mathrm{CH}^0(X) = \mathbf{Z}$$

Recall the complex

$$\mathrm{CH}^b(X) \xrightarrow{\pi_0^*-\pi_1^*} \mathrm{CH}^b(X^2) \xrightarrow{\pi_0^*-\pi_1^*+\pi_2^*} \mathrm{CH}^b(X^3)$$

whose homology we have denoted by $\mathcal{H}^1(X/B, \mathrm{CH}^b)$.

Definition 5.1. ρ is called a *special correspondence* if

$$(\pi_0^* - \pi_1^* + \pi_2^*)(\rho) = 0 \tag{1}$$

$$c(\rho) \neq 0 \mod p \tag{2}$$

As we have seen before, special correspondences appear naturally for splitting varieties of symbols. So what can we do with such correspondences? In this section we derive some first properties of special correspondences.

Let $h\colon \operatorname{Spec} L \to X$ be an L-point of X where L/k is a field extension. We write $X_L = X \times \operatorname{Spec} L$. Let

$$H = (\mathrm{id}_X \times h)^*(\rho) \in \mathrm{CH}^b(X_L)$$

Then

$$c(\rho) = \deg_{X_L/L}(H^{p-1})$$

Lemma 5.2. *One has*

$$\rho_L = H \times X - X \times H$$

in $\mathrm{CH}^b(X_L^2)$

Proof. Just pull back the cocycle condition (1) under

$$\mathrm{id}_X \times \mathrm{id}_X \times h\colon X \times X \times \operatorname{Spec} L \to X \times X \times X$$

∎

Let

$$\rho^j \in \mathrm{CH}^{jb}(X^2)$$

be the j-th power of ρ in the Chow ring of X. Considering ρ^j as an element of $\mathrm{End}(X) = \mathrm{CH}^*(X^2)$ in the category of Chow correspondences, we get a map

$$(\rho^j)_*\colon \mathrm{CH}^r(X) \to \mathrm{CH}^{r-(p-1-j)b}(X)$$
$$(\rho^j)_*(\alpha) = (\pi_0)_*\big(\pi_0^*(\alpha)\rho^j\big)$$

Definition 5.3. Let $0 \leq h \leq p-1$. For $\alpha \in \mathrm{CH}^{bh}(X)$ we define its ρ-degree $d_\rho(\alpha) \in \mathbf{Z}$ by

$$d_\rho(\alpha) \cdot [X] = (\rho^{p-1-h})_*(\alpha)$$

in $\mathrm{CH}^0(X) = [X]\mathbf{Z}$.

For $h = 0$ we simply have

$$d_\rho([X]) = c(\rho)$$

by definition.

Lemma 5.4. *For $\alpha \in \mathrm{CH}^{bh}(X)$ one has*

$$d_\rho(\alpha) = \deg_{X_L/L}(\alpha_L H^{p-1-h})$$

Proof. One has

$$\begin{aligned}(\rho^{p-1-h})_*(\alpha)_L &= (\rho_L^{p-1-h})_*(\alpha_L)\\ &= (\pi_0)_*\big((\alpha_L \times X)(H \times X - X \times H)^{p-1-h}\big)\\ &= \deg(\alpha_L H^{p-1-h})H^0\end{aligned}$$

□

Lemma 5.5. *For $\alpha \in \mathrm{CH}^{bh}(X)$ and $p-1-h \le j \le p-1$ one has*

$$(\rho^j)_*(\alpha)_L \in \mathbf{Z}_{(p)}^\times \cdot d_\rho(\alpha)H^{j+h-(p-1)}$$

Proof. One has

$$\begin{aligned}(\rho^j)_*(\alpha)_L &= (\rho_L^j)_*(\alpha_L)\\ &= (\pi_0)_*\big((\alpha_L \times X)(H \times X - X \times H)^j\big)\\ &= \binom{j}{p-1-h}\deg(\alpha_L H^{p-1-h})H^{j+h-(p-1)}\end{aligned}$$

□

For $\alpha \in \mathrm{CH}^{bh}(X)$ with $1 \le h \le p-1$ the element $(\rho^{p-h})_*(\alpha)$ is in $\mathrm{CH}^b(X)$. Thus

$$\Omega(\alpha) = \alpha\big((\rho^{p-h})_*(\alpha)\big)^{p-1-h}$$

is a zero cycle.

Lemma 5.6.

$$\deg\big(\Omega(\alpha)\big) \in \mathbf{Z}_{(p)}^\times \cdot d_\rho(\alpha)^{p-h}$$

Proof. Over L we have by Lemma 5.5

$$\Omega(\alpha)_L = \alpha_L\big(ud_\rho(\alpha)H\big)^{p-1-h}$$

for some $u \in \mathbf{Z}_{(p)}^\times$. Taking degrees yields the claim (see Lemma 5.4). □

Let

$$I(X) = \deg\big(\mathrm{CH}_0(X)\big) \subset \mathbf{Z}$$

Corollary 5.7. *Suppose $I(X) \subset p\mathbf{Z}$. Then for any $\alpha \in \mathrm{CH}^{bh}(X)$ with $1 \le h \le p-1$ one has*

$$d_\rho(\alpha) \in p\mathbf{Z}$$

Proof. By assumption p divides $\deg\bigl(\Omega(\alpha)\bigr)$. By Lemma 5.6 it follows that p divides some power of $d_\rho(\alpha)$. Since p is prime, the claim follows. □

Corollary 5.8. *Suppose $I(X) \subset p\mathbf{Z}$. Let $\alpha \in \mathrm{CH}^b(X)$ and let*

$$\rho' = \rho + (\pi_0^* - \pi_1^*)(\alpha)$$

Then

$$c(\rho') = c(\rho) \mod p$$

Proof. Just compute over L and use Lemma 5.4 and Corollary 5.7. □

Proposition 5.9. *Let ρ be a special correspondence. Then there exists a projector $\pi \in \mathrm{End}(X) \otimes \mathbf{Z}_{(p)}$ such that*

(1) $\pi_L = c(\rho)^{-1}\rho_L^{p-1} \mod p\,\mathrm{CH}^d(X_L^2) \otimes \mathbf{Z}_{(p)}$

(2) *For $M = (X, \pi)$ one has*

$$M \otimes X = \bigoplus_{i=0}^{p-1} \mathbf{Z}_{(p)}\{bi\} \otimes X$$

in the category of Chow motives over $\mathbf{Z}_{(p)}$.

Proof. [See Section 7 for another way.]

Only a brief sketch for $p = 2$: After replacing ρ by $\rho c(\rho)^{-1}$ we may assume $c(\rho) = 1$. Let

$$\pi' = \rho - X \times (\pi_0)_*(\rho^2)$$

Over L this is a projector, because

$$\pi'_L = H \times X + X \times H$$

Since L can be any residue class field of X, it follows by nilpotence (see [5, Proposition 1]) that there is a projector π with $\pi'_L = \pi_L$.

For general p one proceeds similarly, starting from $\rho^{p-1} \in \mathrm{CH}^d(X^2)$. □

6. Existence of generic splitting varieties

See also [11, Theorem 7.3].

Let

$$u \in H^n_{\mathrm{et}}(k, \mu_p^{\otimes(n-1)})$$

be a symbol and let Z be a not necessarily connected scheme over k such that $u_{\kappa(z)} = 0$ for all $z \in Z$. We call such a scheme a *splitting* *scheme* of u. A smooth splitting scheme is nothing else than a disjoint union of smooth splitting varieties. We will assume smoothness in the following.

As discussed in Section 2, there is a natural element

$$\rho = \rho_Z \in \mathcal{H}^1(Z/k, \mathrm{CH}^b)$$

with

$$b = \frac{p^{n-1} - 1}{p - 1}$$

This element is functorial in Z. Namely, if $f\colon Z' \to Z$ is a morphism, then

$$\rho_{Z'} = f^*(\rho_Z)$$

We are particularly interested in the following situation: Let X be a smooth irreducible proper splitting variety of u of dimension $d = b(p-1) = p^{n-1} - 1$ and let Y be any smooth irreducible splitting variety of u. Let

$$Z = X \cup Y \quad \text{(disjoint union)}$$

We have

$$\mathrm{CH}^b(Z^2) = \mathrm{CH}^b(X^2) \oplus \mathrm{CH}^b(X \times Y) \oplus \mathrm{CH}^b(Y \times X) \oplus \mathrm{CH}^b(Y^2)$$

Let $E = k(Y)$ be the function field of Y. The projection maps

$$\mathrm{CH}^b(Z^2) \to \mathrm{CH}^b(X \times Y) \to \mathrm{CH}^b(X_E)$$

(with $X_E = X \times \operatorname{Spec} E$) induce a map

$$\Psi_{X,E}\colon \mathcal{H}^1(Z/k, \mathrm{CH}^b) \to \mathrm{CH}^b(X_E)/\,\mathrm{CH}^b(X)_E$$

(since $b > 0$ one has $\mathrm{CH}^b(\operatorname{Spec} E) = 0$). Let

$$\rho_{X,E} = \Psi_{X,E}(\rho_Z) \in \mathrm{CH}^b(X_E)/\,\mathrm{CH}^b(X)_E$$

Note that $\rho_{X,E}$ is functorial in X and E.

Choose a representative

$$\tilde{\rho}_{X,E} \in \mathrm{CH}^b(X_E)$$

of $\rho_{X,E}$, let

$$\theta_{X,E} = (\tilde{\rho}_{X,E})^{p-1} \in \mathrm{CH}^d(X_E)$$

and let

$$c(X,E) = \deg_{X_E/E}(\theta_{X,E}) \bmod p \in \mathbf{Z}/p$$

Lemma 6.1. *Assume $I(X) \subset p\mathbf{Z}$. Then*

$$c(X,E) = c(X) \bmod p$$

In particular, $c(X,E)$ does not depend on E or Y at all.

Proof. Let $L = k(X)$ and $K = k(X \times Y)$. By functoriality we have

$$\rho_{X,E} \to \rho_{X,K} \leftarrow \rho_{X,L}$$

under

$$\frac{\mathrm{CH}^b(X_E)}{\mathrm{CH}^b(X)_E} \to \frac{\mathrm{CH}^b(X_K)}{\mathrm{CH}^b(X)_K} \leftarrow \frac{\mathrm{CH}^b(X_L)}{\mathrm{CH}^b(X)_L}$$

It follows that for any representatives

$$\tilde{\rho}_{X,E} \in \mathrm{CH}^b(X_E), \quad \tilde{\rho}_{X,L} \in \mathrm{CH}^b(X_L)$$

there exists $\alpha \in \mathrm{CH}^b(X)$ with

$$(\tilde{\rho}_{X,E})_K = (\tilde{\rho}_{X,L})_K + \alpha_K$$

in $\mathrm{CH}^b(X_K)$.

We next look at $\rho_{X,L}$. Since for

$$Z = X \cup X \quad \text{(disjoint union)}$$

the simplicial schemes $\mathcal{X}(Z/k)$ and $\mathcal{X}(X/k)$ have the same cohomology, the element $\rho_{X,L}$ equals the image of

$$\rho = \rho_X \in \mathcal{H}^1(X/k, \mathrm{CH}^b)$$

in

$$\frac{\mathrm{CH}^b(X_L)}{\mathrm{CH}^b(X)_L}$$

Hence

$$c(X) = \deg_{X_L/L}\big((\tilde{\rho}_{X,L})^{p-1}\big)$$

By Corollary 5.8 we have

$$c(X) = \deg_{X_L/L}\big((\tilde{\rho}_{X,L} + \alpha_L)^{p-1}\big)$$

Hence

$$c(X) = \deg_{X_K/K}\Big(\big((\tilde{\rho}_{X,E})_K\big)^{p-1}\Big) = \deg_{X_E/E}\big((\tilde{\rho}_{X,E})^{p-1}\big)$$

$\square$

Corollary 6.2. *Assume that $I(X) \subset p\mathbf{Z}$ and that $c(X) \neq 0 \pmod{p}$. Then X is a p-generic splitting variety of u.*

Proof. Let $E = k(Y)$ be any splitting field of u. Then

$$\theta_{X,E} \in \mathrm{CH}^d(X_E)$$

is a zero cycle of degree prime to p. $\square$

The core of the argument is entirely due to Voevodsky. On [11, page 37] he defines $\mathcal{X} = \mathcal{X}(Y/k)$, where Y is the union of all splitting varieties up to isomorphism and draws the conclusion in [11, Theorem 7.3].

It is amusing to consider the case $n = 2$. In this case ρ can be defined easily with the classical relation between the kernel of $H^2(k, \mathbf{G}_{\mathrm{m}}) \to H^2\big(k(X), \mathbf{G}_{\mathrm{m}}\big)$ and $\operatorname{Pic}(X)$. Taking for X the Severi-Brauer variety of dimension $p-1$, one may then check by inspection (see Example 2.4) that $c(X) \neq 0 \pmod p$. Corollary 6.2 shows then the p-genericity of the Severi-Brauer variety, without using much of the theory of central simple algebras.

7. Generalities for special correspondences II

7.1. **Preliminary Definitions.** Let X be a smooth variety over a field k.

For $n \geq 0$ and $0 \leq i \leq n$ let

$$\pi_i \colon X^{n+1} \to X^n$$
$$\pi_i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, x_{i+1} \dots, x_n)$$

Then we have complexes

$$\cdots \to \mathrm{CH}^r(X^n) \xrightarrow{\delta_n} \mathrm{CH}^r(X^{n+1}) \xrightarrow{\delta_{n+1}} \mathrm{CH}^r(X^{n+2}) \to \cdots \tag{7.1.1}$$

with

$$\delta_n = \sum_{i=0}^{n} (-1)^i \pi_i^*$$

Here $\mathrm{CH}^r(Z)$ is the classical Chow group of r-codimensional cycles.

We put

$$\mathcal{Z}^n(X/k, \mathrm{CH}^b) = \ker \delta_{n+1}$$
$$\mathcal{H}^n(X/k, \mathrm{CH}^b) = \ker \delta_{n+1} / \operatorname{im} \delta_n$$

We will use the following notations: Let I be an index set. For $i \in I$ let $X_i = X$ be a copy of X and let $X^I = \prod_i X_i$. We will sometimes understand

$$X^{n+1} = X_0 \times \cdots \times X_n$$

where the X_i are copies of X.

For a sequence $i_1, \dots, i_n$ in I let

$$\hat{\pi}_{i_1, \dots, i_n} \colon X^I \to X^n$$
$$\hat{\pi}_{i_1, \dots, i_n}\big((x_i)_{i \in I}\big) = (x_{i_1}, \dots, x_{i_n})$$

For $u \in \mathrm{CH}^*(X^n)$ we write

$$u_{[i_1,\dots,i_n]} = \hat{\pi}^*_{i_1,\dots,i_n}(u) \in \mathrm{CH}^*(X^I)$$

7.2. **Symmetry of special correspondences.** We assume that X is proper of dimension $d = b(p-1)$.

Let $\rho \in \mathrm{CH}^b(X^2)$ be a special correspondence. Let τ be the exchange involution.

Lemma 7.1. $\tau^*(\rho) = -\rho$

Proof. We have

$$\rho_{[1,2]} - \rho_{[0,2]} + \rho_{[0,1]} = 0$$

in $\mathrm{CH}^b(X^3)$ by the cocycle condition. Pulling back with the diagonals $X^2 \to X^3$, $(x,y) \mapsto (x,x,y)$, (x,y,x) yields

$$\rho_{[0,1]} - \rho_{[0,1]} + \rho_{[0,0]} = 0, \quad \rho_{[1,0]} - \rho_{[0,0]} + \rho_{[0,1]} = 0$$

which gives

$$\rho_{[0,0]} = 0, \quad \rho_{[1,0]} + \rho_{[0,1]} = 0$$

□

7.3. **Construction of the motive for $p = 2$.** I first used nilpotence to get the projector, see the proof of Proposition 5.9. Meanwhile I noticed that there is a much more direct argument.

For an illustration, we first consider the case $p = 2$. Then $b = d$.

We assume $c(\rho) = 1$. Let

$$\beta = -(\pi_0)_*(\rho^2) \in \mathrm{CH}_0(X)$$

Then

$$\beta_L = -(\pi_0)_*\big((H_0 - H_1)^2\big) = -(\pi_0)_*(-2H_0H_1) = 2H$$

and therefore

$$\deg(\beta) = 2$$

Lemma 7.2.

$$\rho \circ \rho = -\rho + \beta \times X$$
$$\rho \circ \rho \circ \rho = \rho$$

Proof. One has by Lemma 7.1:

$$(\pi_1)_*(\rho) = (\pi_0 \circ \tau)_*(\rho) = -(\pi_0)_*(\rho) = -c(\rho)[X] = -[X]$$

One has in $\mathrm{CH}^d(X_0 \times X_2)$ with

$$\pi_1 \colon X_0 \times X_1 \times X_2 \to X_0 \times X_2$$
$$(x_0, x_1, x_2) \to (x_0, x_2)$$

and $X_i = X$:

$$\begin{aligned}(\rho\circ\rho)_{[0,2]} &= (\pi_1)_*(\rho_{[0,1]}\rho_{[1,2]})\\ &= (\pi_1)_*\big(\rho_{[0,1]}(\rho_{[0,2]}-\rho_{[0,1]})\big)\\ &= (\pi_1)_*(\rho_{[0,1]})\rho_{[0,2]} - (\pi_1)_*\big((\rho_{[0,1]})^2\big)\\ &= -\rho_{[0,2]} + \beta_{[0]}\end{aligned}$$

This proves the first claim. For the second claim note that

$$(\beta\times X)\circ\rho = \beta\times X$$

for any β. Thus

$$\rho\circ\rho\circ\rho = -\rho\circ\rho + (\beta\times X)\circ\rho = \rho - \beta\times X + \beta\times X = \rho$$

□

Corollary 7.3. *$\rho\circ\rho$ is a projector.* □

Note that

$$(\rho\circ\rho)_L = H\times X + X\times H$$

so $\rho\circ\rho$ is the desired projector.

7.4. **Correspondences generated by ρ.** We work over $\mathbf{Z}_{(p)}$.

In this section we show that the correspondences in $\mathrm{CH}^*(X^n)$ generated by a special correspondence $\rho\in\mathrm{CH}^b(X^2)$ via products, (simplicial) pull backs and push forwards form for every n a free $\mathbf{Z}_{(p)}$-module. It is denoted by R_n.

We consider similarly the correspondences in $\mathrm{CH}^*(X_L^n)$ generated by $H\in\mathrm{CH}^b(X_L)$ (where $\rho_L = H\times X - X\times H$). The resulting ring is denoted by S_n.

It turns out that the natural map $R_n\to S_n$ is injective. This way, identities for correspondences for X generated by ρ can be easily verified by passing to L.

The rings $R_n\to S_n$ are independent of the weight of the symbol. In fact, one has

$$R_n = \mathrm{CH}^*(X^n),\quad S_n = \mathrm{CH}^*\big((\mathbf{P}^{p-1})^n\big)$$

where X is a nonsplit Severi-Brauer variety of dimension $p-1$.

The following discussion is simple in nature, but a bit tedious and tiring. Voevodsky's framework of motives over simplicial schemes indicates that there should be simpler organization and explanation.

For $i = 0,\,\dots,\,p-1$ let

$$\beta_i' = (\pi_0)_*(\rho^{p-1+i})\in\mathrm{CH}^{bi}(X)$$

Then

$$(\beta_i')_L = (\pi_0)_*\big((H_0 - H_1)^{p-1+i}\big) = c(\rho)\binom{p-1+i}{i}(-1)^i H^i$$

Note that for $1 \le i \le p-1$ one has

$$\binom{p-1+i}{i} \in p\mathbf{Z}_{(p)}^\times$$

We put for $1 \le i \le p-1$

$$\beta_i = c(\rho)^{-1}(-1)^i \frac{p}{\binom{p-1+i}{i}}\beta_i' \in \mathrm{CH}^{bi}(X) \otimes \mathbf{Z}_{(p)}$$

so that

$$(\beta_i)_L = pH^i$$

We put

$$\beta_0 = c(\rho)^{-1}\beta_0' = [X]$$

Let

$$\varepsilon = c(\rho)^{-1}\beta_{p-1} \in \mathrm{CH}^d(X) \otimes \mathbf{Z}_{(p)}$$

The zero cycle ε has degree p.

For $i = 0, \dots, p-1$ let

$$\alpha_i = (\rho^i)_*(\varepsilon) \in \mathrm{CH}^{bi}(X)$$

Note that

$$(\alpha_i)_L = pH^i$$

In particular $\alpha_0 = p[X]$.

Lemma 7.4. *In the Chow ring* $\mathrm{CH}^*(X)$ *one has*

$$\alpha_i\alpha_j = p\alpha_{i+j}$$

Proof. Note that $\varepsilon_{[1]}\varepsilon_{[2]}\rho_{[1,2]} = 0$ by dimension reasons. Hence

$$\begin{aligned}(\alpha_i\alpha_j)_{[0]} &= (\pi_{1,2})_*(\varepsilon_{[1]}\varepsilon_{[2]}\rho^i_{[0,1]}\rho^j_{[0,2]})\\ &= (\pi_{1,2})_*(\varepsilon_{[1]}\varepsilon_{[2]}\rho^i_{[0,1]}\rho^j_{[0,1]})\\ &= \alpha_0\alpha_{i+j}\end{aligned}$$

$\square$

Lemma 7.5. *For $i = 0, \dots, p-1$, the element α_i is in the $\mathbf{Z}_{(p)}$-module generated by the elements*

$$\beta_j\beta_{i-j}, \qquad 0 \le j \le i$$

Proof. For a certain unit $u \in \mathbf{Z}_{(p)}^{\times}$ one has

$$\begin{aligned} u(\alpha_i)_{[0]} &= (\pi_{1,2})_*(\rho_{[0,1]}^i \rho_{[1,2]}^{2p-2}) \\ &= (\pi_{1,2})_*\big(\rho_{[0,1]}^i(\rho_{[0,1]} + \rho_{[0,2]})^{2p-2}\big) \\ &= \sum_{j+k=2p-2} \binom{2p-2}{j} (\pi_1)_*(\rho_{[0,1]}^{i+j})(\pi_2)_*(\rho_{[0,2]}^k) \end{aligned}$$

□

Corollary 7.6. *For $i = 1, \ldots, p-1$ one has $\alpha_i = \beta_i$.*

Proof. Note that

$$(\alpha_i)_L = pH^i = (\beta_i)_L$$

Since H is not a torsion element, it suffices to show that $\alpha_i = u\beta_i$ for some $u \in \mathbf{Z}_{(p)}$.

We proceed by induction on i. Suppose $\alpha_j = \beta_j$ for $1 \leq j < i$. By Lemma 7.5, α_i is a linear combination of $\beta_0\beta_i = \beta_i$ and two-fold products $\beta_j\beta_{i-j}$ with $1 \leq j < i$. By induction we have $\alpha_j = \beta_j$, and by Lemma 7.4, we have $\beta_j\beta_{i-j} = p\alpha_i$. Hence α_i is a linear combination of β_i and $p\alpha_i$. □

For $n \geq 1$ let

$$S_n = \mathbf{Z}_{(p)}[H_1, \ldots, H_n] \subset \mathrm{CH}^*(X_L^n) \otimes \mathbf{Z}_{(p)}$$

be the subring generated by the $H_i = H_{[i]}$. The ring S_n has the presentation

$$\mathbf{Z}_{(p)}[H_1, \ldots, H_n]/(H_1^p, \ldots, H_n^p)$$

In particular, it is free of rank p^n.

We define

$$R_n \subset \mathrm{CH}^*(X^n) \otimes \mathbf{Z}_{(p)}$$

as the submodule generated by the monomials

$$(\beta_i)_{[1]}\rho_{[1,2]}^{h_2} \cdots \rho_{[1,n]}^{h_n}$$

with $0 \leq i \leq p-1$, $0 \leq h_k \leq p-1$, $k = 2, \ldots, n$.

The image of R_n in $\mathrm{CH}^*(X_L^n) \otimes \mathbf{Z}_{(p)}$ is the subring

$$\mathbf{Z}_{(p)}[H_2 - H_1, \ldots, H_n - H_1] + pS_n \subset S_n$$

(note that $(H_i - H_1)^p = 0 \bmod p$). It is easy to see that the specified monomials form a $\mathbf{Z}_{(p)}$-basis of R_n. Furthermore, $R_n \to S_n$ is injective.

Lemma 7.7. *R_n contains the elements $(\beta_i)_{[k]}$ for $k = 1, \ldots, n$.*

Proof. We have

$$\begin{aligned}(\beta_i')_{[2]} - (\beta_i')_{[1]} &= (\pi_0)_*(\rho_{[0,2]}^{p-1+i} - \rho_{[0,1]}^{p-1+i}) \\ &= (\pi_0)_*\big((\rho_{[0,1]} + \rho_{[1,2]})^{p-1+i} - \rho_{[0,1]}^{p-1+i}\big) \\ &= \sum_{k=1}^{p-1+i} \binom{p-1+i}{k} (\pi_0)_*(\rho_{[0,1]}^{p-1+i-k})\rho_{[1,2]}^k \\ &= \sum_{k=1}^{i} \binom{p-1+i}{k} (\beta_{i-k}')_{[1]}\rho_{[1,2]}^k\end{aligned}$$

□

Lemma 7.8.

$$pc(\rho)\rho = (\pi_0^* - \pi_1^*)(\beta_1')$$

Proof. Just specialize the previous computations to the case $i = 1$. □

Corollary 7.9. *For any* $[\rho] \in \mathcal{H}^1(X/k, \mathrm{CH}^b)$ *one has*

$$pc(\rho)[\rho] = 0$$

□

Lemma 7.10. R_2 *contains the element* $\rho_{[1,2]}^p$.

Proof. This is the most tricky part.

Note that $\rho^{2p-1} = 0$ by dimension reasons. We have

$$\begin{aligned}c(\rho)\rho_{[1,2]}^p &= (\pi_0)_*(\rho_{[0,2]}^{p-1}\rho_{[1,2]}^p) \\ &= (\pi_0)_*\big(\rho_{[0,2]}^{p-1}(\rho_{[0,2]} - \rho_{[0,1]})^p\big) \\ &= (-1)^p(\pi_0)_*(\rho_{[0,2]}^{p-1}\rho_{[0,1]}^p) + p\Omega \\ &= (-1)^p(\pi_0)_*\big((\rho_{[0,1]} + \rho_{[1,2]})^{p-1}\rho_{[0,1]}^p\big) + p\Omega \\ &= \sum_{i=0}^{p-2} u_i(\beta_{i+1}')_{[1]}\rho_{[1,2]}^{p-1-i} + p\Omega\end{aligned}$$

Here $u_i \in \mathbf{Z}$ and Ω is a linear combination of the elements

$$\gamma_i = (\pi_0)_*(\rho_{[0,2]}^{p-1+i}\rho_{[0,1]}^{p-i})$$

with $i = 1, \ldots, p-1$.

Hence it suffices to show

$$p\gamma_i \in R_2$$

The element ε is a zero cycle of degree p. Since we work over $\mathbf{Z}_{(p)}$, we may assume that k has no extensions of degree prime to p. Hence we may assume that there exists a field L with $[L:k] = p$ and a point $\operatorname{Spec} L \to X$. (The following use of the field L of degree p may and should be replaced by a more formal discussion using the cycle ε directly.)

Then

$$\begin{aligned} p(\gamma_i)_{[1,2]} &= N_{L/k}\big(((\gamma_i)_{[1,2]})_L\big) \\ &= N_{L/k}\big((\pi_0)_*\big((H_0-H_2)^{p-1+i}(H_0-H_1)^{p-i}\big)\big) \end{aligned}$$

Since $i \geq 1$ and $H^p = 0$, the element $p(\gamma_i)_{[1,2]}$ is a linear combination of the elements

$$pN_{L/k}(H_1^k H_2^{p-k})$$

with $k = 1, \ldots, p-1$. We have

$$\big((\beta_k)_{[1]}\rho_{[1,2]}^{p-k}\big)_L = pH_1^k(H_1-H_2)^{p-k}$$

Since $H^p = 0$, these elements generate the same module as the elements

$$pH_1^k H_2^{p-k}$$

Therefore the element $p(\gamma_i)_{[1,2]}$ is a linear combination of the elements

$$p(\beta_k)_{[1]}\rho_{[1,2]}^{p-k}$$

But those elements are obviously in R_2. □

We summarize:

Corollary 7.11. *R_n is the ring generated over $\mathbf{Z}_{(p)}$ by $(\beta_i)_{[k]}$, $(\alpha_i)_{[k]}$, $\rho_{[k,l]}$ with $1 \leq k,l \leq n$ and $0 \leq i \leq p-1$.* □

Corollary 7.12. *The family of subrings $R_n \subset \mathrm{CH}^*(X^n)$ is closed under external products $\mathrm{CH}^*(X^n) \times \mathrm{CH}^*(X^m) \to \mathrm{CH}^*(X^{n+m})$. It is also closed under pull backs and push forwards along any simplicial morphism $X^n \to X^m$.* □

Let

$$\Pi_\rho = c(\rho)^{-1}\sum_{i=0}^{p-1} H_0^i H_1^{p-1-i} \in S_2 \subset \mathrm{CH}^*(X_L^2)$$

One has $\Pi_\rho \in \mathrm{CH}^d(X_L^2) = \operatorname{End}(X_L)$. Since $c(\rho) = \deg(H^{p-1})$, it is easy to see that Π_ρ is an idempotent when considered as an element of $\operatorname{End}(X_L)$.

Note that

$$(\rho_{[0,1]}^{p-1})_L = \Pi_\rho \mod pS_2$$

Hence $\Pi_\rho \in R_2 \subset \mathrm{CH}^*(X^2)$. In fact, one has $\Pi_\rho \in \mathrm{CH}^d(X^2) = \operatorname{End}(X)$.

It is also an idempotent of $\mathrm{End}(X)$, since $R_2 \to S_2$ is injective and the R_n are closed under composition operations.

Definition 7.13. The element $\Pi_\rho \in R_n$ is called the projector of ρ.

The Chow motive

$$M = M_\rho = (X, \Pi_\rho)$$

is called the motive of ρ.

Proposition 7.14. *One has*

$$X \otimes M = \bigoplus_{i=0}^{p-1} X \otimes \mathbf{Z}_{(p)}\{bi\}$$

in the category of Chow motives over $\mathbf{Z}_{(p)}$.

Proof. Let $f\colon Y \to X$ be a morphism. For $0 \le i \le p-1$ let

$$\varphi_i\colon\ \mathrm{CH}^h(Y) \to \mathrm{CH}^{h+bi}(Y \times X)$$

$$\varphi_i(\alpha) = f^*(\alpha)(f \times \mathrm{id}_X)^*(\rho^i)$$

$$\Phi = \sum_{i=0}^{p-1} \varphi_i\colon\ \bigoplus_{i=0}^{p-1} \mathrm{CH}^{h-bi}(Y) \to \mathrm{CH}^h(Y \times X)$$

and

$$\psi_i\colon\ \mathrm{CH}^{h+bi}(Y \times X) \to \mathrm{CH}^h(Y)$$

$$\psi_i(\beta) = (\pi_X)_*\big(\beta(f \times \mathrm{id}_X)^*(\rho^{p-1-i})\big)$$

$$\Psi = \sum_{i=0}^{p-1} \psi_i\colon\ \mathrm{CH}^h(Y \times X) \to \bigoplus_{i=0}^{p-1} \mathrm{CH}^{h-bi}(Y)$$

Then for $\alpha \in \mathrm{CH}^h(Y)$ one has

$$\begin{aligned}(\psi_j \circ \varphi_i)(\alpha) &= (\pi_X)_*(\alpha_Y \rho_{Y,X}^{p-1-j+i}) \\ &= \alpha f^*\big((\pi_1)_*(\rho_{[0,1]}^{p-1-j+i})\big)\end{aligned}$$

Hence $\Psi \circ \Phi$ is a triangular matrix with invertible elements on the diagonal, and therefore an isomorphism.

Note that $(\mathrm{id}_Y \times \Pi_\rho) \circ \Phi = \Phi$. It remains to show that

$$\Phi\colon\ \bigoplus_{i=0}^{p-1} \mathrm{CH}^{h-bi}(Y) \to \mathrm{CH}^h(Y \times M_\rho)$$

is surjective. The projection $Y \times M_\rho \to Y$ induces a spectral sequence (see [4])

$$E_2^{r,s} = A^r(Y, A^s(M_\rho)) \Longrightarrow A^{r+s}(Y \times M_\rho)$$

Since $\mathrm{CH}^s((M_\rho)_L) = H^i\mathbf{Z}_{(p)}$ for $s = bi$ and $\mathrm{CH}^s((M_\rho)_L) = 0$ otherwise, the claim follows. $\square$

8. Computing $c(X)$

In the following we refer to the notations in [6]. All Chow groups are understood with coefficients $\mathbf{Z}/p$.

8.1. **Recalling the R_i.** See [6, Section 6, p. 15–16]. We switch from the index n in [6] to m, to avoid conflicts below.

Given forms (S, H_i, α_i), $i = 1, \ldots, m$, we defined forms

$$(R_i/R_{i+1}, J_i, \gamma_i), \qquad (R_i/R_{i+1}, J'_i, \gamma'_i)$$

We write $R = R_1$ and assume $S = \operatorname{Spec} k$. Then we have a sequence of bundles

$$R = R_1 \to R_2 \to \cdots \to R_m = S = \operatorname{Spec} k$$

with relative dimensions $\dim(R_i/R_{i+1}) = p^{i+1} - p^i$. Thus

$$\dim R = p^m - p = p(p^{m-1} - 1)$$

Remark 8.1. The extension $R(\sqrt[p]{\gamma_1})/R$ is a model for the extension Y^p/C^pY for a norm variety Y of the symbol $\{\alpha_1, \ldots, \alpha_m\}$, where $C^pY = (Y^p)/(\mathbf{Z}/p)$ is the p-th cyclic symmetric product of Y.

We denote:

$$s_i = c_1(J_i) \in \mathrm{CH}^1(R_i) \subset \mathrm{CH}^1(R)$$
$$t_i = c_1(J'_i) \in \mathrm{CH}^1(R_i) \subset \mathrm{CH}^1(R)$$

We have by definition

$$R_i/R_{i+1} = \mathcal{C}_p = \mathcal{C}_p(\alpha_1, \ldots, \alpha_i, \gamma_{i+1})$$

8.2. **Recalling the $\mathcal{C}_r$.** See [6, Section 5, p. 10]. We fix the index i above and put $n = i + 1$.

Let us recall some parts of the definition of $\mathcal{C}_p$. We have:

$$\mathcal{C}_r = \mathcal{C}_r(\alpha_1, \ldots, \alpha_{n-1}, \beta) = (S_r/S_{r-1}, L_r, \beta_r), \qquad r \geq -1.$$

With the notations from [6] we have a sequence of bundles

$$R_i = \mathcal{C}_p = S_p \to S_{p-1} \to \cdots \to S_0 = S' = R_{i+1}$$

with relative dimensions $\dim S_r/S_{r-1} = p^{n-1} - p^{n-2}$.

Remark 8.2. S_r/S_{r-1} parameterizes one of the semi-elementary moves in subsymbols. One uses p moves of the given type, which are in total parameterized by $S_p/S_0 = R_i/R_{i+1}$. In particular,

$$\dim(R_i/R_{i+1}) = p(p^{n-1} - p^{n-2}) = p(p^i - p^{i-1})$$

as mentioned before (recall $n = i + 1$).

In [6, Proof of Theorem 5.2, p. 13] we defined certain elements

$$x_r = c_1(L_r)^{p^{i-1}} \in \mathrm{CH}^{p^{i-1}}(S_r), \qquad r \geq -1$$
$$z_r = c_1(K'_{n-1,r})^{p^{i-1}} \in \mathrm{CH}^{p^{i-1}}(S_r), \qquad r \geq 1$$

subject to the relations

$$x_{-1} = 0$$
$$x_0 = c_1(L)^{p^{i-1}} \in \mathrm{CH}^{p^{i-1}}(S')$$
$$x_r = x_{r-2} + z_r, \qquad r \geq 1$$
$$z_r^p = z_r x_{r-1}^{p-1}$$

Here $L = J_{i+1}$ is the line bundle for γ_{i+1}. Hence

$$x_0 = s_{i+1}^{p^{i-1}}$$

We have x_r, $z_r \in \mathrm{CH}^{p^{i-1}}(S_r)$. Moreover

$$(S_r \to S_{r-1})_*(z_r^i) = \begin{cases} 0 & \text{for } i = 0, \ldots, p-2 \\ 1 & \text{for } i = p-1 \end{cases} \tag{8.2.1}$$

We have to sharpen some computations for the rings $R_r(a,b)$ of [6, Section 5].

Lemma 8.3. *One has*

$$x_r^{p^2-1} = x_r^{p-1}$$

in $R_r(0,1)$.

Proof. By [6, Corollary 5.5] one has

$$z_{r+1}^{p^2} = z_{r+1}^p$$

in $R_{r+1}(0,1)$. Since $z_{r+1}^p = z_{r+1}x_r^{p-1}$ we have

$$z_{r+1}x_r^{p^2-1} = z_{r+1}x_r^{p-1}$$

Since $R_{r+1}(0,1)$ is the free $R_r(0,1)$-module with basis z_{r+1}^i, $0 \leq i \leq p-1$, the claim follows. □

Remark 8.4. This proof of Lemma 8.3 is a bit strange since one uses the ring $R_{r+1}(a,b)$ to get an identity for $R_r(a,b)$.

Alternatively, one may argue within $R_r(a,b)$ as follows. By [6, Proposition 5.4], the ring $R_r(a,b)$ is isomorphic to a product of rings of the form

$$K_k = \mathbf{F}_p[v_1, \ldots, v_k]/(v_1^p, \ldots, v_k^p), \quad k \geq 0.$$

An element $u \in K_k$ can be written as $u = a + v$ with $a \in \mathbf{F}_p$ and $v^p = 0$. Then

$$u^{p^2-1} = (a+v)^{p(p-1)}u^{p-1} = a^{p-1}u^{p-1}$$

If u is a unit, then $a \neq 0$ and $a^{p-1} = 1$. Thus, if u is a unit, or if $u = 0$, we have

$$u^{p^2-1} = u^{p-1}$$

One may now inspect the isomorphisms

$$R_r(a,b) \simeq \prod_{i \in \mathbf{F}_p} R_{r-1}(b, ib+a)$$

(for $b \neq 0$) resp.

$$R_r(a,0) \simeq \mathbf{F}_p[z_1]/(z_1^p) \otimes R_{r-1}(0,1)$$

from the proof of [6, Proposition 5.4] to see that x_r will in each component either go to 0 or to $\mathit{unit} \cdot x_{r-1}$. This way the claim follows by induction on r.

Remark 8.5. Meanwhile I found a much much simpler way to prove [6, Corollary 5.5] and Lemma 8.3.

Namely one has

$$x_r^p x_{r-1} - x_r x_{r-1}^p = 0 \tag{8.2.2}$$

$$\sum_{i=0}^{p} (x_r^{p-1})^i (x_{r-1}^{p-1})^{p-i} = 1 \tag{8.2.3}$$

in $R_r(0,1)$.

Proof of (8.2.2). Let

$$f(x,y) = x^p - xy^{p-1}, \quad g(x,y) = yf(x,y) = x^p y - xy^p$$

Then

$$\begin{aligned} f(x_r, x_{r-1}) &= (z_r + x_{r-2})^p - (z_r + x_{r-2})x_{r-1}^{p-1} \\ &= (z_r x_{r-1}^{p-1} + x_{r-2}^p) - (z_r + x_{r-2})x_{r-1}^{p-1} \\ &= x_{r-2}^p - x_{r-2}x_{r-1}^{p-1} = f(x_{r-2}, x_{r-1}) \end{aligned}$$

Hence

$$g(x_r, x_{r-1}) = g(x_{r-2}, x_{r-1}) = -g(x_{r-1}, x_{r-2})$$

and

$$\pm g(x_r, x_{r-1}) = g(x_0, x_{-1}) = g(1,0) = 0$$

□

Proof of (8.2.3). Let

$$h(x,y)=\sum_{i=0}^{p}(x^{p-1})^i(y^{p-1})^{p-i}$$

Then

$$h(x,y)=f(x,y)^{p-1}+y^{(p-1)p}$$

Hence, by the above,

$$h(x_r,x_{r-1})=h(x_{r-2},x_{r-1})=h(x_{r-1},x_{r-2})=\cdots=h(1,0)=1$$

□

One finds now

$$\begin{aligned} x_r^{p-1}&=x_r^{p-1}h(x_r,x_{r-1})\\ &=x_r^{p^2-1}+x_r^{p-1}f(x_{r-1},x_r)^{p-1}\\ &=x_r^{p^2-1}+g(x_{r-1},x_r)^{p-1}=x_r^{p^2-1}\end{aligned}$$

Moreover one has

$$u^{p^2}=u^p$$

for all $u\in R_r(0,1)$, because this holds for all the ring generators x_s of $R_r(0,1)$ over $\mathbf{F}_p$.

The ring R' is the homogeneous variant $R_p(0,x_0)$ of $R_p(0,1)$, see the lines before [6, Corollary 5.7].

Corollary 8.6. *One has*

$$x_r^{p^2-1}=x_r^{p-1}x_0^{p^2-p}$$

in R'. □

Lemma 8.7. *Let $\pi_i\colon R_i\to R_{i+1}$ be the projection. Then for $h,k\ge 0$ one has*

$$(\pi_i)_*(x_{p-1}^{(p-1)k}x_p^{(p-1)h})=\begin{cases}0 & \textit{if } h=0 \textit{ or } k+h<p\\ x_0^{(p-1)(k+h-p)} & \textit{else}\end{cases}$$

Proof. If $h=0$ or $k+h<p$, the claim follows by dimension reasons. Otherwise we may by Corollary 8.6 reduce to the case $h\le p$. As in the proof of [6, Lemma 5.3], one sees then that

$$x_p^{(p-1)h}$$

has the same leading term as

$$x_p^{(p-1)}x_{p-1}^{(p-1)(h-1)}$$

We may therefore assume $h=1$.

Again by Corollary 8.6 reduce to the case $k \leq p$. And again as in the proof of [6, Lemma 5.3], one sees then that

$$x_p^{(p-1)} x_{p-1}^{(p-1)k}$$

has the same leading term as

$$x_p^{(p-1)} x_{p-1}^{(p-1)} x_{p-2}^{(p-1)(k-1)}$$

Iterating this, we end up with

$$x_p^{(p-1)} x_{p-1}^{(p-1)} \cdots x_1^{(p-1)} x_0^{(p-1)(k+1-p)}$$

This proves the claim. $\square$

8.3. **The final computation.** By Corollary 3.4 we want to compute the number

$$c = \deg\Big(\big(\hat{Q}\big((\gamma_1') \cdots (\gamma_{m-1}') \cup \alpha\big)\big)^{p-1}\Big) \quad \bmod p$$

Here $T \to B \subset R$ is the norm-1 torus defined by the form (R_1, J_1, γ_1) and $\alpha \in H^{1,1}(T, \mathbf{Z}/p)$ is its standard class.

The torus $T \to B$ is an open subscheme of the projective bundle $\mathbf{P}(A) \to R$. Let

$$w \in \mathrm{CH}^1\big(\mathbf{P}(A)\big)$$

be the first Chern class of the tautological line bundle. Then

$$\mathrm{CH}^*\big(\mathbf{P}(A)\big) = \mathrm{CH}^*(R)[w]/(w^p - ws_1^{p-1})$$

One finds

$$Q_0(\alpha) = w|T$$

in $\mathrm{CH}^1(T)$.

By Lemma 4.1 we have (mod p)

$$c = \Big(\sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \prod_{i=1}^{m} (\beta_{\sigma(i)})^{p^{i-1}}\Big)^{p-1}$$

in $\mathrm{CH}_0\big(\mathbf{P}(A)\big) = \mathbf{Z}$ with $\beta_1 = t_1, \ldots, \beta_{m-1} = t_{m-1}$ and $\beta_m = w$. From

$$w^p = ws_1^{p-1}$$

we get

$$c = w^{p-1}\Big(\sum_{\sigma \in S_m} \varphi_\sigma\Big)^{p-1}$$

where

$$\varphi_\sigma = \mathrm{sgn}(\sigma) t_{m-1}^{p^{\sigma(1)-1}} t_{m-2}^{p^{\sigma(2)-1}} \cdots t_1^{p^{\sigma(m-1)-1}} s_1^{p^{\sigma(m)-1}-1}$$

Hence

$$c = \Big(\sum_{\sigma \in S_m} \varphi_\sigma\Big)^{p-1}$$

in $\mathrm{CH}_0(R)=\mathbf{Z}$.

We have $s_1=x_p$, $t_1=x_{p-1}$ (with $i=1$ above). By Lemma 8.7 we have

$$(R_1\to R_2)_*(x_{p-1}^{(p-1)k}x_p^{(p-1)h})=\begin{cases}0 & \text{if } h=0 \text{ or } k+h<p\\ s_2^{(p-1)(k+h-p)} & \text{else}\end{cases}$$

Note that if we replace formally in the sum

$$\sum_{\sigma\in S_m}\varphi_\sigma$$

the variables s_1, t_1 with s_2, we get 0. Thus the same statement holds also for

$$\Big(\sum_{\sigma\in S_m}\varphi_\sigma\Big)^{p-1}$$

For

$$\omega=(\sigma_1,\dots,\sigma_{p-1})\in(S_m)^{p-1}$$

let

$$\Phi_\omega=\varphi_{\sigma_1}\cdots\varphi_{\sigma_{p-1}}$$

If $\sigma_1(m)=\cdots=\sigma_{p-1}(m)=1$, then (use (8.2.1) for $r=p$ and $i=0$)

$$(R_1\to R_2)_*(\Phi_\omega)=0$$

Otherwise

$$(R_1\to R_2)_*(\Phi_\omega)$$

is formally obtained from Φ_ω by replacing the variables s_1, t_1 with s_2 and then dividing by $s_2^{p(p-1)}$. Hence we have (up to sign)

$$c=\Big(\sum_{\sigma\in S_m,\,\sigma(m)=1}\mathrm{sgn}(\sigma)t_{m-1}^{p^{\sigma(1)-1}}\cdots t_2^{p^{\sigma(m-2)-1}}s_2^{p^{\sigma(m-1)-1}-1}\Big)^{p-1}$$

in $\mathrm{CH}_0(R_2)=\mathbf{Z}$. Or:

$$\begin{aligned}c&=\Big(\sum_{\sigma\in S_{m-1}}\mathrm{sgn}(\sigma)t_{m-1}^{p^{1+\sigma(1)-1}}\cdots t_2^{p^{1+\sigma(m-2)-1}}s_2^{p^{1+\sigma(m-1)-1}-1}\Big)^{p-1}\\&=\Big(\sum_{\sigma\in S_{m-1}}\mathrm{sgn}(\sigma)(t_{m-1}^p)^{p^{\sigma(1)-1}}\cdots(t_2^p)^{p^{\sigma(m-2)-1}}(s_2^p)^{p^{\sigma(m-1)-1}-1}\Big)^{p-1}\end{aligned}$$

We have $s_2^p=x_p$, $t_2^p=x_{p-1}$ (with $i=2$ above). By Lemma 8.7 we have

$$(R_2\to R_3)_*(x_{p-1}^{(p-1)k}x_p^{(p-1)h})=\begin{cases}0 & \text{if } h=0 \text{ or } k+h<p\\ (s_3^p)^{(p-1)(k+h-p)} & \text{else}\end{cases}$$

Proceeding further this way, we end up with

$$c = \Big(\sum_{\sigma \in S_1} \operatorname{sgn}(\sigma)\Big)^{p-1} = 1$$

8.4. **Further computations.** The remarks of this section have been found some weeks after typing the Subsection 8.3. In the end they will simplify the presentation of the computation of $c(X)$ considerably.

We keep the notations from Subsection 8.3.

Let

$$H = \hat{Q}\big((\gamma_1') \cdots (\gamma_{m-1}') \cup \alpha\big) \in \mathrm{CH}^b(T)$$

Let

$$\overline{s}_i = s_i^{p^{i-1}} \in \mathrm{CH}^{p^{i-1}}(R_i) \subset \mathrm{CH}^{p^{i-1}}(R)$$
$$\overline{t}_i = t_i^{p^{i-1}} \in \mathrm{CH}^{p^{i-1}}(R_i) \subset \mathrm{CH}^{p^{i-1}}(R)$$

There is the following remarkable decomposition of H:

Lemma 8.8. *One has (up to sign)*

$$H = w \prod_{i=1}^{m-1} \big(\overline{t}_i^p - \overline{t}_i \overline{s}_i^{p-1}\big)$$

Proof. Once this statement is formulated, it can be easily proved by the arguments used before in Subsection 8.3. In fact, the proof is much less tedious. We omit the details for now. □

Corollary 8.9. $c(X) = 1 \bmod p$.

Proof. This is now easy: Note that

$$\overline{t}_i^p - \overline{t}_i \overline{s}_i^{p-1} = f(\overline{t}_i, \overline{s}_i)$$

where f is as in the proof of (8.2.2). The proof of (8.2.2) yields

$$\big(\overline{t}_i^p - \overline{t}_i \overline{s}_i^{p-1}\big)^{p-1} + (\overline{s}_i)^{(p-1)p} = h(\overline{t}_i, \overline{s}_i) = \overline{s}_{i+1}^{p-1}$$

From this one gets

$$(R_i \to R_{i+1})_*\big(\big(\overline{t}_i^p - \overline{t}_i \overline{s}_i^{p-1}\big)^{p-1}\big) = \pm 1$$

Plugging this into the formula of Lemma 8.8 yields the claim. □

Lemma 8.8 was found by a mere computation. I don't have yet a geometric explanation of it.

Here is a related remark, which also deserves some explanation:

Lemma 8.10. *One has*

$$(S_{r+1} \to S_r)_*(x_{r+1}^{2p-1}) = f(x_{r-1}, x_r)$$

9. Relations with characteristic numbers

The material of this section is not needed for the proof of the Bloch-Kato conjecture.

Lemma 9.2 of this section was inspired by [3, Lemma 4.4].

Let

$$S_\bullet = S_0 + S_1 + \cdots \colon \mathrm{CH}_*(X) \otimes \mathbf{Z}/p \to \mathrm{CH}_*(X) \otimes \mathbf{Z}/p$$
$$S_h \colon \mathrm{CH}_j(X) \otimes \mathbf{Z}/p \to \mathrm{CH}_{j-h(p-1)}(X) \otimes \mathbf{Z}/p$$

and

$$P^\bullet = P^0 + P^1 + \cdots \colon \mathrm{CH}^*(X) \otimes \mathbf{Z}/p \to \mathrm{CH}^*(X) \otimes \mathbf{Z}/p$$
$$P^h \colon \mathrm{CH}^j(X) \otimes \mathbf{Z}/p \to \mathrm{CH}^{j+h(p-1)}(X) \otimes \mathbf{Z}/p$$

be the Steenrod operations of Brosnan [1]. The two variants are related by

$$S_\bullet(\alpha) = P^\bullet(\alpha) \cdot b_\bullet(TX)$$

where $b_\bullet(-)$ is the characteristic class with

$$b_\bullet(V) = \sum_{i\geq 0} b_{i(p-1)}(V)$$
$$b_\bullet(V \oplus W) = b_\bullet(V) b_\bullet(W)$$
$$b_\bullet(L) = \big(1 + c_1(L)^{p-1}\big)^{-1}$$

for vector bundles V, W and line bundle L. (For $p = 2$ one gets the Segre classes $b_\bullet(V) = c_\bullet(-V)$.)

Let

$$b = \frac{p^n - 1}{p - 1}$$

Let X be a smooth proper irreducible variety of dimension

$$d = b(p-1) = p^n - 1$$

Theorem 9.1. *Suppose $p = 2$ or $n = 1$.*

Suppose $I(X) \subset p\mathbf{Z}$. Suppose further that there exists a special correspondence $\rho \in \mathrm{CH}^b(X^2)$. Then

$$\frac{b_d(X)}{p} \neq 0 \mod p$$

The same statement holds without the restriction $p = 2$ or $n = 1$, see below.

Proof. For Y/k proper with

$$I(Y) = \deg_{Y/k}\big(\mathrm{CH}_0(Y)\big) \subset p\mathbf{Z}$$

let

$$\mathrm{DEG}_Y\colon\ \mathrm{CH}_0(Y)/p \to p\mathbf{Z}/p^2\mathbf{Z}$$
$$\mathrm{DEG}_Y(u \bmod p) = \deg_{Y/k}(u) \bmod p^2$$

for $u \in \mathrm{CH}_0(Y)$.

We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_d(X^2) \arrow[r, "S_d"] \arrow[d, "(\pi_0)_*"] & \mathrm{CH}_0(X^2)/p \arrow[r, "\mathrm{DEG}_{X^2}"] \arrow[d, "(\pi_0)_*"] & p\mathbf{Z}/p^2\mathbf{Z} \arrow[d, equal] \\
\mathrm{CH}_d(X) \arrow[r, "S_d"] & \mathrm{CH}_0(X)/p \arrow[r, "\mathrm{DEG}_X"] & p\mathbf{Z}/p^2\mathbf{Z}
\end{tikzcd}$$

One has

$$S_d([X]) = b_d(TX)$$

and

$$(\pi_0)_*\big((\rho)^{p-1}\big) = c(\rho)[X]$$

One has

$$S_\bullet(\rho^{p-1}) = P^\bullet(\rho^{p-1})b_\bullet(TX^2) = P^\bullet(\rho)^{p-1}b_\bullet(TX^2)$$

For $n = 1$ we have $b = 1$. In this case one has

$$P^\bullet(\rho) = \rho + \rho^p$$

Hence

$$S_\bullet(\rho^{p-1}) = \big(\rho^{p-1} + (p-1)\rho^{2(p-1)}\big)\big(1 + b_d(TX_0) + b_d(TX_1)\big)$$

Therefore

$$\begin{aligned} c(\rho)b(X) = {} & \big(1 + (-1)^{p-1}\big)c(\rho)b(X) \\ & + (-1)^{p-1}(p-1)\binom{2(p-1)}{p-1}c(\rho)^2 \mod pI(X) \end{aligned}$$

This proves the claim.

For $p = 2$ we have $b = d$. It suffices to show that if we represent $S_d(\rho)$ by an integral zero cycle, then the degree of this zero cycle is not divisible by 4.

We have

$$S_d(\rho) = \rho b_d(TX^2) + P^1(\rho)b_{d-1}(TX^2) + \cdots + P^d(\rho) \tag{9.0.1}$$

For the last term we have $P^d(\rho) = \rho^2 \bmod 2$ and one has

$$\deg(\rho^2) = -2c(\rho)^2$$

It suffices to show that all the other terms on the right hand side of (9.0.1) have integral representatives with degrees divisible by 4.

Let $X_0 = X_1 = X$ and write $X^2 = X_0 \times X_1$. Let F be the function field of X.

For the first term on the right hand side of (9.0.1) we have over F by dimension reasons

$$\begin{aligned}\rho_F b_d(TX^2) &= (H \times X - X \times H) \sum_{j+r=d} b_j(TX_0) b_r(TX_1) \\ &= \big(H \times b_d(TX_1)\big) - \big(b_d(TX_0) \times H\big)\end{aligned}$$

This zero cycle has obviously degree 0.

As for the remaining terms: Let $1 \leq i \leq d-1$. Note that for $H \in \mathrm{CH}^d(X_F) \otimes \mathbf{Z}/2$ we have $P^i(H) = 0$ by dimension reasons. Since $\rho_F = H \times X - X \times H$ we get

$$P^i(\rho)_F = 0 \tag{9.0.2}$$

in $\mathrm{CH}^{d+i}(X_F^2) \otimes \mathbf{Z}/2$.

Let $\beta \in \mathrm{CH}^{d+i}(X^2)$ be an integral representative of $P^i(\rho)$. We have to show that $\deg\big(\beta b_{d-i}(TX^2)\big) \in \mathbf{Z}$ is 4-divisible. By (9.0.2) there exists $\gamma \in \mathrm{CH}^{d+i}(X_F^2)$ such that $\beta_F = 2\gamma$. It remains to show that $\deg\big(\gamma b_{d-i}(TX^2)\big) \in \mathbf{Z}$ is 2-divisible. Now

$$b_{d-i}(TX^2) = \sum_{j+r=d-i} b_j(TX_0) b_r(TX_1)$$

It remains to show that all the numbers $\deg\big(\gamma b_j(TX_0) b_r(TX_1)\big) \in \mathbf{Z}$ are 2-divisible. We have $j + r = d - i > 0$. We assume $r > 0$ (the case $j > 0$ is similar). Then

$$\deg\big(\gamma b_j(TX_0) b_r(TX_1)\big) = \deg_{X_{1F}/F}\big((\pi_0)_*\big(\gamma b_j(TX_0)\big) b_r(TX_1)\big)$$

This number is 2-divisible because of the following Lemma 9.2. □

Lemma 9.2. *Let $p = 2$. Suppose that there exists a special correspondence $\rho \in \mathrm{CH}^b(X^2)$. Let $r > 0$. Let $\alpha \in \mathrm{CH}^r(X)$ and $\beta \in \mathrm{CH}^{d-r}(X_F)$. Then*

$$\deg_{X_F/F}(\alpha_F \beta) \in I(X) \otimes \mathbf{Z}_{(2)}$$

Proof. Let $\varphi \in \mathrm{CH}^{d-r}(X^2)$ be a preimage of β under the natural map

$$\mathrm{CH}^{d-r}(X^2) \to \mathrm{CH}^{d-r}(X \times \operatorname{Spec} F)$$

and consider the zero cycle

$$\omega = \rho(\alpha \times X)\varphi \in \mathrm{CH}^{2d}(X^2)$$

Its degree is in $I(X^2) = I(X)$. We compute its degree over F. One has

$$\omega_F = (H \times X)(\alpha_F \times X)\varphi_F - (X \times H)(\alpha_F \times X)\varphi_F$$

The first term vanishes because $H\alpha_F = 0$ by dimension reasons. As for the second term note that

$$(\pi_0)_*\big((\alpha_F \times X)\varphi_F\big) = \deg_{X_F/F}(\alpha_F\beta)[X]$$

Hence

$$\deg(\omega) = -c(\rho)\deg_{X_F/F}(\alpha_F\beta)$$

□

Here is the variant of Lemma 9.2 for any p.

Let

$$J(X) = \{\, n \in \mathbf{Z} \mid n^{p-1} \in I(X) \,\}$$

If $I(X) \subset p\mathbf{Z}$, then $J(X) \subset p\mathbf{Z}$.

Lemma 9.3. *Let $r > 0$. Suppose that there exists a special correspondence $\rho \in \mathrm{CH}^b(X^2)$. Let $\alpha \in \mathrm{CH}^r(X)$ and $\beta \in \mathrm{CH}^{d-r}(X_F)$. Then*

$$\deg_{X_F/F}(\alpha_F\beta) \in J(X) \otimes \mathbf{Z}_{(p)}$$

Proof. Let $\varphi \in \mathrm{CH}^{d-r}(X^2)$ be a preimage of β under the natural map

$$\mathrm{CH}^{d-r}(X^2) \to \mathrm{CH}^{d-r}(X \times \operatorname{Spec} F)$$

Then

$$(\pi_0)_*(\alpha_{[0]}\varphi_{[0,1]}) = \deg_{X_F/F}(\alpha_F\beta)[X_1]$$

Since

$$(\pi_1)_*([X_1]\rho_{[1,2]}^{p-1}) = c(\rho)[X_2]$$

we get

$$\begin{aligned} c(\rho)\deg(\alpha_F\beta)[X_2] &= (\pi_{0,1})_*(\alpha_{[0]}\varphi_{[0,1]}\rho_{[1,2]}^{p-1}) \\ &= (\pi_{0,1})_*\big(\alpha_{[0]}\varphi_{[0,1]}(\rho_{[0,2]} - \rho_{[0,1]})^{p-1}\big) \\ &= \sum_{i=0}^{p-1}\binom{p-1}{i}(-1)^i(\pi_0)_*\big(\rho_{[0,2]}^{p-1-i}(\pi_1)_*(\alpha_{[0]}\varphi_{[0,1]}\rho_{[0,1]}^i)\big) \\ &= \sum_{i=1}^{p-1}\binom{p-1}{i}(-1)^i d_\rho\big((\pi_1)_*(\alpha_{[0]}\varphi_{[0,1]}\rho_{[0,1]}^i)\big)[X_2] \end{aligned}$$

Here the term for $i = 0$ vanishes. Namely, since $r > 0$ one has $(\pi_1)_*(\varphi_{[0,1]}) = 0$ by dimension reasons and therefore $(\pi_1)_*(\alpha_{[0]}\varphi_{[0,1]}) = \alpha_{[0]}(\pi_1)_*(\varphi_{[0,1]}) = 0$. The claim follows from Corollary 5.7. □

In the following we are going to prove Theorem 9.1 without the restriction $p = 2$ or $n = 1$. We assume that we are given a special correspondence $\rho \in \mathrm{CH}^b(X^2)$. We also assume $I(X) \subset p\mathbf{Z}$. Moreover, as before, we use the notations $\rho_L = H \times X - X \times H$ with $L = F = k(X)$.

Definition 9.4. For $i \geq 0$ define

$$e_i(\rho) \in \mathbf{Z}/p$$

as follows. Define $h_i \in \mathbf{Q}$ by

$$i(p-1) + b(h_i+1) = b(p-1)$$

If h_i is a nonnegative integer, we set

$$e_i(\rho) = \deg\bigl(P^i(H)H^{h_i}\bigr) \in \mathbf{Z}/p$$

Otherwise we set $e_i(\rho) = 0$.

For $i = 0$ we get $h_0 = p - 2$ and

$$e_0(\rho) = c(\rho)$$

Note that if $i > 0$, then $h_i < p - 2$.

Lemma 9.5. *For $i > 0$ one has*

$$e_i(\rho) = 0$$

Proof. Let

$$\alpha = (\pi_0)_*\bigl(P^i(\rho)\rho^{h_i+1}\bigr) \in \mathrm{CH}^b(X) \otimes \mathbf{Z}/p$$

Then

$$\alpha_L = (\pi_0)_*\Bigl(\bigl(P^i(H_0) - P^i(H_1)\bigr)(H_0 - H_1)^{h_i+1}\Bigr)$$

Since $h_i + 1 < p - 1$ one has

$$\alpha_L = u e_i(\rho) H$$

for some unit $u \in \mathbf{Z}_{(p)}^{\times}$. The claim follows from Corollary 5.7. $\square$

We consider the elements

$$(\theta_i)_{[0,1]} = c(\rho)^{-2}(\pi_{2,3})_*\bigl(P^i(\rho_{[2,3]})\rho_{[0,2]}^{p-1}\rho_{[1,3]}^{p-1}\bigr) \in \mathrm{CH}^{b+i(p-1)}(X^2) \otimes \mathbf{Z}/p$$

and

$$\omega_i = c(\rho)^{-1}(\pi_1)_*\bigl(P^i(\rho)\rho^{p-1}\bigr) \in \mathrm{CH}^{b+i(p-1)}(X) \otimes \mathbf{Z}/p$$

Lemma 9.6.

$$P^i(\rho) = \theta_i - (\pi_0^* - \pi_1^*)(\omega_i)$$

Proof.

$$\begin{aligned}(\theta_i)_{[0,1]} &= c(\rho)^{-2}(\pi_{2,3})_*\Bigl(\bigl(P^i(\rho_{[0,1]}) - P^i(\rho_{[0,2]}) + P^i(\rho_{[1,3]})\bigr)\rho_{[0,2]}^{p-1}\rho_{[1,3]}^{p-1}\Bigr)\\ &= P^i(\rho_{[0,1]}) - (\omega_i)_{[0]} + (\omega_i)_{[1]}\end{aligned}$$

$\square$

Lemma 9.7. *For $i > 0$ one has*

$$(\theta_i)_L = 0$$

in $\mathrm{CH}^{b+i(p-1)}(X_L^2) \otimes \mathbf{Z}/p$.

Proof.

$$(\theta_i)_L = c(\rho)^{-2}(\pi_{2,3})_*\big(\big(P^i(H_2) - P^i(H_3)\big)(H_0 - H_2)^{p-1}(H_1 - H_3)^{p-1}\big)$$

The claim follows from Lemma 9.5. □

Lemma 9.8. *Let $j, k > 0$ with $j + k + rb = 2d$ for some $r \le p - 1$. Let $\alpha \in \mathrm{CH}^j(X)$, $\beta \in \mathrm{CH}^k(X)$. Then*

$$\deg\big(\pi_0^*(\alpha)\pi_1^*(\beta)\rho^r\big) \in J(X)^2 \otimes \mathbf{Z}_{(p)}$$

Proof. Computing this degree over L, one gets essentially $d_\rho(\alpha)d_\rho(\beta)$. The claim follows from Corollary 5.7. □

Theorem 9.9. *Suppose $I(X) \subset p\mathbf{Z}$. Suppose further that there exists a special correspondence $\rho \in \mathrm{CH}^b(X^2)$. Then*

$$\frac{b_d(X)}{p} \ne 0 \mod p$$

Proof. Let A be the $2d$-dimensional component (so A is a zero cycle) of

$$\begin{aligned} S_\bullet(\rho^{p-1}) &= P^\bullet(\rho)^{p-1} b_\bullet(TX^2) \\ &= \big(\rho + P^1(\rho) + \cdots + P^{b-1}(\rho) + \rho^p\big)^{p-1} b_\bullet(TX_0) b_\bullet(TX_1) \end{aligned}$$

The zero cycle A is a sum of terms

$$U = P^{i_1}(\rho)\cdots P^{i_{p-1}}(\rho) b_j(TX_0) b_k(TX_1)$$

of total degree $2d$. We consider various cases.

Case 1. $i_1 = \cdots = i_{p-1} = 0$. In this case

$$U = \rho^{p-1}\big(b_d(TX_0) + b_d(TX_1)\big)$$

The degree of this integral zero cycle is $(1 + (-1)^{p-1})c(\rho)b_d(X)$.

Case 2. After a permutation of the i_j one has $i_1 = \cdots = i_{p-2} = 0$, $i_{p-1} = b$. (So this case appears exactly $p - 1$ times). In this case

$$U = \rho^{2(p-1)}$$

The degree of this integral zero cycle is $upc(\rho)^2$ for some p-unit u.

It suffices to show that for all other cases the zero cycle U has an integral representative with degree divisible by p^2.

After plugging the formula of Lemma 9.6 into U we get zero cycles of the form

$$V = \theta_{i_1}\cdots\theta_{i_r}\pi_0^*(\alpha)\pi_1^*(\beta)$$

with $\alpha \in \mathrm{CH}^j(X)$, $\beta \in \mathrm{CH}^k(X)$. The α, β are products of the $b_j(TX)$ and of integral representatives of the ω_i.

We choose integral representatives $\tilde{\theta}_{i_j}$ of θ_{i_j} and put

$$\tilde{V} = \tilde{\theta}_{i_1} \cdots \tilde{\theta}_{i_r} \pi_0^*(\alpha)\pi_1^*(\beta)$$

which is an integral zero cycle. We have to show that $\deg(\tilde{V})$ is p^2-divisible in all the cases not coming from Case 1 or Case 2.

Case 3. $r \geq 2$ and $i_1, i_2 > 0$. In this case one computes the degree over L. The p^2-divisibility follows from Lemma 9.7.

Case 4. $j = k = 0$. This case appears only when $r = p - 1$ (because all the ω_i are positive dimensional). But since we have excluded Case 1 and Case 2, we will be in Case 3.

Case 5. $i_1 = \cdots = i_r = 0$. Then $r < p - 1$, because otherwise we would come from Case 1.

Suppose $j = 0$. Then

$$(\pi_0)_*(V) = (\pi_0)_*(\rho^r)\beta = 0$$

since $(\pi_0)_*(\rho^r) = 0$.

Similarly for $k = 0$.

Suppose $j, k > 0$. Then

$$V = \rho^r \pi_0^*(\alpha)\pi_1^*(\beta)$$

The p^2-divisibility follows Lemma 9.8.

There remains the following case:

Case 6. $k > 0$ or $j > 0$, $r \geq 1$, $i_1 \geq 1$. We assume $k > 0$. Let

$$W = (\pi_0)_*\big(\tilde{\theta}_{i_1} \cdots \tilde{\theta}_{i_r} \pi_0^*(\alpha)\big)$$

Then

$$W_L = 0 \mod p$$

by Lemma 9.7.

The p^2-divisibility follows Lemma 9.3.

□

Here is a very general fact about smooth proper varieties X of dimension $d = b(p - 1)$.

Lemma 9.10. *Suppose that $d \neq p^m - 1$ for all m. Then*

$$b_d(X) \in pI(X) + p^2\mathbf{Z}_{(p)}$$

Proof. First recall that

$$b_d(X) \in p\mathbf{Z}$$

for all $d > 0$ and X. By the multiplicativity property of the series b_i, this implies

$$b_d(Y \times Z) \in p^2\mathbf{Z}$$

if $\dim Y$, $\dim Z > 0$.

Let Ω^* be the complex cobordism ring. Suppose that $d \neq p^m - 1$ for all m. Let $x_0 \in \Omega^d$ with $s_d(x_0) \neq 0 \bmod p$. Consider the characteristic class

$$f(V) = b_d(V) - b_d(x_0)s_d(x_0)^{-1}s_d(V)$$

over $\mathbf{Z}_{(p)}$. Then $f(x_0) = 0$. Moreover $f(yz) \in p^2\mathbf{Z}_{(p)}$ for positive dimensional y, $z \in \Omega^*$. By the known structure of Ω^*, this implies

$$f(x) \in p^2\mathbf{Z}_{(p)}$$

for $x \in \Omega^d$. By the Hattori-Stong theorem and by Riemann-Roch it follows that

$$f(X) \in p^2\mathbf{Z}_{(p)}$$

for any X over any field. The claim follows from

$$b_d(X) = f(X) + \frac{b_d(x_0)}{p}s_d(x_0)^{-1}ps_d(X)$$

□

Example 9.11. Let $p = 2$ and $d = 2$. Then

$$b_2 = c_1^2 - c_2 = (c_1^2 + c_2) - 2c_2$$

and therefore

$$b_2(X) = 4\,\mathrm{Todd}(X) - 2c_2(X)$$

Corollary 9.12. *Suppose $I(X) \subset p\mathbf{Z}$. Suppose further that there exists a special correspondence $\rho \in \mathrm{CH}^b(X^2)$. Then*

$$d = p^m - 1$$

for some m.

For the case of quadrics see [2, Theorem 6.1], [3, Theorem 5.1].

Proof. This is clear from Theorem 9.9 and Lemma 9.10. □

Here is the variant of Lemma 9.10 for $d = p^m - 1$.

Lemma 9.13. *Suppose that $d = p^m - 1$ for some m. Then*

$$b_d(X) = \pm s_d(X) \mod \big(pI(X) + p^2\mathbf{Z}_{(p)}\big)$$

Proof. Let $x_0 \in \Omega^d$ with $s_d(x_0) = p$. Consider the characteristic class

$$f(V) = b_d(V) - \frac{b_d(x_0)}{p} s_d(V)$$

As in the proof of Lemma 9.10 one gets

$$f(X) \in p^2\mathbf{Z}$$

We have

$$b_d(X) = f(X) + \frac{b_d(x_0)}{p} s_d(X)$$

The number

$$\frac{b_d(x_0)}{p} \mod p$$

is the coefficient of P^b when expressing q_m in the P^i (by some well known formula). This coefficient is equal to ± 1 . □

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