# BETWEEN QUADRATIC AND SYMMETRIC BILINEAR FORMS

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`preliminary version`

Contents



*Date*: September 13, 2022; Sept. 18: added first part; Sept. 27: added Remark 3.6.

## Introduction

This is work in progress. Currently the text has two parts.

The second (older) part (see page 10) contains an introduction which explains the basic idea in detail. The relation with group schemes of order 2 is indicated. Moreover for the rank 2 case there is a formula for the Dickson invariant for symmetries of a non-degenerate "form" in $M(V)$. It takes values in the corresponding group scheme.

The first (newer) part gives a somewhat different definition of the modules $M(V)$ with other notations. The half-determinant is defined. The method used for the half-determinant might be also interesting for the classical case of odd-dimensional quadratic forms. No group schemes in this part yet.

## §1. Preliminaries

Let $R$ be a ring (commutative, with 1) and let $V$ be a locally free $R$-module of finite rank $n$.

Let further

$$S_2V \subset V^{\otimes 2} \to S^2V$$

be the $R$-modules of symmetric 2-tensors: If we let $\mathbf{Z}/2\mathbf{Z}$ act on $V^{\otimes 2}$ by the switch involution

$$\tau\colon V^{\otimes 2} \to V^{\otimes 2}$$
$$\tau(x \otimes y) = y \otimes x$$

then

$$S_2V = H^0\big(\mathbf{Z}/2\mathbf{Z}, V^{\otimes 2}\big) = \big(V^{\otimes 2}\big)^\tau = \{\, z \in V^{\otimes 2} \mid \tau(z) = z \,\}$$
$$S^2V = H_0\big(\mathbf{Z}/2\mathbf{Z}, V^{\otimes 2}\big) = \big(V^{\otimes 2}\big)/(1-\tau)\big(V^{\otimes 2}\big)$$

and there is the exact sequence

$$0 \to S_2V \to V^{\otimes 2} \xrightarrow{1-\tau} V^{\otimes 2} \to S^2V \to 0$$

If $W = V^\vee$ is the dual, then $S_2V$ is the module of symmetric bilinear forms on $W$ and $S^2V$ is the module of quadratic forms on $W$:

$$S_2V = (S^2W)^\vee, \qquad S^2V = (S_2W)^\vee$$

Elements in $S^2V$ are denoted as usual as polynomials $x^2$, $xy$, for elements in $S_2V$ we use the notations

$$x^{[2]} = x \otimes x$$
$$x * y = x \otimes y + y \otimes x$$

for $x$, $y \in V$.

Let

$$S_2V \xrightarrow{j} S^2V \xrightarrow{k} S_2V$$

denote the standard morphisms, given by projection and symmetrization

$$j\colon S_2V \subset V^{\otimes 2} \to S^2V \qquad x \otimes y \mapsto xy$$
$$k\colon S^2V \to S_2V \subset V^{\otimes 2} \qquad xy \mapsto x * y$$

respectively. Thus

$$j(x^{[2]}) = x^2 \qquad\qquad k(x^2) = 2x^{[2]}$$
$$j(x * y) = 2xy \qquad\qquad k(xy) = x * y$$

The compositions of $j$, $k$ in either way are multiplication by 2:

$$kj = 2 \cdot \mathrm{id}_{S_2V}, \qquad jk = 2 \cdot \mathrm{id}_{S^2V}$$

The morphism $j$ associates to a symmetric bilinear form $h$ (on $W$) the quadratic form $q(v) = h(v,v)$, the morphism $k$ associates to a quadratic form $q$ the symmetric bilinear form $h(v,w) = q(v+w) - q(v) - q(w)$.

## §2. The modules $M_{\lambda,\mu}(V)$

Let $\lambda$, $\mu \in R$ be elements of the base ring subject to

$$\lambda\mu = 2$$

We define the $R$-module

$$M(V) = M_{\lambda,\mu}(V) \subset S_2V \oplus S^2V$$

as the set of all pairs

$$(h, q) \in S_2V \oplus S^2V$$

with

$$j(h) = \mu q$$
$$\lambda h = k(q)$$

For

$$f = (h, q) \in M(V)$$

we call its components

$$h_f = h, \qquad q_f = q$$

the associated (symmetric) bilinear form resp. the associated quadratic form.

The classical cases are

- The quadratic form case: $\lambda = 1$. Here $\mu = 2$ and

  $$M_{1,2}(V) = S^2V$$

  Any $f \in M_{1,2}(V)$ is of the form

  $$f = \big(k(q), q\big)$$

  with $q \in S^2V$.
- The (symmetric) bilinear form case: $\mu = 1$. Here $\lambda = 2$ and

  $$M_{2,1}(V) = S_2V$$

  Any $f \in M_{2,1}(V)$ is of the form

  $$f = \big(h, j(h)\big)$$

  with $h \in S_2V$.

**(2.1) Lemma.** *Let $x_i$ be a basis of $V$. Then the elements*

$$(\mu x_i^{[2]}, x_i^2), \qquad (x_i * x_j, \lambda x_i x_j) \quad (i < j)$$

*form a basis of $M_{\lambda,\mu}(V)$. In particular,*

$$\operatorname{rank} M_{\lambda,\mu}(V) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} = \operatorname{rank} S_2 V = \operatorname{rank} S^2 V$$

*Proof*: The conditions for $(h, q) \in M(V)$ can be separated according to the basis $x_i^{[2]}$, $x_i * x_j$ of $S_2 V$ and $x_i^2$, $x_i x_j$ of $S^2 V$. From

$$j(x_i^{[2]}) = x_i^2$$
$$k(x_i x_j) = x_i * x_j$$

it follows that the coefficients of $x_i^{[2]}$ in $h$ and of $x_i x_j$ in $q$ are determined by the corresponding coefficients in $q$ (with factor $\mu$) and in $h$ (with factor $\lambda$), respectively.

To conclude the proof one notes that the given elements are indeed in $M(V)$ (this uses $\lambda\mu = 2$). □

For an element

$$f = \sum_i a_i (\mu x_i^{[2]}, x_i^2) + \sum_{i<j} a_{ij} (x_i * x_j, \lambda x_i x_j)$$

in $M(V)$ we use the abbreviated notations

$$f = \mu \bullet \sum_i a_i x_i^{[2]} + \sum_{i<j} a_{ij}\, x_i * x_j$$
$$f = \sum_i a_i x_i^2 + \lambda \bullet \sum_{i<j} a_{ij} \lambda x_i x_j$$

We call them the bilinear resp. quadratic form notation. Here $\mu \bullet$ is to be understood as a mere symbol with index $\mu$. Similarly for the quadratic form notation. The omitted elements $\lambda$ resp. $\mu$ with $\lambda\mu = 2$ are thought to be present in the background.

The $x_i$ can be any elements of $V$. For instance, in quadratic form notation one has the rules

$$(x + y)^2 = x^2 + y^2 + \lambda \bullet \mu xy$$
$$\lambda \bullet x(x + y) = \lambda x^2 + \lambda \bullet xy$$

with $x, y \in V$.

2.1. **The determinant.** First a general remark. For a 2-tensor $h \in V^{\otimes 2}$ the determinant

$$\det(h) = (\Lambda^n V)^{\otimes 2}$$

is defined for instance as follows. Let

$$h' \colon V^\vee \to V$$
$$h'(\alpha) = (\alpha \otimes 1)(h)$$

be the associated duality morphism. As for any morphism between modules of equal rank one can take its determinant (= highest exterior power). This leads to the definition

$$\det(h) = \Lambda^n(h') \in \mathrm{Hom}\big(\Lambda^n V^\vee, \Lambda^n V\big) = (\Lambda^n V^\vee)^\vee \otimes \Lambda^n V = (\Lambda^n V)^{\otimes 2}$$

If $V = R^n$, then $\det(h)$ is the determinant of the corresponding $n \times n$-matrix.

**(2.2) Definition.** The determinant of

$$f \in M_{\lambda,\mu}(V)$$

is defined as the determinant of its associated bilinear form:

$$\det(f) = \det(h_f) \in (\Lambda^n V)^{\otimes 2}$$

with $n = \operatorname{rank} V$.

So for $f = (h, q)$ one simply has $\det(f) = \det(h)$. This is like for a usual quadratic form $q$ where one takes the determinant of the associated bilinear form $k(q)$.

2.2. **The discriminant ($n$ even).** For $\operatorname{rank} V = 2k$ one defines the discriminant by the usual sign change

$$\mathrm{disc}(f) = (-1)^k \det(h_f) \in (\Lambda^n V)^{\otimes 2}$$

For instance, if $n = 2$ and

$$f = ax^2 + by^2 + \lambda \bullet cxy$$

then

$$\mathrm{disc}(f) = -\det\begin{pmatrix} \mu a & c \\ c & \mu b \end{pmatrix} = c^2 - \mu^2 ab$$

**(2.3) Remark.** For quadratic forms a finer invariant is the discriminant algebra $D(q)$ (Loos 1997 [4]). In the non-degenerate even-dimensional case $D(q)$ is a quadratic etale algebra (the center of the Clifford algebra). In other words, $D(q)$ is a $\mathbf{Z}/2\mathbf{Z}$-torsor. For a symmetric bilinear form $h$ there is no $\mathbf{Z}/2\mathbf{Z}$-torsor (unless 2 is invertible), but one can form the quadratic extension $R[t]/(t^2 \pm \det(h))$. If $h$ is non-degenerate ($\det(h)$ is invertible), this is a $\mu_2$-torsor. One may speculate that for non-degenerate $f \in M_{\lambda,\mu}(V)$ there is a $G$-torsor $D(f)$ where $G$ is the group scheme of order 2 associated with the pair $(\lambda, \mu)$ (Tate-Oort 1970 [5]).

## §3. The half-determinant ($n$ odd)

Let $f \in M_{\lambda,\mu}(V)$. In general, the matrix $H$ of the associated bilinear form $h_f$ (with respect to a basis) has the factor $\mu$ on each diagonal element:

**(3.1)** $$H = \mu D + X + X^t$$

where $D$ is a diagonal matrix (the diagonal part of $f$) and $X$ is a proper upper diagonal matrix (the off-diagonal part of $f$).

Assume $n = 2k + 1$. Then the determinant of the alternating matrix $X - X^t$ vanishes. Since $\lambda\mu = 2$ one has

$$H = \mu(D + \lambda X^t) + (X - X^t)$$

Consider this expression over the polynomial ring

$$R_0 = \mathbf{Z}[\mu, \lambda, D_i, X_{ij}]$$

Since

$$\det(H)|_{\mu=0} = \det(X - X^t) = 0$$

it follows that

$$\det(H) = \mu P(\mu, \lambda, D, X)$$

for some universal polynomial $P \in R_0$.

**(3.2) Definition.** The half-determinant of $f$ is defined as

$$\operatorname{hdet}(f) = P(\mu, \lambda, D, X) \in R$$

for any representation (3.1) of $h_f$.

**(3.3) Remark.** The method of dividing a universal polynomial by $\mu$ is the same as the usual method to define the half-determinant of a usual odd-dimensional quadratic form (where one divides by 2), see for instance Conrad 2014 [3, Proposition C.1.4, p. 304].

One hand this is a quick way to convince oneself that hdet exists. But if one is honest, it is also bit tiring. Namely in principle one should argue that the definition does not depend on the choice of basis. One also should make precise the underlying specialization argument. Not to forget functoriality.

But there is a better way, as we will describe now.

At first we do not assume that $n$ is odd.

For a 2-tensor $h \in V^{\otimes 2}$ one defines for any $0 \leq r \leq n$ the tensor

$$\Lambda^r(h) \in (\Lambda^r V)^{\otimes 2}$$

by taking the exterior powers of its duality morphism $h' \colon V^\vee \to V$:

$$\Lambda^r(h) = \Lambda^r(h') \in \operatorname{Hom}\bigl(\Lambda^r V^\vee, \Lambda^r V\bigr) = (\Lambda^r V^\vee)^\vee \otimes \Lambda^r V = (\Lambda^r V)^{\otimes 2}$$

See section 2.1, where the case $r = n$ was considered.

**(3.4) Remark.** A little excursion here: One may also define $\Lambda^r(h)$ via the Pfaffian exponential

$$\operatorname{Pf} \colon \Lambda^2(V \oplus V) \to \Lambda^{\text{even}}(V \oplus V)$$
$$\operatorname{Pf}(\omega) = \sum_{r \geq 0} \frac{\omega^r}{r!}$$

(which is integrally defined). Its diagonal part yields the $\Lambda^r(h)$. Namely, restricting Pf to the middle term of

$$\Lambda^2(V \oplus V) = (\Lambda^2 V \otimes R) \oplus (V \otimes V) \oplus (R \otimes \Lambda^2 V)$$

yields the desired maps

$$(\Lambda^r)_{r \geq 0} \colon V \otimes V \to \bigoplus_{r \geq 0} \Lambda^r V \otimes \Lambda^r V \subset \Lambda^{\text{even}}(V \oplus V)$$

This is undoubtedly the best way to define the $\Lambda^r(h)$, however it needs some preparations.

One observes that if $h$ is symmetric, so is $\Lambda^r(h)$. So for $h \in S_2V$ one has

$$\Lambda^r(h) \in S_2(\Lambda^r V)$$

(In the case $r = n$ this can go without mentioning, because $\operatorname{rank} \Lambda^n V = 1$.) Hence there is an element in

$$\Lambda^r(h) \in S_2(\Lambda^r V) = S_2\big(\Lambda^{n-r}(V^\vee)\big) \otimes (\Lambda^n V)^{\otimes 2}$$

In the particular case $r = n - 1$ we get an element

$$\Lambda^{n-1}(h) \in S_2(V^\vee) \otimes (\Lambda^n V)^{\otimes 2}$$

But then any quadratic form

$$q \in S^2 V = \big(S_2(V^\vee)\big)^\vee$$

can be evaluated on it.

**(3.5) Definition.** For $f \in M_{\lambda,\mu}(V)$ one defines

$$\operatorname{hdet}'(f) = q_f\big(\Lambda^{n-1}(h_f)\big) \in (\Lambda^n V)^{\otimes 2}$$

with $n = \operatorname{rank} V$

That's a neat definition, isn't it? One gets the functoriality for free.

**(3.6) Remark.** On the other hand, one should not obscure matters by the functorial style. Let $A$ be a $n \times n$-matrix. Consider

$$\operatorname{hdet}'(A) = \operatorname{trace}\big(A(A + A^t)^\#\big)$$

Here $A^t$ is the transpose and $B^\#$ denotes the adjunct (with $BB^\# = \det(B) \cdot 1$). If $B$ is symmetric, so is $B^\#$. Since

$$\operatorname{trace}\big((D - D^t)C\big) = 0$$

for symmetric $C$, it follows that

$$\operatorname{hdet}'(A) = \operatorname{hdet}'(A + D - D^t)$$

Hence $\operatorname{hdet}'(A)$ depends only on its associated quadratic form. Moreover

$$\begin{aligned} 2 \cdot \operatorname{hdet}'(A) &= \operatorname{hdet}'(A) + \operatorname{hdet}'(A^t) \\ &= \operatorname{trace}\big((A + A^t)(A + A^t)^\#\big) \\ &= n \cdot \det(A + A^t) \end{aligned}$$

This displays the essence of the next Lemma.

**(3.7) Lemma.** *One has*

$$\mu \operatorname{hdet}'(f) = n \det(f)$$

*Proof*: Let $W = V^\vee$. Consider

$$u \in S_2 W \xrightarrow{i} W^{\otimes 2}$$

where $i$ is the inclusion. For $h \in V^{\otimes 2}$ one has

$$\langle h, i(u) \rangle = \langle j(h), u \rangle$$

The same remark holds after tensoring with the line bundle $(\Lambda^n V)^{\otimes 2}$. Then if

$$i(u) = \Lambda^{n-1}(h) \in W^{\otimes 2} \otimes (\Lambda^n V)^{\otimes 2} = (\Lambda^{n-1} V)^{\otimes 2}$$

one finds

$$\langle h, i(u)\rangle = h \wedge \Lambda^{n-1}(h) = n\Lambda^n(h) = n \det(h)$$

where the $\wedge$-product is taken on each tensor factor. The factor $n$ stems from the fact that the composition of the natural maps

$$\Lambda^n V \xrightarrow{\delta} V \otimes \Lambda^{n-1} V \xrightarrow{\wedge} \Lambda^n V$$

is multiplication with $n$. (Admittedly this deserves some more details, which are for now left as an exercise.)

For the proof of the Lemma it suffices now to note $j(h_f) = \mu q_f$. $\square$

**(3.8) Lemma.** *If $n$ is even, then*

$$\operatorname{hdet}'(f) = \lambda \frac{n}{2} \det(f)$$

*Proof*: Omitted. (There is no use of $\operatorname{hdet}'$ for even $n$.) $\square$

If $n$ is odd, then it is relatively prime to $\mu$, since $\mu$ divides $2 = \lambda\mu$. One can therefore exploit Lemma 3.7 as follows.

**(3.9) Definition.** Assume $n$ is odd, $n = 2k + 1$. The *half-determinant* of $f \in M_{\lambda,\mu}(V)$ is defined as

$$\operatorname{hdet}(f) = \operatorname{hdet}'(f) - \lambda\, k \det(f) = \lambda(k+1)\det(f) - \operatorname{hdet}'(f)$$

Obviously one has

$$\mu \operatorname{hdet}(f) = \det(f)$$

**(3.10) Remark.** The name "half-determinant" refers to the quadratic form case with $\lambda = 1$ and $\mu = 2$.

As mentioned, this case is very well known (via universal division by 2). However the method described here (juggling with exterior powers) is perhaps new even in this case—I am not aware of a reference.

In the case $\mu = 1$ the construction is vain: For a bilinear form the "half-determinant" in our wording is just its determinant.

**(3.11) Remark.** We cannot resist to mention another method for the case $n = 3$ (for classical quadratic forms $q \in S^2 V$ only). In this case one can use the Clifford algebra. Let again $W = V^\vee$ be the dual.

There is the linear morphism

$$\theta\colon \Lambda^3 W \to C_1(q) \subset C(q)$$
$$\theta(u \wedge v \wedge w) = uvw - wvu$$

Indeed, if $u = v$, then $uv = vu = q(u)$ and the right hand side vanishes. Similarly for $v = w$. Hence $\theta$ is well defined.

Inspection of the generic case $q = ax^2 + bxy$ shows that the morphism $\theta$ maps to the center $Z(q)$ of $C(q)$. Moreover one finds (exercise!)

$$\theta(\omega)^2 = \langle \operatorname{hdet}(q), \omega^2\rangle$$

This is a priori an equation in $C_0(q)$ but takes place in the submodule $1 \cdot R$. It means that $\theta^2$ is a morphism

$$\theta^2\colon (\Lambda^3 W)^{\otimes 2} \to R \subset C_0(q)$$

which equals

$$\mathrm{hdet}(q) \in (\Lambda^3 V)^{\otimes 2}$$

Write

$$Z(q) = Z_0(q) \oplus Z_1(q), \qquad Z_i(q) = Z(q) \cap C_i(q)$$

If $\mathrm{hdet}(q)$ is invertible, then $Z_0(q) = R$ and $\theta$ is the well known isomorphism $\Lambda^3 W \to Z_1(q)$.

`Obsolete version. To be removed later but currently essentially disjoint from the first part.`

## Preface (OBSOLETE)

Is the Euclidean metric a quadratic form or a symmetric bilinear form? Length or angles? This text is dedicated to all people who can't decide between quadratic forms and symmetric bilinear forms. To anybody who doesn't distinguish between these concepts: please quit immediately.

## Introduction (OLD VERSION)

Let $R$ be a ring (commutative, with 1) and $V$ be a locally free $R$-module of finite rank $n$. Denote by

$$B(V) = \{\, h\colon V \times V \to R \mid h \text{ is } R\text{-bilinear and symmetric}\,\}$$
$$Q(V) = \{\, q\colon V \to R \mid q \text{ is a quadratic form over } R\,\}$$

the $R$-modules of symmetric bilinear resp. quadratic forms on $V$. Let further

$$j\colon B(V) \to Q(V)$$
$$j(h)(v) = h(v,v)$$

and

$$k\colon Q(V) \to B(V)$$
$$k(q)(v,w) = q(v+w) - q(v) - q(w)$$

be the natural maps associating to a bilinear form its associated quadratic form and vice versa. In terms of a basis $x_i$ of the dual of $V$, the morphisms $j$ and $k$ have the description

$$j(x_i \otimes x_i) = x_i^2 \qquad k(x_i^2) = 2x_i \otimes x_i$$
$$j(x_i \otimes x_j + x_j \otimes x_i) = 2x_i x_j \qquad k(x_i x_j) = x_i \otimes x_j + x_j \otimes x_i$$

In particular, the compositions of $j$, $k$ in either way are multiplication by 2:

$$kj = 2 \cdot \mathrm{id}_{B(V)}, \qquad jk = 2 \cdot \mathrm{id}_{Q(V)}$$

Now let $\lambda$, $\mu \in R$ be elements of the base ring subject to

$$\lambda\mu = 2$$

Consider the $R$-module

$$M(V) = M_{\lambda,\mu}(V) = \frac{B(V) \oplus Q(V)}{j_\lambda\big(B(V)\big) + k_\mu\big(Q(V)\big)}$$

where

$$j_\lambda(h) = \big(-\lambda h, j(h)\big)$$
$$k_\mu(q) = \big(k(q), -\mu q\big)$$

In terms of a basis $x_i$ as above, the $R$-module $M(V)$ has basis

$$(x_i \otimes x_i, 0), \qquad (0, x_i x_j) \quad (i < j)$$

(It is obvious that these elements are generators, the independence requires $\lambda\mu = 2$.) In particular

$$\operatorname{rank} M(V) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} = \operatorname{rank} B(V) = \operatorname{rank} Q(V)$$

The knowledgeable reader will have already thought about the group schemes

$$G = G_{\lambda,\mu} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1-\mu x \end{pmatrix} \,\middle|\, x^2 = \lambda x \right\}$$

of order 2 (Tate-Oort 1970 [5]). Indeed, one may associate a module $M_G(V)$ directly to $V$ and a group scheme $G$ of order 2 (however I don't have yet a geometric interpretation of the construction).

Here are descriptions in basic cases:

- The case $\mu = 1$. Then $\lambda = 2$, $G = \mu_2$ and $M_{2,1}(V) = B(V)$.
- The case $\lambda = 1$. Then $\mu = 2$, $G = \mathbf{Z}/2\mathbf{Z}$ and $M_{1,2}(V) = Q(V)$.
- The case $\lambda = \mu = 0$. Here we are in characteristic 2. The group scheme is the kernel of the Frobenius

$$G = \alpha_2 = \ker\left(\mathbf{G}_{\mathrm{a}} \xrightarrow{x \mapsto x^2} \mathbf{G}_{\mathrm{a}}\right)$$

on the additive group. For the module one has

$$M_{0,0}(V) = W^{(2)} \oplus \Lambda^2 W$$

where $W = V^\vee$ is the dual of $V$,

$$W^{(2)} = \varphi_*(W) = W \otimes_{R,\varphi} R$$

is the Frobenius image of $W$ (with $\varphi(a) = a^2$ used as change of base rings) and $\Lambda^2 W$ is the second exterior power of $W$ (the module of alternating 2-forms on $V$).

What started as a little game out of curiosity, opened a huge box: What about generalizing most of quadratic form theory to $G$-twisted forms, i.e., elements $f \in M(V)$?

This exposition[1] starts with some obvious preliminaries (like the generalized orthogonal group and the group of similarities). Then we present for odd $n$ the generalized half-determinant of a form $f \in M(V)$. Moreover, there is (so far only for $n = 2$) a homomorphism

$$D\colon \mathrm{O}(f) \to G$$

generalizing the Dickson invariant. (I believe that this generalizes to all even $n$.)

[1] Not yet finished!

## Appendix to the introduction

The idea of considering a family of modules $M$ between $B$ and $Q$ appeared when looking at the classical topic of conic sections (ellipses, hyperbolas, etc.).

It seems worthwhile to try to present the theory of say ellipses (with its focus points, directrix and Dandelin spheres) over more general base rings $R$ than just the real numbers $\mathbf{R}$. I always found the standard expositions as too "real"—one never knows where exactly the ordering of $\mathbf{R}$ is needed. How to describe the focus points algebraically? Do they form an extension of degree 2 or (together with the pair of imaginary focus points on the minor axis) an extension of degree 4? Here another fascinating topic is: The (complex) square of a central ellipse is again an ellipse (Arnold 1990 [1, Appendix 1, p. 95–96])

One may expect that the classical considerations over $\mathbf{R}$ extend to any ring $R$ with 2 invertible. However one immediately runs into the following question: An ellipse is given by two binary quadratic forms $p$ (the Euclidean metric) and $q$ (the leading part of the polynomial describing the ellipse). The two axis of the ellipse correspond to a common diagonalization of $p$ and $q$ and are given by, well, not by another quadratic form, but by a symmetric bilinear form, namely

$$p \wedge q \in \Lambda^2 Q(V) = B(V) \otimes \Lambda^2 V$$

(here $\operatorname{rank} V = 2$, thus $\operatorname{rank} B(V) = \operatorname{rank} Q(V) = 3$). On the other hand, if we start with symmetric bilinear forms $p$, $q$, then we get a quadratic form:

$$p \wedge q \in \Lambda^2 B(V) = Q(V) \otimes \Lambda^2 V$$

Which one is it? Even if one assumes that 2 is invertible, it is clear that the (equivalent) presentations in the two settings will look different (with formulas having 2's at different places). At the moment I tend to the second setting. About the degeneration to characteristic 2 I have no idea, maybe that doesn't lead to anything interesting.

## §4. Preliminaries

Nothing here yet.

## §5. The half-determinant

Nothing here yet.

## §6. The Dickson invariant (n = 2)

At least some formulas are here.

If $q$ is a non-degenerate quadratic form of even dimension, then for $g \in \mathrm{O}(q)$ there is the Dickson invariant

$$D(g) \in \mathbf{Z}/2\mathbf{Z}$$

often defined via the action on the center of the Clifford algebra. The image of $D(g)$ under $\mathbf{Z}/2\mathbf{Z} \to \mu_2$ is the determinant of $g$. Hence, if 2 is invertible, then

$$D(g) = \frac{1 - \det(g)}{2}$$

This gives a guide on how to try to define $D(g)$ for symmetries of a generalized form $f \in M_{\lambda,\mu}(V)$. So far I checked that this works fine in dimension 2. Here are the resulting formulas. We work with similitudes right away.

Let $V = R^2$. We present a symmetric bilinear form by its symmetric matrix, as usual. Quadratic forms are presented by upper triangular matrices (corresponding to a bilinear form representing the quadratic form).

Let

$$f = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\right] \in M_{\lambda,\mu}(V)$$

with $a$, $b$, $c \in R$.

There are the associated bilinear form

$$h_f = \begin{pmatrix} \mu a & c \\ c & \mu b \end{pmatrix}$$

and quadratic form

$$q_f = ax_1^2 + bx_2^2 + \lambda c x_1 x_2$$

With respect to the homomorphisms

$$\mathbf{Z}/2\mathbf{Z} \to G_{\lambda,\mu} \to \mu_2$$

one has $h_f \mapsto \mu f$ and $f \mapsto q_f$, respectively.

6.1. **The discriminant.** The (generalized) discriminant of $f \in M(V)$ is defined as

$$\operatorname{disc}(f) = -\det(h_f) = c^2 - \mu^2 ab$$

Note that

$$\operatorname{disc}(q_f) = \lambda^2 \operatorname{disc}(f) = (\lambda c)^2 - 4ab$$

(The discriminant of a quadratic form is up to sign the determinant of the associated bilinear form. The same happens for "forms" in $M(V)$).

6.2. **The action of** $\mathrm{GL}_2(R)$**.** Let

$$g = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{GL}_2(R)$$

Its action on $f = [h, q]$ can be read from the action on $B(V)$ and $Q(V)$:

$$\begin{aligned} g(h) = g^t h g &= \begin{pmatrix} x & z \\ y & t \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax^2 + by^2 & axy + bzt \\ axy + bzt & ay^2 + bt^2 \end{pmatrix} \\ g(q) &= \begin{pmatrix} x & z \\ y & t \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = c\begin{pmatrix} xz & xt \\ yz & yt \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} g(f) &= g\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} ax^2 + by^2 + \lambda cxz & 0 \\ 0 & ay^2 + bt^2 + \lambda cyt \end{pmatrix}, \begin{pmatrix} 0 & \mu(axy + bzt) + c(xt + yz) \\ 0 & 0 \end{pmatrix}\right] \end{aligned}$$

6.3. **The Dickson invariant.** Assume that $\operatorname{disc}(f)$ is invertible.

Let
$$g = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{GL}_2(R)$$
be a similitude of $f$:
$$g(f) = uf$$
for some (unique) $u \in \mathbf{G}_{\mathrm{m}}(R)$.

Then
$$\det(g)^2 = u^2$$
(as for usual binary quadratic forms) and we have the element
$$D'(g) = \frac{\det(g)}{u} \in \mu_2$$
(if $u = 1$ this is the determinant of $g \in \mathrm{O}(f)$).

We wish to lift $D'(g)$ with respect to
$$\begin{gathered} G_{\lambda,\mu} \to \mu_2 \\ \operatorname{Spec} R[x]/(x^2 - \lambda x) \to \operatorname{Spec} R[t]/(t^2 - 1) \\ x \mapsto 1 - \mu x \end{gathered}$$
Briefly, to do so, one takes
$$D(g) = \frac{1 - D'(g)}{\mu}$$
over the base ring $R_0[\mu^{-1}]$ with
$$R_0 = \mathbf{Z}[\lambda, \mu]/(\lambda\mu - 2)$$
and tries to get rid of the denominator $\mu$.

Since we assume that
$$\operatorname{disc}(f) = c^2 - \mu^2 ab$$
is invertible, locally at least one of $\mu$, $c$ is invertible.

So one has get rid of the denominator $\mu$ only when $c$ is invertible. It turns out that this is straightforward. One finds:

The (generalized) Dickson invariant of $g$ is
$$D(g) = \begin{cases} \dfrac{1 - D'(g)}{\mu} & \text{if } \mu \text{ is invertible} \\[2ex] -\dfrac{axy + bzt + \lambda cyz}{(xt - yz)c} & \text{if } c \text{ is invertible} \end{cases}$$
Note that the other denominator $xt - yz = \det(g)$ is invertible anyway.

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