CONSTRUCTION OF SPLITTING VARIETIES

MARKUS ROST

Contents

Date: April, 1998.

1. Preliminaries, Conventions, and Notations

- The ground field k has characteristic 0. We fix a prime p. We assume $\mu_p \subset k$.
- By a scheme or a variety X (over k) we mean a separated scheme of finite type $\pi_X\colon X \to \operatorname{Spec} k$.
- If X is a smooth variety, then TX denotes the tangent bundle of X.
- Let V be vector bundle over X. We denote by $\pi_V\colon \mathbb{P}(V) \to X$ the projective bundle associated to V. Moreover

  $$\mathbb{L}(V) \to \pi_V^* V$$

  denotes the tautological line bundle on $\mathbb{P}(V)$.

  For the fiber tangent bundle $T\big(\mathbb{P}(V)/X\big)$ one has

  $$T\big(\mathbb{P}(V)/X\big) = \pi_V^* V \otimes \mathbb{L}(V)^\vee / \mathcal{O}_{\mathbb{P}(V)}.$$
- Let V be vector (or an affine) bundle over X. We denote by $\mathbb{A}(V) \to X$ the associated scheme V.
- By a *form* we understand a triple $(T/S, L, \alpha)$ where $T \to S$ are schemes, L is line bundle on T and $\alpha \in H^0(T, L^{\otimes -p})$ is a form of degree p on L.

  There is a natural homomorphism $\mu_p \to \operatorname{Aut}(T/S, L, \alpha)$ induced from the standard action of $\mathbb{G}_{\mathrm{m}}$ on L.
- Let $(\operatorname{Spec} k, L, \alpha)$ be a nonzero form and let $u \in L$ be a basis vector. Then the p-power class

  $$\{\alpha\} = \{\alpha(u)\} \in K_1 k/p = k^*/(k^*)^p$$

  is independent on the choice of u.
- Let $(T/S, L, \alpha)$ and let Γ be a finite group acting on $(T/S, L, \alpha)$ (i.e., there is given a homomorphism $\Gamma \to \operatorname{Aut}(T/S, L, \alpha)$). We say that $(T/S, L, \alpha)$ is an *admissable* Γ*-form* if the following conditions hold:
  - α is nonzero on an open dense subscheme of T.
  - Γ has only finitely many fixed points on T (a fixed point is a point $P \in T$ with $gP = P$ for all $g \in G$).
  - At each fixed point P the form α is nonzero.
- For vector bundles V, V' on schemes X/S resp. X'/S we denote by $V \boxplus_S V'$ the *exterior direct sum*, given by the sum of the pull backs to $X \times_S X'$. Similarly we denote by $V \boxtimes_S V'$ the *exterior tensor product*, given by the tensor product of the pull backs.
- For forms $(T/S, L, \alpha)$ and $(T'/S, L', \alpha')$ we denote by

  $$(T/S, L, \alpha) \boxtimes_S (T'/S, L', \alpha') = \big((T \times_S T')/S, L \boxtimes_S L', \alpha \boxtimes_S \alpha'\big)$$

  their *exterior product*, with the form defined by

  $$(\alpha \boxtimes_S \alpha')(u \boxtimes_S u') = \alpha(u)\alpha'(u')$$

  for sections u, u' of L, L', respectively.

  If $(T/S, L, \alpha)$ and $(T'/S, L', \alpha')$ are admissable Γ-forms, then $(T/S, L, \alpha) \boxtimes_S (T'/S, L', \alpha')$ is an admissable Γ-form.
- Let (S, H_i, α_i), $i = 1, \ldots, n$, be admissable Γ-forms and let $P \in S$ be a k-rational fixed point. We say that P is *twisting* for the family $(S, H_i, \alpha_i)_i$, if the homomorphism

  $$\Gamma \to \mu_p^n = \prod_{i=1}^n \operatorname{Aut}(H_i|P, \alpha_i|P)$$

is surjective.

- By a *cellular* variety we mean a variety which admits a stratification by affine spaces. The motive of a cellular variety is the direct sum of powers of the Tate motive L, with a summand $L^{\otimes i}$ for each i-cell. If X and Y are cellular, then $X \times Y$ is cellular and one has
$$\mathrm{CH}_*(X \times Y) = \mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y).$$
- Let L be a line bundle L on a smooth and proper variety X over k of dimension $d \geq 0$. We write
$$\delta(L) = \deg\bigl(c_1(L)^d\bigr) \in \mathbb{Z}.$$
Here
$$\deg\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} k) = \mathbb{Z}$$
is the degree map. If $d = 0$ we understand by $\delta(L)$ the degree of X as a finite extension of k.

  If V is a vector space of dimension n, then
$$\delta\bigl(\mathbb{L}(V)\bigr) = \deg\Bigl(c_1\bigl(\mathbb{L}(V)\bigr)^{n-1}\Bigr) = (-1)^{n-1}.$$
- The *index* I_X of a proper variety is
$$I_X = \deg\bigl(\mathrm{CH}_0(X)\bigr) \subset \mathbb{Z}$$
- If p is a prime, a field k is called *p-special* if $\operatorname{char} k \neq p$ and if k has no finite field extensions of degree prime to p.
- Let (S, L, α) be a form. We consider the bundle of algebras
$$A = A(S, L, \alpha) = TL/I$$
over R. Here TL is the tensor algebra of L and I is the ideal subsheaf generated by
$$\lambda^{\otimes p} - \alpha(\lambda)$$
for local sections λ of L. A a is bundle of commutative algebras of degree p. Note that
$$A = \bigoplus_{i=0}^{p-1} L^{\otimes i}$$
as vector bundles. We denote by
$$N_A\colon A \to \mathcal{O}_S$$
the norm of the algebra A.
- We use the notation
$$\mathrm{Cyclic}^p(Z) = (Z^p)/(\mathbb{Z}/p).$$

2. Consequences of Voevodsky's work

In this paper p always is a prime, k is a field with $\operatorname{char} k = 0$ and $K_n^{\mathrm{M}} k$ denotes Milnor's n-th K-group of k. Let

$$h_{(n,p)}\colon K_n^{\mathrm{M}} k/p \to H^n_{\text{ét}}(k, \mu_p^{\otimes n}),$$
$$\{a_1, \dots, a_n\} \mapsto (a_1, \dots, a_n).$$

be the norm residue homomorphism.

2.1. Voevodsky's theorem. V. Voevodsky announced in October 1996 the following theorem:

Theorem (Voevodsky). *Let p be a prime and let m be a natural number.*

Suppose that for every subfield $k \subset \mathbb{C}$ containing the p-th roots of unity and for every sequence of elements $a_1, \dots, a_n \in k^*$, $2 \le n \le m$, there exists a smooth projective variety X over k such that:

(V_1) $\{a_1, \dots, a_n\}_{k(X)} = 0$ *in* $K_n^{\mathrm{M}} k(X)/p$.

(V_2) *X has dimension $d = p^{n-1} - 1$.*

(V_3) *In the category of Chow motifs over $k(X)$ with $\mathbb{Z}_{(p)}$-coefficients there exist an effective object Y such that*

$$X_{k(X)} = L^{\otimes 0} \oplus (Y \otimes L).$$

Here L denotes the Tate motive.

(V_4) *The characteristic number $s_d\big(X(\mathbb{C})\big) \in \mathbb{Z}$ is not divisible by p^2.*

(V_5) *The sequence*

$$\coprod_{x \in X_{(1)}} K_2 \kappa(x) \xrightarrow{d} \coprod_{x \in X_{(0)}} K_1 \kappa(x) \xrightarrow{N_X} K_1 k$$

is exact. Here $N_X = \sum \mathrm{N}_{\kappa(x)|k}$.

Then one has:

(BK) *The Bloch-Kato conjecture holds in weight m and* $\operatorname{mod} p$, *i.e., the norm residue homomorphism $h_{(m,p)}$ is bijective. (for all fields k with $\operatorname{char} k \neq p$)*

(S) *For $n \le m$, for elements $a_1, \dots, a_n \in k^*$, and for a smooth projective variety X satisfying* (V_1)–(V_5), *the sequence*

$$\coprod_{x \in X_{(0)}} K_1 \kappa(x) \xrightarrow{N} K_1 k \xrightarrow{b \mapsto (a_1, \dots, a_n, b)} H^{n+1}_{\text{ét}}(k, \mu_p^{\otimes(n+1)})$$

is exact.

2.2. A degree formula for $s_d(X)$. We fix a prime p be a prime and a number d of the form $d = p^n - 1$.

Let X, Y be irreducible smooth proper varieties over k with $\dim Y \le \dim X = d$ and let $f\colon X \to Y$ be a morphism. Define $\deg f$ as follows: If $\dim f(X) < \dim X$, then $\deg f = 0$. Otherwise $\deg f \in \mathbb{N}$ is the degree of the extension of the function fields:

$$f_*([X]) = \deg f \cdot [Y].$$

Theorem 1 ("Degree formula")**.**

$$\big(s_d(X)/p\big) = (\deg f)\big(s_d(Y)/p\big) \mod I_Y$$

This is a consequence of algebraic cobordism theory. One uses the spectrum Φ considered in [4].

Corollary 2. *The class*

$$s_d(X)/p \bmod I_X \in \mathbb{Z}/I_X$$

is a birational invariant

2.3. On a higher degree formula. All of what I am saying in the next lines are mainly guesses from my poor knowledge of Morava K-theories and algebraic cobordism. Everything has to be checked.

Let Φ_r be the Φ-construction of [4], iterated r-times, i.e., Φ_r is a tower consisting of $\Sigma^{2id,id}H_{\mathbb{Z}/p}$, $i = 0, \ldots, r$ with all intermediate towers of length 2 being a suspension of Φ. Then the Thom class lifts to $\mathrm{MU} \to \Phi_r$ and for X of dimension rd we have a fundamental class

$$[X] \in \pi_{2rd,rd}(X \wedge \Phi_r).$$

Define $t(X) \in \mathbb{Z}/p$ as the image of $[X]$ in

$$\pi_{2rd,rd}(\operatorname{Spec} k \wedge \Phi_r) = \pi_{2rd,rd}(\operatorname{Spec} k \wedge \Sigma^{2rd,rd}H_{\mathbb{Z}/p}) = \mathbb{Z}/p$$

From the known structure of Morava K-theories it follows that (perhaps up to multiplication with a number prime to p)

$$t(X_1 \times X_2 \times \cdots \times X_r) = \big(s_d(X_1)/p\big)\big(s_d\big(X_2\big)/p\big) \cdots \big(s_d(X_r)/p\big) \mod p$$

if $\dim X_i = d$.

Furthermore, let Ψ be the fibre of $\Phi_r \to H_{\mathbb{Z}/p}$ and define $J(X) \subset \mathbb{Z}$ as the image of

$$\pi_{2rd,rd}(X \wedge \Psi) \to \pi_{2rd,rd}(\operatorname{Spec} k \wedge \Psi) = \pi_{2rd,rd}(\operatorname{Spec} k \wedge \Sigma^{2rd,rd}H_{\mathbb{Z}/p}) = \mathbb{Z}/p$$

Note that $\Psi = \Sigma^{2d,d}\Phi_{r-1}$.

Then the "higher degree formula" is

$$t(X) = t(Y)(\deg f) \mod J(Y) \tag{1}$$

for any $f\colon X \to Y$ with X, Y, smooth proper of dimension rd. It should be possible to show this in the same way as for the degree formula for s_d/p.

Moreover, one should have

$$J(X) = J(Y), \qquad \text{if } \deg f \text{ is prime to } p \tag{2}$$

by a transfer argument.

I guess that the following is true:

Let X_i, $i = 1, \ldots, r$ be of dimension d and suppose that $I_{(X_{i\,F_i})} \subset p\mathbb{Z}$ for all i, where

$$F_i = k(X_1 \times \cdots \times \widehat{X}_i \times \cdots \times X_r).$$

Let $X = X_1 \times \cdots \times X_r$. Then

$$J(X) \subset p\mathbb{Z} \tag{3}$$

In the case of curves ($d = 1 = 2^1 - 1$) one has (?)

$$J(X) = (\pi_X)_*(K_0(X)^{(1)})$$

where

$$K_0(X)^{(1)} = \ker\big(K_0(X) \to \mathrm{CH}^0(X)\big)$$

and $\pi_X : X \to \operatorname{Spec} k$ is the structure map for X.

In this case (3) is not difficult to show:

Proof for $d = 1$ and $r = 2$: One has an exact sequence

$$\coprod_{x \in X_{1(0)}} K_0(X_{2\kappa(x)}) \to K_0(X_1 \times X_2)^{(1)} \to K_0(X_{2k(X_1)})^{(1)} \to 0$$

Push forward along $\pi' : X_1 \times X_2 \to X_1$ maps this sequence into the sequence

$$\mathrm{CH}_0(X_1) \to K_0(X_1) \to K_0(k(X_1)) \to 0$$

Since the index of $X_{2k(X_1)}$ is 2-divisible, we see that

$$\pi'_*(K_0(X_1 \times X_2)^{(1)}) \subset \mathrm{CH}_0(X_1) + 2K_0(k(X_1))$$

The claim (3) follows since I_{X_1} is 2-divisible. □

It should be possible to extend this reasoning to the general case (?).

In my application one has $r = p$ and the X_i are of the following type: Let $a_m \in k_0^*$ be such that $\{a_1, \dots, a_n\}$ is a nontrivial symbol, let $k = k_0(t_1, \dots, t_p)$ and let X_i/k be a norm variety for the symbol $\{a_1, \dots, a_n, t_i\}$. We may take X_i to be defined over $k_0(t_i)$. Note that then each of the fields $k_0(t_i)(X_i)$ has a k_0-place, hence the field F_i has a $k_0(t_i)$-place, whence $\{a_1, \dots, a_n, t_i\}$ is nontrivial over F_i. Therefore the index of X_{iF_i} is p.

3. The Conner-Floyd theorem: computing $s_d(X)$

In this section we indicate how one can get information about $s_d(X)$ from a $(\mathbb{Z}/p)^{n-1}$-action on X with isolated fixed points.

We assume that p is odd and $k \subset \mathbb{C}$. For odd p the Chern number $s_d(X) = s_d(TX)$ of a complex variety is also an Pontrjagin number of the underlying differentiable manifold $M = X(\mathbb{C})$. Therefore the number $s_d(X)$ can be computed in terms of the class $[M]$ of M in the oriented cobordism ring.

In order to compute this number for certain norm varieties we use the following theorem of Conner and Floyd: ([2]).

Theorem 3. *Let $d = p^n-1$, let $G = (\mathbb{Z}/p)^n$, and let M be an oriented differentiable manifold. Suppose that there exist a fixed point free G-action on M. Then the class of M in the oriented cobordism ring Ω_* lies in the ideal $I_{n-1,p}$ generated by Milnor the base elements $M_{0,p} = p \cdot \mathrm{Point}$, $M_{1,p}$, $\ldots$, $M_{n-1,p}$ $(\dim M_{i,p} = p^i - 1)$.*

Corollary 4. *Let $d = p^n - 1$, let $G = (\mathbb{Z}/p)^n$, and let M be an oriented differentiable manifold of (real) dimension $2d$. Suppose that there exist an fixed point free G-action on M. Then $s_d(M)$ is divisible by p^2.*

Proof. This follows from $s_d(M) \in p\mathbb{Z}$ for all M of dimension $2d$ and $s_d(M_1 \times M_2) = 0$ if $d > \dim X_i > 0$. □

Using canonical desingularization [1] and the Conner-Floyd theorem one finds:

Corollary 5. *Let p be odd, let X, Y be proper varieties with Y smooth. Suppose that $G = (\mathbb{Z}/p)^n$ acts on X and Y such that the fixed point schemes $\mathcal{F}_X$ and $\mathcal{F}_Y$ are of dimension 0 and suppose that $\mathcal{F}_X \subset X_{\mathrm{reg}}$. Suppose further that the families of G-representations $(T_P X)_{P \in \mathcal{F}_X(\mathbb{C})}$, $(T_P Y)_{P \in \mathcal{F}_Y(\mathbb{C})}$ are isomorphic. Then there exist a smooth proper variety $\widetilde{X}$ together with a birational isomorphism $\widetilde{X} \to X$ such that $\widetilde{X}(\mathbb{C})$ and $Y(\mathbb{C})$ represent the same element in $\Omega_*/I_{n-1,p}$.*

In particular, if $\dim X = \dim Y = d = p^n - 1$, then

$$s_d(\widetilde{X}) = s_d(Y) \bmod p^2$$

This consequence is extremely useful to compute the birational invariant of Corollary 2.

Proof. By canonical desingularization [1] we may assume that X is smooth. Let Z be the multifold connected sum of the differentiable manifolds $X(\mathbb{C})$ and $-Y(\mathbb{C})$, build by glueing together pairs of fixed points with isomorphic G-normal structures. Since S^{2d} and $S^1 \times S^{2d-1}$ are bordant, one has $[Z] = [X] - [Y]$ for the cobordism classes. On Z we have a fixed point free G-action, and the Conner-Floyd theorem shows $[Z] \in I_{n-1,p}$. □

If the families of G-representations $(T_P X)_{P \in \mathcal{F}_X(\mathbb{C})}$, $(T_P Y)_{P \in \mathcal{F}_Y(\mathbb{C})}$ are isomorphic, we say that X and Y are *G-fixed point equivalent.*

4. The forms $\mathcal{A}(\alpha_1,\dots,\alpha_n)$ ("algebras")

Given a scheme S and forms (S,H_i,α_i), $i=1,\dots,m$, we define forms

$$\mathcal{A}(\alpha_1,\dots,\alpha_n)=(P_n/S,K_n,\Phi_n),\qquad 0\le n\le m.$$

For $n=0$ we put

$$\begin{aligned}P_0&=S,\\ K_0&=\mathcal{O}_S,\\ \Phi_0(t)&=t^p.\end{aligned}$$

Suppose $(P_{n-1}/S,K_{n-1},\Phi_{n-1})$ is defined. We consider the 2-dimensional vector bundle

$$V_n=\mathcal{O}_{P_{n-1}}\oplus H_n\boxtimes_S K_{n-1}$$

on P_{n-1}, and the form

$$\varphi_n\colon V_n\to\mathcal{O}_{P_{n-1}}$$

on V_n defined by

$$\varphi_n(t-u\otimes v)=t^p-\alpha_n(u)\Phi_{n-1}(v)$$

for sections t, u, v of $\mathcal{O}_{P_{n-1}}$, H_n, K_{n-1}, respectively.

Let $(P_{n-1,j},V_{n,j},\varphi_{n,j})$, $j=1,\dots,p-1$ be copies of (P_{n-1},V_n,φ_n). We put

$$(P_n/S,K_n,\Phi_n)=(P_{n-1}/S,K_{n-1},\Phi_{n-1})\boxtimes_S\mathop{\boxtimes_S}_{j=1}^{p-1}\big(\mathbb{P}(V_{n,j}),\mathbb{L}(V_{n,j}),\varphi_{n,j}\big).$$

We assume now that $S=\operatorname{Spec}k$ and list the most important properties of the forms (P_n,K_n,Φ_n).

Lemma 6. *The variety P_n is smooth, proper, cellular, connected, and of dimension p^n-1.*

Proof. Indeed, P_n is an iterated projective bundle. The computation of the dimension is clear for $n=0$ and for $n>0$ we find

$$\begin{aligned}\dim P_n&=\dim P_{n-1}+(p-1)(1+\dim P_{n-1})\\ &=(p^{n-1}-1)+(p-1)p^{n-1}=p^n-1\end{aligned}$$

by induction on n. □

Lemma 7. $\delta(K_n)=(-1)^n \bmod p$.

Proof. This is clear for $n=0$. Let

$$\begin{aligned}u_n&=c_1(K_n)&&\in\mathrm{CH}^1(P_n),&&n\ge0,\\ u_{n-1,j}&=c_1(K_{n-1,j})&&\in\mathrm{CH}^1(P_{n-1,j}),&&n\ge1,\ j=1,\dots,p-1,\\ z_{n,j}&=c_1\big(\mathbb{L}(V_{n,j})\big)&&\in\mathrm{CH}^1\big(\mathbb{P}(V_{n,j})\big),&&n\ge1,\ j=1,\dots,p-1.\end{aligned}$$

For $n\ge1$ let

$$\widehat{P}_n=P_{n-1}\times\prod_{j=1}^{p-1}P_{n-1,j}$$

Then

$$\mathrm{CH}^*(\widehat{P}_n) = \mathrm{CH}^*(P_{n-1}) \otimes \bigotimes_{j=1}^{p-1} \mathrm{CH}^*(P_{n-1,j})$$

and

$$\mathrm{CH}^*(P_n) = \frac{\mathrm{CH}^*(\widehat{P}_n)[z_{n,j};\ j = 1, \dots, p-1]}{\langle z_{n,j}^2 - z_{n,j} u_{n-1,j};\ j = 1, \dots, p-1 \rangle}.$$

Moreover

$$u_n = u_{n-1} + \overline{z}_n, \quad \text{with} \quad \overline{z}_n = \sum_{j=1}^{p-1} z_{n,j}.$$

Note that

$$u_{n-1}^{p^{n-1}} = u_{n-1,j}^{p^{n-1}} = 0, \quad z_{n,j}^{p^{n-1}+1} = 0$$

by dimension reasons. Hence, calculating mod p,

$$u_n^{p^{n-1}} = (u_{n-1} + \overline{z}_n)^{p^{n-1}} = u_{n-1}^{p^{n-1}} + \overline{z}_n^{p^{n-1}} = \overline{z}_n^{p^{n-1}}.$$

One finds (using Lemma 8 below)

$$\begin{aligned} u_n^{p^n - 1} &= u_n^{p^{n-1}-1} u_n^{p^{n-1}(p-1)} = u_n^{p^{n-1}-1} \overline{z}_n^{p^{n-1}(p-1)} \\ &= u_n^{p^{n-1}-1} \big(z_{n,1}^{p^{n-1}} + z_{n,2}^{p^{n-1}} + \dots + z_{n,p-1}^{p^{n-1}} \big)^{p-1} \\ &= -u_{n-1}^{p^{n-1}-1} z_{n,1}^{p^{n-1}} z_{n,2}^{p^{n-1}} \cdots z_{n,p-1}^{p^{n-1}} \\ &= -u_{n-1}^{p^{n-1}-1} z_{n,1} u_{n,1}^{p^{n-1}-1} z_{n,2} u_{n,2}^{p^{n-1}-1} \cdots z_{n,p-1} u_{n,p-1}^{p^{n-1}-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \delta(K_n) &= -\delta(K_{n-1}) \big(-\delta(K_{n-1,1}) \big) \big(-\delta(K_{n-1,2}) \big) \cdots \big(-\delta(K_{n-1,p-1}) \big) \\ &= -\delta(K_{n-1}) \bmod p. \end{aligned}$$

whence the claim. □

Lemma 8. *Let R be a ring over $\mathbb{F}_p$ and let $v_1, v_2, \dots, v_{p-1} \in R$, be elements with $v_1^2 = v_2^2 = \dots = v_{p-1}^2 = 0$. Then*

$$(v_1 + v_2 + \dots + v_{p-1})^{p-1} = -v_1 v_2 \cdots v_{p-1}.$$

Proof. Note that $(p-1)! = -1 \bmod p$. □

The construction of (P_n, K_n, Φ_n) is functorial in the forms (S, H_i, α_i). In particular the group

$$\Gamma_n = \mu_p^n \subset \prod_{i=1}^{n} \mathrm{Aut}(S, H_i, \alpha_i)$$

acts on (P_n, K_n, Φ_n).

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \dots, n$.

Lemma 9. *(P_n, K_n, Φ_n) is an admissable Γ_n-form. All fixed points are k-rational.*

Proof. By induction on n. Suppose that $(P_{n-1}, K_{n-1}, \Phi_{n-1})$ is an admissable Γ_{n-1}-form. It suffices to show that $(\mathbb{P}(V_n), \mathbb{L}(V_n), \varphi_n)$ is an admissable Γ_n-form. It is easy to see that φ_n is generically nonzero. Every Γ_n-fixed point on $\mathbb{P}(V_n)$ lies over a Γ_{n-1}-fixed point $P \in P_{n-1}$. It suffices to show that the fibre $(\operatorname{Spec}\kappa(P), \mathbb{L}(V_n)|P, \varphi_n|P)$ is an admissable Γ-form where

$$\Gamma = \operatorname{Aut}(S, H_n, \alpha_n) = \ker(\Gamma_n \to \Gamma_{n-1}).$$

This is easy to see: If $(\operatorname{Spec} k, H, \alpha)$ is a nonzero form over k, then

$$\mu_p = \operatorname{Aut}(\operatorname{Spec} k, H, \alpha)$$

has in $\mathbb{P}(k \oplus H)$ only the two fixed points $\mathbb{P}(0 \oplus H)$ and $\mathbb{P}(k \oplus 0)$. The form $\varphi(t - u) = t^p - \alpha(u)$ is nonzero on the lines $t = 0$ and $u = 0$. □

Lemma 10. *Let $\eta_n \in P_n$ be the generic point. Then*

$$\{\alpha_1, \ldots, \alpha_n, \Phi_n(\eta_n)\} = 0 \in K^{\mathrm{M}}_{n+1} k(P_n)/p.$$

Proof. By induction on n. Suppose that

$$\{\alpha_1, \ldots, \alpha_{n-1}, \Phi_{n-1}(\eta_{n-1})\} = 0 \in K^{\mathrm{M}}_n k(P_{n-1})/p.$$

One has

$$\Phi_n(\eta_n) = \Phi_{n-1}(\eta_{n-1}) \cdot \prod_{j=1}^{p-1} \big(1 - \alpha_n \Phi_{n-1,j}(\eta_{n-1,j})\big).$$

Hence it suffices to show

$$\{\alpha_1, \ldots, \alpha_n, 1 - \alpha_n \Phi_{n-1,j}(\eta_{n-1,j})\} \in K^{\mathrm{M}}_{n+1} k(P_n)/p$$

for each $j = 1, \ldots, p-1$. This follows from $\{a, 1 - ab\} = -\{b, 1 - ab\}$. □

Remark 1. Given the forms $(\operatorname{Spec} k, H_i, \alpha_i)$, form the vector space

$$A_n = \bigoplus_{j_1,\ldots,j_n=0}^{p-1} H_1^{\otimes j_1} \otimes \cdots \otimes H_n^{\otimes j_n}.$$

One has $\dim A_n = p^n$. On A_n there is the form

$$\Theta_n = \bigoplus_{j_1,\ldots,j_n=0}^{p-1} (-\alpha_1)^{\otimes j_1} \otimes \cdots \otimes (-\alpha_n)^{\otimes j_n}$$

Consider the form $(\mathbb{P}(A_n), \mathbb{L}(A_n), \Theta_n)$. If $p = 2$, this form satisfies all the properties of (P_n, K_n, Φ_n) listed above (up to a sign in the computation of $\delta\big(\mathbb{L}(A_n)\big)$). If $p > 2$, all properties of (P_n, K_n, Φ_n) are also valid, except for the splitting of the symbol. If $n = 1$, $n = 2$, or $n = p = 3$, one may define on A_n an algebra structure with norm form Θ'_n in such a way that $(\mathbb{P}(A_n), \mathbb{L}(A_n), \Theta'_n)$ satisfies all the properties. The (P_n, K_n, Φ_n) form an approximation to these algebras, with the advantage, that (P_n, K_n, Φ_n) can be constructed for all p and n.

5. The forms $\mathcal{B}(\alpha_1, \ldots, \alpha_n)$ ("relative algebras")

Let $n \geq 1$. Given forms (S, H_i, α_i), $i = 1, \ldots, n-1$, and $(S'/S, L, \beta)$, we define a form

$$\mathcal{B}(\alpha_1, \ldots, \alpha_{n-1}, \beta) = (P'_n/S', K'_n, \Phi'_n)$$

as follows. Let $(P_{n-1}/S, K_{n-1}, \Phi_{n-1})$ be as in section 4. Put

$$\overline{P}_{n-1} = S' \times_S P_{n-1}$$

We consider the 2-dimensional vector bundle

$$\overline{V}_n = \mathcal{O}_{\overline{P}_{n-1}} \oplus L \boxtimes_S K_{n-1}$$

on $\overline{P}_{n-1}$, and the form

$$\overline{\varphi}_n \colon \overline{V}_n \to \mathcal{O}_{\overline{P}_{n-1}}$$

on $\overline{V}_n$ defined by

$$\overline{\varphi}_n(t - u \otimes v) = t^p - \beta(u)\Phi_{n-1}(v)$$

for sections t, u, v of $\mathcal{O}_{\overline{P}_{n-1}}$, L, K_{n-1}, respectively.

Let

$$(\overline{P}_{n-1,j}, \overline{V}_{n,j}, \overline{\varphi}_{n,j}, K_{n-1,j}, P_{n-1,j}),\ j = 1, \ldots, p-1$$

be copies of $(\overline{P}_{n-1}, \overline{V}_n, \overline{\varphi}_n, K_{n-1}, P_{n-1})$. We put

$$(P'_n/S', K'_n, \Phi'_n) = \boxtimes_{S'}{}_{j=1}^{p-1} \big(\mathbb{P}(\overline{V}_{n,j}), \mathbb{L}(\overline{V}_{n,j}), \overline{\varphi}_{n,j}\big).$$

We assume now that $S = \operatorname{Spec} k$ and list the most important properties of the forms (P'_n, K'_n, Φ'_n).

Lemma 11. *The variety P'_n is smooth and proper over S', and of relative dimension $p^n - p^{n-1}$. If S' is cellular, so is P'_n. The fibres of S/S' are connected.*

Proof. Note that P'_n/S' is an iterated projective bundle. Moreover

$$\dim P'_n/S' = (p-1)(\dim P_{n-1} + 1) = p^n - p^{n-1}$$

by Lemma 6. □

Let

$$\begin{aligned} u'_n &= c_1(K'_n) &&\in \mathrm{CH}^1(P'_n), \\ u_{n-1,j} &= c_1(K_{n-1,j}) &&\in \mathrm{CH}^1(P_{n-1,j}), \\ v_n &= c_1(L) &&\in \mathrm{CH}^1(S'). \end{aligned}$$

Lemma 12. *One has*

$$u_n'^{\,p^n} = u_n'^{\,p^{n-1}} v_n^{p^n - p^{n-1}} \bmod p.$$

If $S' = \operatorname{Spec} k$, then

$$\delta(K'_n) = \deg(u_n'^{\,p^n - p^{n-1}}) = -1 \bmod p.$$

Proof. Let

$$\widehat{\overline{P_n}} = S' \times \prod_{j=1}^{p-1} P_{n-1,j}$$

Then

$$\mathrm{CH}^*(\widehat{\overline{P_n}}) = \mathrm{CH}^*(S') \otimes \bigotimes_{j=1}^{p-1} \mathrm{CH}^*(P_{n-1,j})$$

and

$$\mathrm{CH}^*(P'_n) = \frac{\mathrm{CH}^*(\widehat{\overline{P_n}})[z_{n,j};\ j=1,\dots,p-1]}{\langle z_{n,j}^2 - z_{n,j}(v_n + u_{n-1,j});\ j=1,\dots,p-1\rangle}.$$

Moreover

$$u'_n = \overline{z}_n, \quad \text{with} \quad \overline{z}_n = \sum_{j=1}^{p-1} z_{n,j}.$$

Recall that $u_{n-1,j}^{p^{n-1}} = 0$. Calculating mod p, one finds

$$\begin{aligned}
{u'_n}^{p^n} &= \overline{z}_n^{p^n} \\
&= z_{n,1}^{p^n} + \dots + z_{n,p-1}^{p^n} \\
&= z_{n,1}^{p^{n-1}}(v_n + u_{n-1,1})^{p^{n-1}(p-1)} + \dots + z_{n,p-1}^{p^{n-1}}(v_n + u_{n-1,p-1})^{p^{n-1}(p-1)} \\
&= z_{n,1}^{p^{n-1}}(v_n^{p^{n-1}} + u_{n-1,1}^{p^{n-1}})^{(p-1)} + \dots + z_{n,p-1}^{p^{n-1}}(v_n^{p^{n-1}} + u_{n-1,p-1}^{p^{n-1}})^{(p-1)} \\
&= z_{n,1}^{p^{n-1}} v_n^{p^{n-1}(p-1)} + \dots + z_{n,p-1}^{p^{n-1}} v_n^{p^{n-1}(p-1)} \\
&= \overline{z}_n^{p^{n-1}} v_n^{p^{n-1}(p-1)} = {u'_n}^{p^{n-1}} v_n^{p^{n-1}(p-1)}.
\end{aligned}$$

This proves the first claim.

Suppose $v_n = 0$. Then $z_{n,j}^{p^{n-1}+1} = 0$. One finds mod p (using Lemma 8)

$$\begin{aligned}
{u'_n}^{p^{n-1}(p-1)} &= \big(z_{n,1}^{p^{n-1}} + z_{n,2}^{p^{n-1}} + \dots + z_{n,p-1}^{p^{n-1}}\big)^{p-1} \\
&= -z_{n,1}^{p^{n-1}} z_{n,2}^{p^{n-1}} \cdots z_{n,p-1}^{p^{n-1}} \\
&= -z_{n,1} u_{n-1,1}^{p^{n-1}-1} z_{n,2} u_{n-1,2}^{p^{n-1}-1} \cdots z_{n,p-1} u_{n-1,p-1}^{p^{n-1}-1}
\end{aligned}$$

Since $\delta(K_{n-1}) \neq 0 \bmod p$, it follows that

$$\begin{aligned}
\delta(K'_n) &= -\big(-\delta(K_{n-1,1})\big)\big(-\delta(K_{n-1,2})\big) \cdots \big(-\delta(K_{n-1,p-1})\big) \\
&= -1 \bmod p,
\end{aligned}$$

whence the second claim. □

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \dots, n-1$. Let Γ be a finite group, let $\Gamma \to \Gamma_{n-1}$ be an epimorphism and let $\Gamma \to \mathrm{Aut}(S', L, \beta)$ be a homomorphism. Thus Γ acts on all the forms $(\mathrm{Spec}\, k, H_i, \alpha_i)$, $i = 0, \dots, n-1$, and (S', L, β).

Lemma 13. *Suppose that (S', L, β) is an admissable Γ-form with all fixed points k-rational. Moreover suppose that each fixed point is twisting for the forms*

$$(S, H_i, \alpha_i),\ i = 1, \dots, n-1, \text{ and } (S', L, \beta).$$

Then (P'_n, K'_n, Φ'_n) is an admissable Γ-form with all fixed points k-rational.

Proof. This follows as for Lemma 9. ☐

Lemma 14. *Suppose that S' is irreducible. Let $\eta_n \in P_n$ be the generic point. Then*

$$\{\alpha_1, \dots, \alpha_{n-1}, \beta(\eta_n), \Phi_n(\eta_n)\} = 0 \in K^{\mathrm{M}}_{n+1} k(P_n)/p.$$

Proof. This follows as for Lemma 10. ☐

Remark 2. Given the form (S', L, β) one may define the "Kummer algebra"

$$A = A(S', L, \beta) = L^{\otimes 0} \oplus L^{\otimes 1} \oplus \cdots \oplus L^{\otimes p-1}$$

with the product given by the natural multiplication in the tensor algebra using the form $\beta\colon L^{\otimes p} \to L^{\otimes 0}$ to reduce the degree mod p. One finds

$$\mathrm{CH}^*\big(\mathbb{P}(A)\big) \otimes \mathbb{F}_p = \mathrm{CH}^*(S') \otimes \mathbb{F}_p[x]/\langle x^p - x^{p-1}y\rangle$$

with $x = c_1\big(\mathbb{L}(A)\big)$ and $y = c_1(L)$.

Hence we have a homomorphism

$$R = \mathbb{F}_p[x]/\langle x^p - x^{p-1}y\rangle \to \mathrm{CH}^*\big(\mathbb{P}(A)\big) \otimes \mathbb{F}_p$$

Lemma 12 shows that there is a homomorphism

$$R \to \mathrm{CH}^*(P'_n) \otimes \mathbb{F}_p, \qquad x \mapsto {u'_n}^{p^{n-1}},\ y \mapsto v_n^{p^{n-1}}.$$

If one thinks in terms of the (in general nonexisting) algebras

$$A_n = A(\alpha_1, \dots, \alpha_{n-1}, \beta)$$

with "subalgebras"

$$A_{n-1} = A(\alpha_1, \dots, \alpha_{n-1}),$$

and one imagines to form something like the projective space $\mathbb{P}_{A_{n-1}}(A_n)$, then one may think of P'_n as an approximation $P'_n \to \mathbb{P}_{A_{n-1}}(A_n)$ with the homomorphism $R \to \mathrm{CH}^*(P'_n) \otimes \mathbb{F}_p$ being the pull back on the Chow rings (if say $S' = \mathbb{P}^\infty$ and with L the universal line bundle).

6. The forms $\mathcal{C}(\alpha_1, \dots, \alpha_n)$ (Chain lemma construction)

Let $n \geq 2$. Given forms (S, H_i, α_i), $i = 1, \dots, n-1$, and $(S'/S, L, \beta)$, we define forms

$$\mathcal{C}_r = \mathcal{C}_r(\alpha_1, \dots, \alpha_{n-1}, \beta) = (S_r/S_{r-1}, L_r, \beta_r), \qquad r \geq -1.$$

For $r = -1$, 0 we put

$$(S_{-1}/S_{-2}, L_{-1}, \beta_{-1}) = (S/S, H_{n-1}, \alpha_{n-1}),$$
$$(S_0/S_{-1}, L_0, \beta_0) = (S'/S, L, \beta).$$

Let $r > 0$ and suppose $\mathcal{C}_{r-2}$ and $\mathcal{C}_{r-1}$ are defined.

Let

$$(P'_{n-1,r}/S_{r-1}, K'_{n-1,r}, \Phi'_{n-1,r}) = \mathcal{B}(\alpha_1, \dots, \alpha_{n-1}, \beta_{r-1})$$

be the form constructed in section 5, starting from (S, H_i, α_i), $i = 1, \dots, n-2$, and $(S_{r-1}/S_{r-2}, L_{r-1}, \beta_{r-1})$. Put

$$(S_r/S_{r-1}, L_r, \beta_r) = (S_{r-2}/S_{r-3}, L_{r-2}, \beta_{r-2}) \boxtimes_{S_{r-2}} (P'_{n-1,r}/S_{r-1}, K'_{n-1,r}, \Phi'_{n-1,r}).$$

We assume now that $S = \operatorname{Spec} k$ and list the most important properties of the forms $(S_r/S_{r-1}, L_r, \beta_r)$.

Lemma 15. *The variety S_r is smooth and proper over S', and of relative dimension $r(p^{n-1} - p^{n-2})$. If S' is cellular, so is S_r. The fibres of S/S' are connected.*

Proof. This follows from Lemma 11. For the dimension note

$$\dim S_r/S_{r-1} = \dim P'_{n-1,r}/S_{r-1} = p^{n-1} - p^{n-2}$$

by Lemma 11. □

Thus if $\dim S' = (p^l - 1)p^n$ for some $\ell \geq 0$, then $\dim S_p = (p^{l+1} - 1)p^{n-1}$.

Theorem 16. *Let $\ell \geq 0$ and suppose that S' is smooth and proper of dimension $(p^l - 1)p^n$. Then*

$$\delta(L_p) = \delta(L) \bmod p.$$

The proof requires some calculations.

Let a, $b \in \mathbb{F}_p$, and let $r \geq 0$ be an integer. In the ring $\mathbb{F}_p[z_1, \dots, z_r]$ let

$$x_{-1} = a,$$
$$x_0 = b,$$
$$x_m = z_m + x_{m-2}, \quad 1 \leq m \leq r.$$

Then

$$x_{2k} = z_{2k} + z_{2k-2} + \dots + z_4 + z_2 + b,$$
$$x_{2k+1} = z_{2k-1} + z_{2k-3} + \dots + z_3 + z_1 + a.$$

We denote by I the ideal generated by

$$z_m^p - z_m x_{m-1}^{p-1}, \quad 1 \leq m \leq r$$

and put

$$R_r(a, b) = \mathbb{F}_p[z_1, \dots, z_r]/I.$$

The elements

$$z^J = z_1^{i_1} \cdots z_r^{i_r}, \quad J = (i_1, \dots, i_r), \quad 0 \leq i_j \leq p-1$$

form an $\mathbb{F}_p$-basis of $R_r(a,b)$. For $u \in R_r(a,b)$ let $c_m(u)$ be the coefficient of $z_1^{p-1} \cdots z_m^{p-1}$.

Lemma 17. *If $1 \le r \le p$ one has $c_r(x_r^{r(p-1)}) = 1$ in $R_r(a,b)$.*

Proof. One has for $1 \le m \le p$:

$$\begin{aligned} x_m^{m(p-1)} &= x_m^{p(m-1)+(p-m)} \\ &= (z_m + x_{m-2})^{p(m-1)+(p-m)} \\ &= (z_m^p + x_{m-2}^p)^{(m-1)}(z_m + x_{m-2})^{(p-m)} \\ &= (z_m x_{m-1}^{p-1} + x_{m-2}^p)^{(m-1)}(z_m + x_{m-2})^{(p-m)}. \end{aligned}$$

Hence for $m \le p$ one has

$$c_m(x_m^{m(p-1)}) = c_{m-1}(x_{m-1}^{(m-1)(p-1)}).$$

The claim follows by induction. □

Proposition 18. *If $(a,b) \neq (0,0)$, then $R_r(a,b)$ is isomorphic to a product of rings of the form*

$$\mathbb{F}_p[v_1,\dots,v_k]/(v_1^p,\dots,v_k^p), \quad k \ge 0.$$

Proof. By induction on $r \ge 0$. The case $r = 0$ is obvious.

Suppose $b \neq 0$. Then the polynomial

$$z_1^p - z_1 x_0^{p-1}$$

is separable with roots $z_1 = ib$, $i \in \mathbb{F}_p$. It follows that we have isomorphism

$$R_r(a,b) \xrightarrow{\sim} \prod_{i \in \mathbb{F}_p} R_r(a,b)/(z_1 - ib).$$

The ring $R_r(a,b)/(z_1 - ib)$ is the quotient of $\mathbb{F}_p[z_2,\dots,z_r]$ by the ideal generated by

$$z_m^p - z_m x_{m-1}^{p-1}, \quad 2 \le m \le r$$

with

$$\begin{aligned} x_0 &= b, \\ x_1 &= ib + a, \\ x_m &= z_m + x_{m-2}, \quad 2 \le m \le r. \end{aligned}$$

Hence $R_r(a,b)/(z_1 - ib) \simeq R_{r-1}(b, ib + a)$. The claim follows from the induction hypothesis.

Suppose $b = 0$. Then $a \neq 0$. In this case we consider the homomorphism

$$\begin{aligned} \varphi\colon \mathbb{F}_p[z_1,\dots,z_r] &\to \mathbb{F}_p[z_1]/(z_1^p) \otimes R_{r-1}(0,1), \\ z_m &\mapsto (a + z_1) \otimes z_{m-1}, \quad 2 \le m \le r, \\ z_1 &\mapsto z_1 \otimes 1. \end{aligned}$$

We claim that $\varphi(I) = 0$. For this it suffices to show

$$\varphi(z_m^p - z_m x_{m-1}^{p-1}) = 0, \qquad 1 \le m \le r.$$

This is obvious for $m=1$. If $m=2$, then

$$\begin{aligned}\varphi(z_2^p - z_2x_1^{p-1}) &= \varphi\big(z_2^p - z_2(z_1+a)^{p-1}\big)\\ &= (a+z_1)^p\otimes z_1^p - \big((a+z_1)\otimes z_1\big)\big(z_1\otimes 1 + 1\otimes a\big)^{p-1}\\ &= (a+z_1)^p\otimes z_1^p - \big((a+z_1)\otimes z_1\big)\big((z_1+a)\otimes 1\big)^{p-1}\\ &= (a+z_1)^p\otimes (z_1^p - z_1) = 0.\end{aligned}$$

If $m=2k\geq 2$, then

$$\begin{aligned}\varphi(z_{2k}^p - z_{2k}x_{2k-1}^{p-1}) &= \varphi\big(z_{2k}^p - z_{2k}(z_{2k-1}+\cdots+z_3+z_1+a)^{p-1}\big)\\ &= (a+z_1)^p\otimes z_{2k-1}^p -\\ -\big((a+z_1)\otimes z_{2k-1}\big)&\big((a+z_1)\otimes z_{2k-2}+\cdots+(a+z_1)\otimes z_2 + z_1\otimes 1 + 1\otimes a\big)^{p-1}\\ &= (a+z_1)^p\otimes z_{2k-1}^p -\\ -\big((a+z_1)\otimes z_{2k-1}\big)&\big((a+z_1)\otimes z_{2k-2}+\cdots+(a+z_1)\otimes z_2 + (a+z_1)\otimes 1\big)^{p-1}\\ &= (a+z_1)^p\otimes \big(z_{2k-1}^p - z_{2k-1}(z_{2k-2}+\cdots+z_2+1)\big)^{p-1}\\ &= (a+z_1)^p\otimes (z_{2k-1}^p - z_{2k-1}x_{2k-2}^{p-1}) = 0.\end{aligned}$$

If $m=2k-1\geq 3$, then

$$\begin{aligned}\varphi(z_{2k-1}^p - z_{2k-1}x_{2k-2}^{p-1}) &= \varphi\big(z_{2k-1}^p - z_{2k-1}(z_{2k-2}+\cdots+z_2)^{p-1}\big)\\ &= (a+z_1)^p\otimes\big(z_{2k-2}^p - z_{2k-2}(z_{2k-3}+\cdots+z_1)^{p-1}\big)\\ &= (a+z_1)^p\otimes (z_{2k-2}^p - z_{2k-2}x_{2k-3}^{p-1}) = 0.\end{aligned}$$

It follows that φ induces a homomorphism

$$\begin{aligned}&\overline{\varphi}\colon R_r(a,b)\to \mathbb{F}_p[z_1]/(z_1^p)\otimes R_{r-1}(0,1),\\ &z_m\mapsto (a+z_1)\otimes z_{m-1},\quad 2\leq m\leq r,\\ &z_1\mapsto z_1\otimes 1.\end{aligned}$$

$\overline{\varphi}$ is obviously surjective. By dimension reasons, $\overline{\varphi}$ must be an isomorphism. Again the claim follows from the induction hypothesis. □

Corollary 19. *$u^{p^2}=u^p$ for all $u\in R_p(0,1)$.* □

Corollary 20. *Let $n\geq 2$, and let $u_n = x_p^{p^n-p}\in R_p(0,1)$. Then $c_p(u_n)=1$.*

Proof. For $n=2$ this is Lemma 17. Moreover, by Corollary 19, the element u_n does not depend on n. □

We rewrite things in a homogenous form. Let x be a variable and let

$$R' = \mathbb{F}_p[x,z_1,\ldots,z_p]/I'$$

where I' is the homogenous ideal generated by

$$z_m^p - z_m x_{m-1}^{p-1},\quad 1\leq m\leq p$$

with

$$\begin{aligned}x_{-1}&=0,\\ x_0&=x,\\ x_m&=z_m+x_{m-2},\quad 1\leq m\leq p.\end{aligned}$$

Then $R'/(x-1)=R_p(0,1)$. Corollaries 19 and 20 yield the following two corollaries:

Corollary 21. *$u^{p^2} = u^p x^{p^2-p}$ for all $u \in R'$.* □

Corollary 22. *Let $n \geq 2$. Then*

$$x_p^{p^n-p} = z_1^{p-1} z_2^{p-1} \cdots z_p^{p-1} x^{p^n-p^2} \bmod x^{p^n-p^2+1} R'$$

Proof. Recall the basis elements $(z^J)_J$ of $R_p(0,1)$ considered above. The elements $(z^J x^{p^n-p-|J|})_J$ form a basis of the homogenous subspace of R' of degree $p^n - p$. It follows that

$$x_p^{p^n-p} = c_p(x_p^{p^n-p}) z_1^{p-1} z_2^{p-1} \cdots z_p^{p-1} x^{p^n-p^2} \bmod \langle z^J x^{p^n-p-|J|};\ |J| < p^2 - p \rangle.$$

But if $|J| < p^2 - p$ then $z^J x^{p^n-p-|J|} \in x^{p^n-p^2+1} R'$. □

Proof of Theorem 16: Let

$$x_r = c_1(L_r)^{p^{n-2}} \in \mathrm{CH}^{p^{n-2}}(S_r), \qquad r \geq -1,$$
$$z_r = c_1(K'_{n-1,r})^{p^{n-2}} \in \mathrm{CH}^{p^{n-2}}(P'_{n-1,r}), \qquad r \geq 1.$$

Then, calculating mod p,

$$x_{-1} = 0,$$
$$x_0 = c_1(L)^{p^{n-2}} \in \mathrm{CH}^{p^{n-2}}(S') \otimes \mathbb{F}_p,$$
$$x_r = x_{r-2} + z_r, \qquad r \geq 1,$$

since

$$c_1(L_r) = c_1(L_{r-2}) + c_1(K'_{n-1,r}).$$

Moreover

$$z_r^p = z_r x_{r-1}^{p-1}$$

by Lemma 12.

We have a homomorphism

$$R'(x) \to \mathrm{CH}^*(S_p) \otimes \mathbb{F}_p, \quad z_m \mapsto z_m, \quad x \mapsto x_0.$$

It follows from Corollary 22 that (mod p)

$$x_p^{p^{\ell+2}-p} = z_1^{p-1} z_2^{p-1} \cdots z_p^{p-1} x_0^{p^{\ell+2}-p^2} \bmod \langle x^{p^{\ell+2}-p^2+1} \rangle$$

Now if $\dim S' = (p^l - 1)p^n$, then $x_0^{p^{l+2}-p^2+1} = 0$. Hence

$$x_p^{p^{\ell+2}-p} = \delta(K'_{n-1,1}) \delta(K'_{n-1,2}) \cdots \delta(K'_{n-1,p-1}) \delta(L) = \delta(L) \bmod p,$$

where the last equation follows from Lemma 12. □

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \ldots, n-1$. Let Γ be a finite group, let $\Gamma \to \Gamma_{n-1}$ be an epimorphism and let $\Gamma \to \mathrm{Aut}(S', L, \beta)$ be a homomorphism. Thus Γ acts on all the forms $(\operatorname{Spec} k, H_i, \alpha_i)$, $i = 0, \ldots, n-1$, and (S', L, β).

Lemma 23. *Suppose that (S', L, β) is an admissable Γ-form, that all fixed points are k-rational and that each fixed point $P \in S'$ is twisting for the forms*

$$(S', H_i, \alpha_i),\ i = 1, \ldots, n-1,\ \text{and}\ (S', L, \beta).$$

Then for all $r \geq 0$, (S_r, L_r, β_r) is an admissable Γ-form, all fixed points are k-rational, and each fixed point $P \in S_r$ is twisting for the forms

$$(S_r, H_i, \alpha_i),\ i = 1, \ldots, n-2,\ (S_r, L_{r-1}, \beta_{r-1}),\ \text{and}\ (S_r, L_r, \beta_r).$$

Proof. Let $P \in S_r$ be a fixed point. By induction we may assume that P is k-rational and that

$$\Gamma \to \mathrm{Aut}(L_{r-2}|P, \beta_{r-2}|P) \times \mathrm{Aut}(L_{r-1}|P, \beta_{r-1}|P) \times \prod_{i=1}^{n-2} \mathrm{Aut}(H_i|P, \alpha_i|P)$$

is surjective. We claim that

$$\Gamma \to \mathrm{Aut}(L_r|P, \beta_r|P) \times \mathrm{Aut}(L_{r-1}|P, \beta_{r-1}|P) \times \prod_{i=1}^{n-2} \mathrm{Aut}(H_i|P, \alpha_i|P)$$

is surjective. Note that $L_r|P = L_{r-2}|P \otimes K_{n-1,r}|P$. The claim follows now from the fact that $\mathrm{Aut}(L_{r-2}|P, \alpha_{r-2}|P)$ acts trivially on $K_{n-1,r}|P$.

The remaining parts of the statement follow from Lemma 13. □

Lemma 24. *Suppose that S' is irreducible. Let $\eta_r \in S_r$ be the generic point. Then*

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_{r-1}(\eta_{r-1}), \beta_r(\eta_r)\} = (-1)^r \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta(\eta_0)\}$$

in $K_n^{\mathrm{M}} k(S_r)/p$.

Proof. We show

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_{r-1}(\eta_{r-1}), \beta_r(\eta_r)\} = \{\alpha_1, \dots, \alpha_{n-2}, \beta_{r-1}(\eta_{r-1}), \beta_r(\eta_{r-2})\}.$$

We have

$$\beta_r(\eta_r) = \beta_r(\eta_{r-2})\Phi'_{n-1,r}.$$

The claim follows now from Lemma 14. □

We will need the following special case:

Corollary 25.

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_p(\eta_p), \beta_{p-1}(\eta_{p-1})\} = \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta(\eta_0)\}$$

in $K_n^{\mathrm{M}} k(S_p)/p$. □

Remark 3. Let $S' = \operatorname{Spec} k$. We think of the symbol

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_p(\eta_p)\}$$

as a family of symbols of weight $n-1$ "between"

$$\{\alpha_1, \dots, \alpha_{n-2}\} \quad \text{and} \quad \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta\},$$

with S_p as parameter space.

Our later considerations indicate that this family is universal over p-special fields. For $n = 2$ we will make this precise, and for $p = 2$ this can be done using Pfister forms. I have no idea how to show this in general. In the case $n = p = 3$ the universality would have important consequences for the classification of groups of type F_4.

7. The forms $\mathcal{K}(\alpha_1, \ldots, \alpha_n)$ (universal families of Kummer splitting fields)

Let $n \geq 1$. Given forms (S, H_i, α_i), $i = 1, \ldots, n$, we define forms

$$\mathcal{K}_i = \mathcal{K}_i(\alpha_1, \ldots, \alpha_n) = (R_i/R_{i+1}, J_i, \gamma_i), \qquad 1 \leq i \leq n,$$
$$\mathcal{K}'_i = \mathcal{K}'_i(\alpha_1, \ldots, \alpha_n) = (R_i/R_{i+1}, J'_i, \gamma'_i), \qquad 1 \leq i \leq n.$$

We put

$$(R_n/R_{n+1}, J_n, \gamma_n) = (S/S, H_n, \alpha_n)$$

and

$$(R_n/R_{n+1}, J'_n, \gamma'_n) = (S/S, \mathcal{O}_S, \tau)$$

with $\tau(t) = t^p$.

Let $i < n$ and suppose that $\mathcal{K}_{i+1}$ is defined.

Recall the forms

$$\mathcal{C}_r = \mathcal{C}_r(\alpha_1, \ldots, \alpha_i, \gamma_{i+1}) = (S_r/S_{r-1}, L_r, \beta_r)$$

defined in section 6. Let $\pi\colon S_p \to S_{p-1}$ be the projection.

We put

$$\mathcal{K}_i = \mathcal{C}_p(\alpha_1, \ldots, \alpha_i, \gamma_{i+1}),$$
$$\mathcal{K}'_i = \pi^*\mathcal{C}_{p-1}(\alpha_1, \ldots, \alpha_i, \gamma_{i+1}).$$

We assume now that $S = \operatorname{Spec} k$ and list the most important properties of the forms $(R_i/R_{i+1}, J_i, \gamma_i)$ and $(R_i/R_{i+1}, J'_i, \gamma'_i)$.

Lemma 26. *The variety R_i is smooth, proper, cellular, and of dimension $p^n - p^i$.*

Proof. This follows from Lemma 15. For the dimension note

$$\dim R_i/R_{i+1} = p^{i+1} - p^i, \qquad i < n$$

by Lemma 15. □

Lemma 27. $\delta(J_i) = 1 \bmod p$.

Proof. By Theorem 16 we have

$$\delta(J_i) = \delta(J_{i+1}) \bmod p.$$

Hence $\delta(J_i) = \delta(J_n) = 1 \bmod p$. □

The construction of $(R_i/R_{i+1}, J_i, \gamma_i)$ is functorial in the forms (S, H_i, α_i). In particular the group

$$\Gamma_n = \mu_p^n \subset \prod_{i=1}^n \operatorname{Aut}(S, H_i, \alpha_i)$$

acts on $(R_i/R_{i+1}, J_i, \gamma_i)$.

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \ldots, n$.

Lemma 28. *The forms $(R_i/R_{i+1}, J_i, \gamma_i)$ are admissable Γ_n-forms, all fixed points are k-rational, and each fixed point $P \in R_i$ is twisting for the forms*

$$(R_i, H_m, \alpha_m),\ m = 1, \ldots, i-1,\ \text{and}\ (R_i, J_i, \gamma_i).$$

Proof. This follows form Lemma 23. □

Lemma 29. *Let $\eta_i \in R_i$ be the generic point. Then, for $1 \leq i < n$,*

$$\{\alpha_1, \dots, \alpha_{i-1}, \gamma_i(\eta_i), \gamma_i'(\eta_i)\} = \{\alpha_1, \dots, \alpha_i, \gamma_{i+1}(\eta_{i+1})\}$$

in $K^{\mathrm{M}}_{i+1} k(R_i)/p$.

Proof. This follows from Lemma 25. □

In particular we have

$$\{\alpha_1, \dots, \alpha_n\} = \{\alpha_1, \gamma_2, \gamma_3', \dots, \gamma_n'\}, \tag{4}$$

$$\{\alpha_1, \gamma_2\} = \{\gamma_1, \gamma_2'\}, \tag{5}$$

$$\{\alpha_1, \dots, \alpha_n\} = \{\gamma_1, \gamma_2', \dots, \gamma_n'\}. \tag{6}$$

We write

$$(R, J, \gamma) = (R_1, J_1, \gamma_1)$$

We denote by $\widetilde{R} \to R$ be the degree p "Kummer extension" corresponding to γ, defined locally by $\mathcal{O}_{\widetilde{R}} = \mathcal{O}_R[t]/\big(t^p - \gamma(\lambda)\big)$ where λ is a local nonzero section of J.

Corollary 30. *The symbol $\{\alpha_1, \dots, \alpha_n\}$ vanishes in the generic point of $\widetilde{R}$.*

Proof. This follows from Lemma 29 (see (6)). □

8. Construction of a norm variety via chain lemma

We fix a p-th root of unity $\zeta \neq 1$.
Let (R, J, γ) be the form of defined at the end of section 7 and let $G = \Gamma_n$.
Moreover let $A = A(R, J, \gamma)$ be the associated algebra bundle, with norm

$$N_A \colon A \to \mathcal{O}_R.$$

Let $b \in k^*$. We define the variety

$$X = X_b = \{\, [x, t] \in \mathbb{P}(A \oplus \mathcal{O}_R) \mid N_A(x) = bt^p \,\}.$$

The G-action extends to a G-action on A, $A \oplus \mathcal{O}_R$, $\mathbb{P}(A \oplus \mathcal{O}_R)$, and X.

Proposition 31. *The variety X has the following properties:*
(1) *X is proper of dimension $d = \dim X = p^n - 1$.*
(2) *One has*

$$\{\alpha_1, \dots, \alpha_n, b\}_{k(X)} = 0 \quad \textit{in} \quad K^{\mathrm{M}}_{n+1} k(X)/p$$

(3) *The fixed point scheme $\mathcal{F}_X$ of the G-action on X is a smooth 0-dimensional subscheme of X contained in the smooth part of X.*
(4) *There exist a proper smooth G-variety Y such that*
  a) *X and Y are G-fixed point equivalent.*
  b) $s_d(Y) \not\equiv 0 \bmod p^2$.

Proof. (1) follows from Lemma 26 and (2) follows from Corrollary 30. For the variety Y we take $Y = \mathbb{P}(A)$ with the natural G-action.

Proof of 4a): We have the map

$$X \xrightarrow{\pi} Y, \quad \pi([x, t]) = [x]$$

The map π is a branched covering of degree p. It should be noted that the map π seems to have no real significance for the applications of the proposition, however it turns out to be useful to compare the fixed points of X and Y.

To compute the fixed point sets $\mathcal{F}_X$ and $\mathcal{F}_Y$ we first note that both lie over $\mathcal{F}_R$. For each $P \in \mathcal{F}_R(k_{\mathrm{sep}})$ let $G_P = \mu_p \subset \mathrm{Aut}(J|P)$ and let X_P, Y_P be the fibres of X resp. Y over P.

Recall that P is twisting by Lemma 28. Therefore the homomorphism

$$G \to G_P$$

is surjective. From this one sees that all fixed points in X are contained in the smooth locus of X.

For $x \in \mathcal{F}_X \cap X_P$, $y \in \mathcal{F}_Y \cap Y_P$ one has G-equivariant decompositions

$$T_x X = T_x X_r \oplus T_P R$$
$$T_y Y = T_y Y_r \oplus T_P R.$$

Therefore, in order to prove 4a), it suffices to show that for each $r \in \mathcal{F}_R(k_{\mathrm{sep}})$ the fibres X_r and Y_r are G_P-fixed point equivalent. Moreover, we may assume that $k = k_{\mathrm{sep}}$. Hence we are reduced to the case $n = 1$, $G = \mu_p$, $R = \operatorname{Spec} k$, $J = k$, $\gamma(\lambda) = \lambda^p$, and $b = 1$.

In this case the G-fixed points of X resp. Y are

$$[\zeta^i, 1, \dots, 1] \qquad 0 \le i \le p-1 \qquad (\text{for } X)$$
$$[0, \dots, 0, \underset{i}{1}, 0, \dots, 0] \qquad 0 \le i \le p-1 \qquad (\text{for } Y)$$

with respect to the coordinates

$$A \oplus k = k \oplus L \oplus \cdots \oplus L^{\otimes p-1} \oplus k = k^{p+1}$$

resp.

$$A = k \oplus L \oplus \cdots \oplus L^{\otimes p-1} = k^p.$$

The fixed points of Y have all the same tangential G-structure, since the cyclic permutation of the coordinates on $Y = \mathbb{P}(A)$ commutes with the G-action. Moreover the map $\pi\colon X \to Y$ induces isomorphisms between the tangent spaces at the fixed points of X and the fixed point $[1, 0, \dots, 0]$ of Y. Hence X and Y have both p fixed points, with all having the same tangential G-structure. (Of course this can be verified also directly by computing the tangential G-structures: they are all isomorphic to the sum of the $p-1$ irreducible representations of G.)

This proves 4a) and along the way we have also seen (3).

Proof of 4b): The tangent bundle of Y decomposes (in $K_0(Y)$) as the sum of the tangent bundle TR of R and the fibre tangent bundle $T(Y/R)$ of the projection $\pi_A\colon Y = \mathbb{P}(A) \to R$. Hence

$$s_d(TY) = s_d\big(\pi_A^*(TR)\big) + s_d\big(T(Y/R)\big).$$

We have

$$s_d\big(\pi_A^*(TR)\big) = \pi_A^*\big(s_d(TR)\big) = 0$$

since $\dim R < d$. Moreover

$$\begin{aligned} s_d\big(T(Y/R)\big) &= s_d\big(\pi_A^*(A) \otimes \mathbb{L}(A)^\vee - \mathcal{O}_R\big) \\ &= s_d\Big(\bigoplus_{i=0}^{p-1} J^{\otimes i} \otimes \mathbb{L}(A)^\vee\Big) \\ &= \sum_{i=0}^{p-1} s_d(J^{\otimes i} \otimes \mathbb{L}(A)^\vee) \\ &= \sum_{i=0}^{p-1} \big(c_1(J^{\otimes i} \otimes \mathbb{L}(A)^\vee)\big)^d. \end{aligned}$$

We put

$$\begin{aligned} x &= -c_1(J) \in \mathrm{CH}^1(R) \\ y &= -c_1(\mathbb{L}(A)) \in \mathrm{CH}^1(Y). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{CH}^*(Y) &= \mathrm{CH}^*(R)[y]/\big(\prod_{i=0}^{p-1}(y - ix)\big) \\ &= \bigoplus_{i=0}^{p-1} y^i\,\mathrm{CH}^*(R) \end{aligned}$$

(by the computation of the Chow ring of projective bundles) and

$$s_d(TY) = \sum_{i=0}^{p-1} (y - ix)^d.$$

In the ring

$$\mathbb{Z}[x,y]/\big(\prod_{i=0}^{p-1}(y-ix)\big)$$

write

$$\sum_{i=0}^{p-1}(y-ix)^d = \sum_{i=0}^{p-1} a_i y^i x^{d-i}, \quad a_i \in \mathbb{Z}.$$

Since $d-i = \dim R + p - 1 - i$, we have

$$s_d(TY) = a_{p-1} y^{p-1} x^{\dim R}$$

Moreover $(\pi_A)_*(y^{p-1}) = [R] \in \mathrm{CH}^0(R)$. Hence

$$s_d(Y) = \deg\big(s_d(TY)\big) = a_{p-1}\deg\big((-c_1(J))^{\dim R}\big).$$

The claim follows now from $\delta(J) = 1 \bmod p$ (Lemma 27) and from the following Lemma 32. □

Lemma 32. *Let p be a prime, $Z = \mathbb{Z}/p^2$, and let*

$$S = Z[y]/\big(\prod_{i=0}^{p-1}(y-i)\big).$$

For $u \in S$ define $a_i(u) \in Z$ by

$$u = \sum_{i=0}^{p-1} a_i(u) y^i.$$

For $n \geq 1$ let

$$u_n = \sum_{i=0}^{p-1}(y-i)^{p^n-1} \in S.$$

Then $a_{p-1}(u_n) = p$.

Proof. One easily sees $a_{p-1}(u_1) = p$. We show that u_n does not depend on n.

The homomorphism

$$\Phi\colon S \to \prod_{i=0}^{p-1} Z$$
$$y \mapsto (0,1,2,\ldots,p-1)$$

is an isomorphism of rings. Hence it suffices to show that $\Phi(u_n)$ does not depend on n.

This means that for each $j = 0, \ldots, p-1$ the residue class

$$\sum_{i=0}^{p-1}(j-i)^{p^n-1} \bmod p^2$$

is independent of n. In fact, for any integer h one has $h^{p^n-1} = h^{(p-1)} \bmod p^2$. This is obvious if $h = 0 \bmod p$ (if $p \neq 2$). Otherwise $h^{p-1} = 1 \bmod p$ and $h^{(p-1)p} = 1 \bmod p^2$. Then

$$h^{p^n-1} = h^{(p-1)(1+p+\cdots+p^{n-1})} = h^{(p-1)} \bmod p^2.$$

If $p = 2$, then

$$S = Z[y]/(y^2 - y)$$

and the claim is easy to check. □

9. Multiplicativity

A splitting variety of a symbol is called p-generic, if it is a generic splitting variety over any p-special field.

Let Z be a p-generic splitting variety of $\{\alpha_1, \ldots, \alpha_n\}$ of dimension $p^{n-1} - 1$. We assume $\{\alpha_1, \ldots, \alpha_n\} \neq 0$. It follows that $I_Z \subset p\mathbb{Z}$.

Let (R, J, γ) be the form of defined at the end of section 7.

We have diagram of varieties

$$\begin{array}{ccccc}
\mathbb{A}^1 & = & \mathbb{A}^1 & = & \mathbb{A}^1 \\
{\scriptstyle N_A}\uparrow & & {\scriptstyle N_{A'}}\uparrow & & \uparrow{\scriptstyle \text{mult}} \\
\mathbb{A}(A) & \xleftarrow{\bar{g}} & \mathbb{A}(A') & \xrightarrow{\bar{f}} & \text{Cyclic}^p(Z \times \mathbb{A}^1) \\
\downarrow & & \downarrow & & \downarrow \\
R & \xleftarrow{g} & R' & \xrightarrow{f} & \text{Cyclic}^p(Z)
\end{array}$$

Here g is of degree prime to p and f is a morphism. The maps f, g come from the fact that Z has point of degree prime to p over $k(\widetilde{R})$, hence has a $k(\widetilde{R}')$-rational point where R'/R is of degree prime to p. This point defines the map f. The maps f, g are covered by the cyclic extensions of degree p:

$$\begin{array}{ccccc}
\widetilde{R} & \xleftarrow{\hat{g}} & \widetilde{R}' & \xrightarrow{\hat{f}} & Z^p \\
\downarrow & & \downarrow & & \downarrow \\
R & \xleftarrow{g} & R' & \xrightarrow{f} & \text{Cyclic}^p(Z)
\end{array}$$

with $\widetilde{R}' = \widetilde{R}_R R'$.

They are also covered by line bundles:

$$\begin{array}{ccccc}
J & \xleftarrow{\hat{g}} & J' & \xrightarrow{\hat{f}} & (Z^p \times \overline{\mathbb{A}^1})/(\mathbb{Z}/p) \\
\downarrow & & \downarrow & & \downarrow \\
R & \xleftarrow{g} & R' & \xrightarrow{f} & \text{Cyclic}^p(Z).
\end{array}$$

with $\overline{\mathbb{A}^1} \subset \mathbb{A}^p$ the image of $\mathbb{A}^1 \to \mathbb{A}^p$, $t \mapsto t(1, \zeta, \zeta^2, \ldots, \zeta^{p-1})$ with $1 \neq \zeta \in \mu_p$. This diagram induces the maps $\bar{f}$, $\bar{g}$.

Note that $\deg \bar{g} = \deg g$ and $\deg \bar{f} = \deg f$.

The fibre X_t of N_A over the generic point $\text{Spec}\, k(t)$ is a splitting variety of the symbol

$$\{\alpha_1, \ldots, \alpha_n, t\}.$$

One has $I_{X_t} = p\mathbb{Z}$, since $\{\alpha_1, \ldots, \alpha_n, t\} \neq 0$.

We assume now $p \neq 2$. By Proposition 31 (4), Corollary 5, and Corollary 2, one finds that the birational invariant of X_t in

$$\mathbb{Z}/I_{X_t} = \mathbb{Z}/p$$

is nonzero. Since $\deg \bar{g}$ is prime to p, the degree formula implies that the fibre X'_t of $N_{A'}$ has nontrivial invariant. The degree formula shows then that $\deg \bar{f}$ is prime to p, hence $\deg f$ is prime to p.

Now let $K = k(\sqrt[p]{b})$ be a cyclic extension of degree p which splits $\{\alpha_1, \dots, \alpha_n\}$. We assume that k is p-special. It follows that there is a point $\operatorname{Spec} K \to \widetilde{R}$ lying over a rational point $P\colon \operatorname{Spec} k \to R$. Then $b = \gamma(P)$ in $k^*/(k^*)^p$. It follows that

$$\{\alpha_1, \dots, \alpha_n\} = \{\alpha_1, \gamma_2(P), \gamma_3'(P), \dots, \gamma_n'(P)\}, \tag{7}$$

$$\{\alpha_1, \gamma_2(P)\} = \{b, \gamma_2'(P)\}, \tag{8}$$

$$\{\alpha_1, \dots, \alpha_n\} = \{b, \gamma_2'(P), \dots, \gamma_n'(P)\}. \tag{9}$$

(see (4)–(6) after Lemma 29).

Now let $k(\sqrt[p]{b})$, $k(\sqrt[p]{c})$ be two cyclic extensions of degree p which split the symbol $\{\alpha_1, \dots, \alpha_n\}$. Applying the last arguments twice, one finds first $b_i \in k^*$ such that

$$\{\alpha_1, \dots, \alpha_n\} = \{b, b_1, b_2, \dots, b_n\},$$

and then c_i, $c_i' \in k^*$ such that

$$\{b, b_1, b_2, \dots, b_n\} = \{b, c_1, c_2, \dots, c_n\},$$
$$\{b, c_1\} = \{c, c_2'\}.$$

Let $X(b, c_1)$ be the Brauer-Severi variety associated to the symbol $\{b, c_1\}$. It has rational points over $k(\sqrt[p]{b})$ and over $k(\sqrt[p]{c})$. Morover, since Z is a p-generic spliting field, we have a correspondence $X(b, c_1) \to Z$ lying over $\mathbb{Z} \to \mathbb{Z}$ of degree prime to p.

Corollary 33. *Let x, $y \in Z$ be points of degree p and let $\alpha \in \kappa(x)^*$, $\beta \in \kappa(y)^*$. Then there exist $z \in Z$ of degree p and $\gamma \in \kappa(z)^*$, such that*

$$[\alpha] + [\beta] = [\gamma] \quad \text{in} \quad A_0(Z, K_1).$$

Proof. By the previous considerations, and using that $\operatorname{CH}_0(Z_K) = \mathbb{Z}$ whenever $Z(K) \neq \varnothing$, we may reduce to the case of Brauer-Severi variety. In this case the statement is known [3]. □

Corollary 34. *If k is p-special, then the set $\mathcal{V}(N_A)$ of values $\neq 0, \infty$ of the map*

$$N_A \colon \mathbb{A}(A)(k) \to k$$

is a multiplicative subgroup of k^*. □

10. On the norm principle

We fix a p-th root of unity $\zeta \neq 1$.

Let $(\operatorname{Spec} k, I, \epsilon)$ be a nonzero form and let $K = A(\operatorname{Spec} k, I, \epsilon)$ be the associated algebra.

Let (R, J, γ) be the form defined at the end of section 7, let $G = \Gamma_n$, and let $A = A(R, J, \gamma)$ be the associated algebra bundle, with norm

$$N_A \colon A \to \mathcal{O}_R.$$

We denote by

$$N_{A_K} \colon A \otimes K \to \mathcal{O}_R \otimes K$$

the induced map of degree p.

Let $B = \mathcal{O}_R \oplus A \otimes I$. We have a natural inclusion $B \to A \otimes K$. Let

$$M \colon B \to K \otimes_k \mathcal{O}_R$$

be the restriction of N_{A_K} to B.

Let (B_i, R_i, M_i), $i = 1, \ldots, p-1$ be copies of (B, R, M). We put

$$\begin{aligned} U &= \mathbb{P}(A) \times \prod_{i=1}^{p-1} \mathbb{P}(B_i), \\ L &= \mathbb{L}(A) \boxtimes \bigboxtimes_{i=1}^{p-1} \mathbb{L}(B_i), \\ V &= L \oplus \mathcal{O}_U. \end{aligned}$$

Then $\dim U = p^n - 1 + (p-1)p^n = p^{n+1} - 1$ and L is a line bundle on U and V is a 2-dimensional vector bundle on U.

Let $\omega \in K^*$. On V we define a K-valued form

$$\Theta \colon V \to \mathcal{O}_U \otimes K$$

of degree p by

$$\Theta_\omega(u \otimes \otimes_{i=1}^{p-1} u_i + t) = N_A(u) M(u_1) \cdots M(u_{p-1}) - \omega t^p,$$

where u, u_i, t are sections of $\mathbb{L}(A)$, $\mathbb{L}(B_i)$, $\mathcal{O}_U$, respectively.

We define the variety

$$\overline{X}_\omega = \{\, [x] \in \mathbb{P}(V) \mid \Theta_\omega(x) = 0 \,\}.$$

Proposition 35. *There exist an open dense subset $\Omega \subset \mathbb{A}(K)$ such that the variety $\overline{X} = \overline{X}_\omega$ has the following properties:*

(1) $d = \dim X = p(p^n - 1)$.

(2) *One has*

$$\{\alpha_1, \ldots, \alpha_n, \omega\}_{k(\overline{X})} = 0 \quad \text{in} \quad K^{\mathrm{M}}_{n+1} K(\overline{X})/p$$

(3) *The fixed point scheme $\mathcal{F}_{\overline{X}}$ of the G-action on $\overline{X}$ is a smooth 0-dimensional subscheme of $\overline{X}$.*

(4) *There exist a proper smooth G-variety $\overline{Y}$ such that*
   - a) *$\overline{X}$ and $\overline{Y}$ are G-fixed point equivalent.*
   - b) *$\overline{Y}$ is the union of $(p-1)!$ copies of Y^p, where Y is a variety of dimension $d = p^n - 1$ with $s_d(Y) \not\equiv 0 \bmod p^2$.*

Proof. (1) and (2) are obvious from former considerations (for all $\omega \in K^*$).

We have to determine the fixed points on X. Any fixed point lies over a fixed point $P \in R^p$. There are only finitely many such P, and any P is twisting for the bundles $(\operatorname{Spec} k, H_i, \alpha_i)$, $i = 1, \ldots, n$. On $\mathbb{P}(A|P)$ there are exactly p fixed points. The fixed point scheme of $\mathbb{P}(B|P) \simeq \mathbb{P}^p$ consists of a 1-dimensional component

$$\mathcal{P} = \mathbb{P}(k \oplus I \oplus 0 \oplus \cdots \oplus 0) \simeq \mathbb{P}^1$$

and $p-1$ isolated fixed points

$$\mathbb{P}(0 \oplus 0 \oplus 0 \oplus \cdots \oplus J^{\otimes i} \otimes I \oplus \cdots \oplus 0)$$

Hence any of the fixed point components $\mathcal{C}$ in $U|P$ is a product

$$\mathcal{C} = \mathcal{C}_0 \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_{p-1}$$

with $\mathcal{C}_0$ a point, and, for $i > 0$, $\mathcal{C}_i$ a point or $\simeq \mathbb{P}^1$.

Let $s = \dim \mathcal{C}$. If $s < p-1$, then the equation $\Theta_\omega(x) = 0$ has no solution at all, provided ω is generic. If $s = p-1$ and

$$\mathcal{C}_0 = [0, \ldots, \mathbb{L}(A)^{\otimes i}, \ldots, 0]$$

for some $i > 0$, then $\mathbb{P}(V)|\mathcal{C}$ has only two fixed points, which do not lie in X.

Hence fixed points appear only if

$$\mathcal{C} = [1, 0, \ldots, 0] \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_{p-1}$$

with $\mathcal{C}_i \simeq \mathbb{P}^1$. One finds that there exactly $(p-1)!p^p$ fixed points (for generic ω). For this use the following facts:

Let

$$\Phi\colon (\mathbb{P}^1)^{p-1} \to \mathbb{P}^{p-1}$$
$$\Phi\big(([x_i + ty_i])_{i=1,\ldots,p-1}\big) = [\prod_i (x_i + ty_i)]$$

be the "morphism of the fundamental theorem of algebra". Φ is a finite morphism of degree $(p-1)!$. Let

$$\widehat{\Phi}\colon (\mathbb{P}^1)^{p-1} \to \mathbb{P}^{p-1}$$
$$\widehat{\Phi}\big(([x_i + ty_i])_{i=1,\ldots,p-1}\big) = [\prod_i (x_i + ty_i)^p].$$

$\widehat{\Phi}$ is a finite morphism of degree $(p-1)!p^{p-1}$.

Hence there are exactly $(p-1)!p^p$ fixed points over P. These are given by just $(p-1)!$ copies of the fixed point set of $\mathbb{P}(A|P)^p$. The characteristic number of $Y = \mathbb{P}(A)$ has been computed in section 8. □

We assume $\{\alpha_1, \ldots, \alpha_n\}_K \neq 0$. Let ω be the generic element of $\mathbb{A}(K)$. Then $\{\alpha_1, \ldots, \alpha_n, \omega\} \neq 0$.

Let further $T = N_A^{-1}(\omega)$ (the splitting variety over K for $\{\alpha_1, \ldots, \alpha_n, \omega\}$, constructed from the family (R, J, γ) of Kummer splitting fields of $\{\alpha_1, \ldots, \alpha_n\}$).

Let $R_{K/k}(T)$ be the transfer of T. We claim the $R_{K/k}(T)$ has a point of degree prime to p over $k(X_\omega)$.

In fact, by applying Corollary 34 to the ground field $K(X_\omega)$, we see that there exist an extension $H/K(X_\omega)$ of degree prime to p, such that ω is a value of N_A over H. But then there exist an extension $H'/k(X_\omega)$ of degree prime to p, such that ω is a value of N_A over $H' \otimes K$.

Writing $H' = k(W)$ we get maps

$$X_\omega \xleftarrow{\;g\;} W \xrightarrow{\;f\;} R_{K/k}(T)$$

with $(\deg g, p) = 1$.

We would like to conclude that $(\deg f, p) = 1$. To compute $\deg f$ we may do base change $k \to K$, so that $K = k \times \cdots \times k$. Then the diagram looks as

$$X_\omega \xleftarrow{\;g\;} W \xrightarrow{\;f\;} \overline{T} := \prod_{i=1}^{p} N_A^{-1}(\omega_i)$$

with $\omega = (\omega_1, \ldots, \omega_p)$, and $\omega_i \in k$ are p independent generic elements. $(\deg f, p) = 1$ follow from Proposition 35 (4) and the "higher degree formula".

Let us take it for granted, so that $\deg f$ is prime to p. Extension from the generic point of $\mathbb{A}(K)$ to $\mathbb{A}(K)$ provides a diagram

$$\begin{array}{ccccc}
\mathbb{A}(K) & = & \mathbb{A}(K) & = & \mathbb{A}(K) \\
{\scriptstyle N_A M_1 \ldots M_{p-1}} \uparrow & & \uparrow & & \uparrow {\scriptstyle R_{K/k}(N_A)} \\
\mathbb{A}(L) & \xleftarrow{\;\bar{g}\;} & \overline{W} & \xrightarrow{\;\bar{f}\;} & R_{K/k}(\mathbb{A}(A))
\end{array}$$

with $(\deg \bar{g}, p) = (\deg \bar{f}, p) = 1$.

Assume now that k is p-special.

Let $x \in Z_K$ be a point of degree p and let $\delta \in \kappa(x)^*$. Our aim is to show that

$$N_{K/k}([\delta]) \in A_0(Z, K_1)$$

is represented by a sum of elements concentrated in points of Z of degree p (over k). If the symbol is split over K, this is easy to check (assuming $\mathrm{CH}_0(Z_K) = \mathbb{Z}$, when K is a splitting field). So we may assume $\{\alpha_1, \ldots, \alpha_n\}_K \neq 0$.

Let $\omega = N_{\kappa(x)/K}(\delta) \in K^*$. By multiplying ω with some p-th power in $(K^*)^p$, we may arrange that ω lies in any given open dense subset of $\mathbb{A}(K)$.

As we have seen, there is a K-rational point $P \in R$ such that

$$(R, J, \gamma)|P$$

represents the Kummer extension $\kappa(x)/K$. Hence ω is a value of N_A over K. Hence ω is in the image of $R_{K/k}(\mathbb{A}(A))$ under $R_{K/k}(N_A)$. Since $(\deg \bar{f}, p) = 1$, ω is also in the image of $\mathbb{A}(L)$ under $N_A M_1 \ldots M_{p-1}$.

The p projections $\mathbb{A}(L) \to R$ (via $\mathbb{P}(A)$, $\mathbb{P}(B_i)$) give us p points in R, whence p points $z_i \in Z$ of degree p. Furthermore we have elements $\delta_i \in \kappa(z_i)$ such that

$$\delta = N_{\kappa(z_0)/k}(\delta_0) N_{K \otimes \kappa(z_1)/K}(1 + z_1 \sqrt[p]{\epsilon}) \cdots N_{K \otimes \kappa(z_{p-1})/K}(1 + z_{p-1} \sqrt[p]{\epsilon})$$

in K^*.

By multiplicativity we see that

$$[\delta] = [\delta_0]_K + [1 + z_1 \sqrt[p]{\epsilon}] + \cdots + [1 + z_{p-1} \sqrt[p]{\epsilon}].$$

in $A_0(Z_K, K_1)$. But then

$$N_{K/k}([\delta]) = p[\delta_0] + [N_{K \otimes \kappa(z_1)/\kappa(z_1)}(1 + z_1 \sqrt[p]{\epsilon})] + \ldots$$

is concentrated in the points z_i. $\square$???$\square$

References

[1] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302.

[2] E. E. Floyd, *Actions of* $(Z_p)^k$ *without stationary points*, Topology **10** (1971), 327–336.

[3] A. S. Merkurjev and A. A. Suslin, *The group of* K_1*-zero-cycles on Severi-Brauer varieties*, Nova J. Algebra Geom. **1** (1992), no. 3, 297–315.

[4] V. Voevodsky, *The Milnor conjecture*, preprint, Max-Planck-Institute for Mathematics, Bonn, 1996, http://www.math.uiuc.edu/K-theory/0170, 1996.