FIELDS OF U-INVARIANT 9

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ABSTRACT. Let F be a field of characteristic $\neq 2$. The u-invariant of the field F is defined as the maximal dimension of anisotropic quadratic forms over F. It is well known that the u-invariant cannot be equal to 3, 5, or 7. We construct a field F with u-invariant 9. It is the first example of a field with odd u-invariant > 1. The proof uses the computation of the third Chow group of projective quadrics X_ϕ corresponding to quadratic forms ϕ. We compute $\mathrm{CH}^3(X_\phi)$ completely except for the case $\dim \phi = 8$. In our computation we use the results of B. Kahn, M. Rost, and R. Sujatha on the unramified cohomology and the third Chow group of quadrics ([KRS1]). We compute the unramified cohomology $H^4_{nr}(F(\phi)/F)$ for all forms of dimension ≥ 9. We apply our results to prove several conjectures. In particular, we prove a conjecture of Bruno Kahn on the classification of forms of height 2 and degree 3 for all fields of characteristic zero.

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0. Introduction

Let F be a field of characteristic $\neq 2$, and let $\phi = a_1x_1^2 + \cdots + a_nx_n^2$ be a quadratic form over F. We always assume that the forms under consideration are nondegenerate. The form ϕ is called isotropic if the homogeneous equation $a_1x_1^2 + \cdots + a_nx_n^2 = 0$ has a nontrivial solution. Otherwise, the form ϕ is called anisotropic. The u-invariant of the field F is defined as the maximal dimension of anisotropic quadratic forms over F:

$$u(F) = \sup\{\dim\phi \mid \phi \text{ is an anisotropic form over } F\}.$$

Since many questions about a quadratic form can be reduced to that about its anisotropic part, $u(F)$ is a fundamental measure of the complexity of quadratic form theory over F. However, $u(F)$ is often very difficult to determine, and its value is unknown for many fields. Nevertheless, in the following cases the u-invariant is known:

- If F is a quadratically closed field, then $u(F) = 1$. In particular, $u(\mathbb{C}) = 1$.
- If F is a formally real field, then $u(F) = \infty$. In particular, $u(\mathbb{Q}) = \infty$ and $u(\mathbb{R}) = \infty$.
- If F is a finite field, then $u(F) = 2$.
- If F is a local field, then $u(F) = 4$.
- If F is a nonreal global field, then $u(F) = 4$. In particular, $u(\mathbb{Q}(i)) = 4$.
- For any field F, we have $u(F((t_1))((t_2))\dots((t_n))) = 2^n u(F)$. In particular, $u(\mathbb{C}((t_1))((t_2))\dots((t_n))) = 2^n$.
- If $F = \mathbb{C}(t_1, \dots, t_n)$ denotes the field of rational functions in n variables over $\mathbb{C}$, then $u(F) = 2^n$.

Note that in all examples listed above the value of the u-invariant is always a power of 2 or infinite. This observation was probably a reason for the well-known conjecture of Kaplansky (1953) that only powers of 2 are possible for finite values of the u-invariant. It is known that Kaplansky's conjecture is true for finitely generated fields over algebraically closed or finite fields (see [Kah3]). More precisely, if F is such a field, then it is proved that $u(F) = 2^{cd(F)}$, where $cd(F)$ is the cohomological dimension of F. The problem concerning the u-invariant of finitely generated fields of transcendence degree $\neq 0$ over number or local fields seems to be very difficult. For example, the finiteness of the u-invariant is known only in the case when F is of transcendence degree 1 over a local nondyadic field (1998, [HvG] and [PS]). In a series of papers, R. Elam and T. Y. Lam studied the u-invariant for fields satisfying some additional hypotheses ([Elm, EL1, EL2]). In particular, Kaplansky's conjecture was proved for all linked fields.

In ([M2], 1989), A. Merkurev disproved this conjecture by constructing a field F with $u(F) = 6$ (see also [Lam2]). Later, he proved that the u-invariant of a field can be any even number ([M3], 1991). It should be noted that the basic idea of his proof (index reduction formula) cannot be used to construct a field with odd u-invariant. It was still an open problem whether the u-invariant can take odd values other than 1. However, it is known that the u-invariant never equals 3, 5, or 7 (see e.g., [Lam1, Prop. 4.8]). The principal result of this paper is the following

Theorem 0.1. *There exists a field F with $u(F) = 9$.*

In fact, we prove a more precise result. To formulate it, we introduce the following notion:

Definition 0.2. We say that ϕ is an *essential* 9-dimensional form if the following conditions hold:

- ϕ is an anisotropic 9-dimensional form;
- ϕ is not a Pfister neighbor;
- $\operatorname{ind} C_0(\phi) \geq 4$.

The role of this notion is illuminated by the following version of Theorem 0.1.

Theorem 0.3. *Let ϕ be a 9-dimensional quadratic form over a field F. Then the following conditions are equivalent:*

(1) *there exists a field extension E/F such that $u(E) = 9$ and the form ϕ_E is anisotropic,*
(2) *ϕ is an essential form.*

Moreover, if these conditions hold, we can construct a field E with the following additional properties:

- *all anisotropic 9-dimensional forms over E are similar to ϕ_E.*
- *E has no nontrivial odd extensions and $cd_2(E) = 3$.*

The basic tool for the proof of Theorem 0.1 is the following

Proposition 0.4. (cf. Theorem 7.3). *Let ϕ be an essential 9-dimensional quadratic form and ψ be a form of dimension ≥ 10. Then $\phi_{F(\psi)}$ is an essential form.*

Using this proposition we can construct a field with u-invariant 9 (cf. [M3, §3] and/or [M2]). Indeed, we start with an arbitrary field k_0 and consider the form $\phi = \langle t_1, \ldots, t_9 \rangle$ over the rational function field $k = k_0(t_1, \ldots, t_9)$. It is easy to show that the k-form ϕ is essential. By iterated passages to function fields of 10-dimensional quadratic forms, one can obtain a field F with no anisotropic 10-dimensional quadratic forms. By Proposition 0.4, the form ϕ_F is still essential. In particular, ϕ_F is an anisotropic 9-dimensional form. Hence, $u(F) = 9$.

Our proof of Proposition 0.4 (and Theorem 0.3) uses the following recent results of B. Kahn, M. Rost, R. Sujatha, and N. Karpenko:

- the results of B. Kahn, M. Rost, and R. Sujatha, concerning the unramified cohomology of quadrics [KRS1].
- a new result of N. Karpenko related to isotropy of 9-dimensional essential forms over the function fields of 9-dimensional forms (see Theorem 1.13).

An essential part of the proof of Proposition 0.4 is based on the computation of the third Chow group and the fourth unramified cohomology group of quadrics. We compute these groups completely for all quadrics of dimension ≥ 7 (i. e., in the case where the corresponding quadratic forms are of dimension ≥ 9).

Theorem 0.5. *Let ϕ be an F-form of dimension ≥ 9. Let $X = X_\phi$ be the projective quadric corresponding to the form ϕ. Then* $\operatorname{Tors}\operatorname{CH}^3(X) = 0$, *with the following exceptions, where* $\operatorname{Tors}\operatorname{CH}^3(X) \simeq \mathbb{Z}/2\mathbb{Z}$:

(9-a) $\phi = \pi \perp \langle d\rangle$, *where π is similar to an anisotropic 3-fold Pfister form (in this case,* $\operatorname{ind} C_0(\phi) = 1$*).*

(9-b) *ϕ is an anisotropic 9-dimensional form with the following properties:* $\operatorname{ind} C_0(\phi) = 2$, $\det\phi \notin D_F(\phi)$, *and ϕ contains no 7-dimensional Pfister neighbors.*

(10-a) $\phi = \pi \perp \mathbb{H}$, *where π is similar to an anisotropic 3-fold Pfister form (in this case, $\phi \in I^2(F)$ and* $\operatorname{ind} C(\phi) = 1$*).*

(10-b) *ϕ is an anisotropic 10-dimensional form such that $\phi \in I^2(F)$ and* $\operatorname{ind} C(\phi) = 2$.

(10-c) *ϕ is an anisotropic 10-dimensional form with nontrivial discriminant $d = d_\pm\phi \notin F^{*2}$ which is similar to a subform of an anisotropic 12-dimensional form $\tau \in I^3(F)$ and such that $\phi_{F(\sqrt{d})}$ is not hyperbolic (in this case,* $\operatorname{ind} C_0(\phi) = 1$*).*

(11-a) *ϕ is an anisotropic 11-dimensional form with* $\operatorname{ind} C_0(\phi) = 1$.

(12-a) *ϕ is an anisotropic 12-dimensional form belonging to $I^3(F)$ (in particular,* $\operatorname{ind} C(\phi) = 1$*).*

Remarks. 1) The statement that the group $\operatorname{Tors}\operatorname{CH}^3(X)$ is either zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ is due to N. Karpenko. He also proved that this group is trivial for all forms of dimension > 12 ([Kar2]).

2) In many cases the triviality of the group $\operatorname{Tors}\operatorname{CH}^2(X_\phi)$ was proved by B. Kahn and R. Sujatha in [KS3].

3) Theorem 0.5 together with the results of N. Karpenko [Kar1] complete the computation of the group $\operatorname{CH}^3(X_\phi)$ for all forms except for the case when ϕ is an 8-dimensional form with nontrivial discriminant.

Theorem 0.5 is closely related to the computation of the fourth unramified cohomology of quadrics. Let $H^n(F)$ be the Galois cohomology group of F with $\mathbb{Z}/2\mathbb{Z}$-coeficients. For a form ϕ, we denote by $\tilde{H}^4_{nr}(F(\phi)/F)$ the homology group of the complex

$$H^4(F) \to H^4(F(X_\phi)) \to \oplus_{x \in X_\phi^{(1)}} H^3(F(x)),$$

where X_ϕ is the projective quadric corresponding to ϕ. This group was studied in detail in [KRS1, KS2, KS3]. Among many other results, it was proved that if ϕ has dimension ≥ 9 and ϕ is not a 4-fold neighbor, then there exists an injective homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\phi)/F) \to \operatorname{Tors}\operatorname{CH}^3(X_\phi)$ (see [KRS1, Th. 6(1) and Prop.3]). The construction of this homomorphism fundamentally uses mixing the Hochschild-Serre and Bloch-Ogus spectral sequences. Another tool, originally due to Bloch, is an exact sequence that relates $\mathcal{H}$-cohomology and $\mathcal{K}$-cohomology (cf. [CT2, 3.6]). It is a very important question to ask when the homomorphism ϵ is an isomorphism.

Theorem 0.6. *Let ϕ be an anisotropic form of dimension ≥ 9 that is not a 4-fold Pfister neighbor. Then the homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\phi)/F) \to \operatorname{Tors}\operatorname{CH}^3(X_\phi)$ is an isomorphism.*

Remarks. 1) Under the hypothesis of the theorem, B. Kahn and R. Sujatha proved that the homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\phi)/F, \mathbb{Q}/\mathbb{Z}(3)) \to \operatorname{Tors}\operatorname{CH}^3(X_\phi)$ is an isomorphism ([KS3, Th. 1(b)]). Moreover, they proved Theorem 0.6 for fields containing all 2-primary roots of unity (see Corollary 1 in [KS3]).

2) Theorem 0.6 together with Theorem 0.5, complete the computation of the groups $\tilde{H}^4_{nr}(F(\phi)/F)$ for all forms ϕ of dimension ≥ 9. Here we note that in the case of 4-fold neighbors (which is excluded from the formulation of Theorem 0.6) the group $\tilde{H}^4_{nr}(F(\phi)/F)$ was computed in [KRS1, Cor. 8(3)].

The following easy corollary of Theorem 0.6 gives a partial answer to the question stated at the beginning of Subsection 1.1 of [KS2].

Corollary 0.7. (cf. [KS2, Theorem 1]). *Let ϕ be a form of dimension ≥ 9. Then the natural homomorphism $\tilde{H}^4_{nr}(F(\phi)/F) \to \tilde{H}^4_{nr}(F(\phi)/F, \mathbb{Q}/\mathbb{Z}(3))$ is surjective.*

In this paper we also prove the following conjectures.

Conjecture 0.8. ([Kah1, p.154]). *Let $\phi \in I^3(F)$ be an anisotropic form of dimension 12. Let ψ be a form of dimension > 12. Then $\phi_{F(\psi)}$ is anisotropic.*

Conjecture 0.9. ([Lag, Conj. 3]). *Let $\phi \in I^2(F)$ be an anisotropic form of dimension 10 with $\operatorname{ind} C(\phi) = 2$. Let ψ be a form of dimension > 10. Then $\phi_{F(\psi)}$ is anisotropic.*

Conjecture 0.10. ([Izh1, Conj. 2.1]). *Let ϕ be a 10-dimensional anisotropic form. Then ϕ has maximal splitting if and only if at least one of the following conditions holds:*

- *ϕ is a Pfister neighbor,*
- *$\phi \simeq \langle\langle a \rangle\rangle \otimes \tau$ with $\dim \tau = 5$.*

The proofs of all results mentioned above depend only on published papers or papers accepted for publication. However, our following result depends on the theory of Voevodsky related to the proof of Milnor's conjecture. Using some recent results announced by Alexander Vishik [Vi3, Vi4], we prove the following conjecture of Bruno Kahn in the case when $n = 3$ and $\operatorname{char} F = 0$:

Conjecture 0.11. ([Kah2, Conj. 7]). *Let ϕ be a nongood form of height 2 and degree n. Then ϕ is of the form $\tau \otimes q$, where τ is an $(n-2)$-fold Pfister form and q is an Albert form.*

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1. Quadratic forms

This section contains some preliminary results concerning quadratic forms. The basic notations are the same as in the books of T. Y. Lam and W. Scharlau ([Lam1], [Sch]). We recall some of them and introduce several additional notations.

Let ϕ be a quadratic form over F. We denote the Witt index of ϕ by $i_W(\phi)$. The anisotropic part of ϕ is denoted by ϕ_{an}. We say that ϕ is Witt equivalent to ψ if ϕ_{an} is isometric to ψ_{an}. We call two forms ϕ and ψ *stably birationally equivalent* if the corresponding quadrics X_ϕ and X_ψ are stably birationally equivalent. It is well known that the forms ϕ and ψ are stably birationally equivalent if and only if $\phi_{F(\psi)}$ and $\psi_{F(\phi)}$ are both isotropic (see, e.g., [Ohm, Sect. 3]). In this paper we deal with the following four equivalence relations on the set of quadratic forms:

$\phi \simeq \psi$ - isometry of the forms ϕ and ψ,

$\phi \sim \psi$ - similarity of the forms ϕ and ψ (i.e., $\phi \simeq k\psi$ for a suitable $k \in F^*$),

$\phi \overset{st}{\sim} \psi$ - stable rational equivalence of the forms ϕ and ψ,

$\phi = \psi$ - Witt equivalence of the forms ϕ and ψ.

The Clifford algebra (resp., the even part of the Clifford algebra) of ϕ will be denoted by $C(\phi)$ (resp., $C_0(\phi)$). We recall that if $\phi \in I^2(F)$, then $C_0(\phi)$ is a semisimple algebra of the form $A \times A$, where A is a central simple F-algebra. Moreover, in this case we have $C(\phi) = M_2(A)$.

For any form ϕ, we define F-algebra $C_0'(\phi)$ as follows:

- if $\dim \phi$ is odd, then $C_0'(\phi) = C_0(\phi)$. In this case, $C_0'(\phi)$ is a central simple F-algebra.
- if $\dim \phi$ is even and $d = d_\pm \phi \notin F^{*2}$, then $C_0'(\phi) = C_0(\phi)$. In this case, $C_0'(\phi)$ is a central simple L-algebra, where L/F is the *discriminant extension* (i.e., $L = F(\sqrt{d})$).
- if $\dim \phi$ is even and $d = d_\pm \phi \in F^{*2}$, then $C_0'(\phi) = A$, where A is a central simple F-algebra such that $C_0(A) = A \times A$.

By definition, $C_0'(\phi)$ is a simple algebra of degree $2^{[(\dim \phi - 1)/2]}$. By Wedderburn's theorem, there exists a division algebra D and an integer $s \geq 0$ such that

$C_0'(\phi) = M_{2^s}(D)$. We define Schur invariants $\operatorname{ind}\phi$ and $i_S(\phi)$ as follows [1]:

$$\operatorname{ind}\phi = \operatorname{ind} C_0'(\phi) = \deg D, \qquad i_S(\phi) = s$$

We say that $\operatorname{ind}\phi$ is the *Schur index* of ϕ. The integer $i_S(\phi)$ will be called the *Schur splitting index* of ϕ. Many properties of the Schur splitting index $i_S(\phi)$ are the same as the properties of the Witt index $i_W(\phi)$. For example, $i_S(\phi \perp m\mathbb{H}) = i_S(\phi) + m$ and $i_S(k\phi) = i_S(\phi)$. Moreover, $i_S(\phi) \geq i_W(\phi)$.

The following lemma is trivial

Lemma 1.1. *Let ϕ be a form over F.*

- *if* $\dim\phi = 2n+1$, *then* $\deg C_0'(\phi) = 2^{i_S(\phi)} \cdot \operatorname{ind}\phi = 2^n$,
- *if* $\dim\phi = 2n$, *then* $\deg C_0'(\phi) = 2^{i_S(\phi)} \cdot \operatorname{ind}\phi = 2^{n-1}$.

The following statement is well known:

Lemma 1.2. *Let E/F be either an odd extension or a unirational extension (i.e., a subfield of a purely transcendental extension). Then for any F-form ϕ, we have*

$$i_W(\phi_E) = i_W(\phi), \qquad \dim(\phi_E)_{an} = \dim\phi_{an},$$
$$i_S(\phi_E) = i_S(\phi), \quad \operatorname{ind}(\phi_E) = \operatorname{ind}(\phi).$$

In particular, the homomorphism $W(F) \to E(E)$ is injective. Moreover, the homomorphism $W(F)/I^n(F) \to W(E)/I^n(E)$ is injective for all n.

Theorem 1.3 ([Ti])**.** *Let F be a field and $F(t)$ be the field of rational functions in one variable over F.*

- *if A is a simple algebra over F, then* $\operatorname{ind} A_{F(t)} = \operatorname{ind} A$,
- *if A is a central simple algebra over F of exponent 2, and d is an element of F^* such that $d \notin F^{*2}$, then* $\operatorname{ind}(A_{F(t)} \otimes (d,t)) = 2\operatorname{ind} A_{F(\sqrt{d})}$.

Corollary 1.4. *Let F be a field and $F(t)$ be the field of rational functions in one variable over F. Let ϕ be a quadratic form over F, and let $\tilde{\phi} = \phi_{F(t)} \perp \langle kt\rangle$ with $k \in F^*$.*

- *if $\dim\phi$ is odd or $\phi \in I^2(F)$, then* $\operatorname{ind}\tilde{\phi} = \operatorname{ind}\phi$,
- *if $\dim\phi$ is even and $\phi \notin I^2(F)$, then* $\operatorname{ind}\tilde{\phi} = 2\operatorname{ind}\phi$.

Sketch of the proof. Using the formulas for Clifford algebras given in [Lam1], one can compute $C_0(\tilde{\phi})$ in terms of $C(\phi)$ or $C_0(\phi)$. After this, the required result follows easily from Tignol's Theorem 1.3. □

The following lemma is a trivial consequence of Merkurev's index reduction formula ([M3]).

Lemma 1.5. *Let ϕ and ψ be forms over F.*

- *If $\psi \in I^3(F)$, then* $\operatorname{ind}\phi_{F(\psi)} = \operatorname{ind}\phi$.
- *If* $\operatorname{ind}\phi \leq 8$ *and* $\dim\psi \geq 9$, *then* $\operatorname{ind}\phi_{F(\psi)} = \operatorname{ind}\phi$.
- *If* $\operatorname{ind}\phi \geq 4$ *and* $\dim\psi \geq 9$, *then* $\operatorname{ind}\phi_{F(\psi)} \geq 4$.

[1] In [H2] the author used the notation $i_S(\phi)$ for the index of $c(\phi) \in \operatorname{Br}(F)$.

We recall Cassels–Pfister subform theorem (which will be used in the sequel without any reference).

Theorem 1.6. [Sch, Ch.4, Th. 5.4(ii)] *Let ϕ be a nonhyperbolic form, and let ψ be a form such that $\phi_{F(\psi)}$ is hyperbolic. Then for any $k \in D_F(\phi) \cdot D_F(\psi)$, we have $k\psi \subset \phi$. In particular,* $\dim \psi \leq \dim \phi$.

In what follows, we use the following very specific consequence of this theorem.

Corollary 1.7. *Let ψ be a form over F, and let d be an element of F such that $d \notin F^{*2}$. Let K be*

- *either an odd extension of F,*
- *or a unirational extension of F,*
- *or $K = F(\psi)$, where ψ is a form with* $\dim \psi > \dim \phi$.

Then the form $\phi_{K(\sqrt{d})}$ is hyperbolic if and only if the form $\phi_{F(\sqrt{d})}$ is hyperbolic.

Proof. If K/F is odd or unirational, then the extension $K(\sqrt{d})/F(\sqrt{d})$ is also odd or unirational. In these cases, the result follows immediately from Lemma 1.2. Now, we assume that $K = F(\psi)$ with $\dim \psi > \dim \phi$. Let $L = F(\sqrt{d})$. By Theorem 1.6, the form $\phi_{K(\sqrt{d})} \simeq \phi_{L(\psi)}$ is hyperbolic if and only if the form $\phi_{F(\sqrt{d})} \simeq \phi_L$ is hyperbolic. $\square$

The following theorem we will call *Pfister's theorem on 10-dimensional forms in I^3* (or simply Pfister's theorem).

Theorem 1.8. (see [Pf2, Satz 14 and Zautsats], also [Sch, Ch.2, Th.14.4] or [H3, Th. 2.9]). *Let $\phi \in I^3(F)$ and* $\dim \phi = 10$. *Then ϕ is isotropic and can be written in the form $\phi \simeq \pi \perp \mathbb{H}$ for some $\pi \in GP_3(F)$.*

Corollary 1.9. *Let $\phi \in I^3(F)$. We have*

(a) *if* $\dim \phi_{an} < 12$, *then there exists $\pi \in GP_3(F)$ such that ϕ and π are Witt equivalent,*
(b) *if* $\dim \phi = 12$, *then $i_W(\phi_{F(\phi)}) \geq 2$,*
(c) *if* $\dim \phi = 12$ *and ϕ is nonhyperbolic, then $(\phi_{F(\phi)})_{an} \in GP_3(F(\phi))$.*

The following theorems concern the so-called *forms with maximal splitting.*

Theorem 1.10 ([H1, Kah1, H2, Izh1])**.** *Let ϕ be an anisotropic form of dimension $2^n + m$, where $0 < m \leq 2^n$. Then*

- $1 \leq i_1(\phi) \leq m$ *(in the case where $i_1(\phi) = m$, we say that ϕ has maximal splitting),*
- *if ϕ is a Pfister neighbor, then $i_1(\phi) = m$ (i.e., ϕ has maximal splitting),*
- *if $n \geq 3$, $m \geq 2^n - 5$, and ϕ has maximal splitting, then ϕ is an $(n+1)$-fold Pfister neighbor,*
- *if $n \geq 3$, $m = 2^n - 6$, $\phi \in I^2(F)$, and ϕ has maximal splitting, then ϕ is an $(n+1)$-fold Pfister neighbor.*

The first two statements of this theorem (as well as the notion of forms with maximal splitting) are due to Detlev Hoffmann [H1]. The proofs of the third and fourth statements follow easily from the results of Bruno Kahn [Kah1, Remark after Th.4]. Besides, an elementary proof of the third statement can be found in [Izh1], and an elementary proof of the fourth statement is given in [H2] and [Izh1].

Theorem 1.11 ([H1, Izh4])**.** *Let ϕ be an anisotropic form with $2^n < \dim\phi \leq 2^{n+1}$. Suppose that ϕ has maximal splitting (e.g., $\dim\phi = 2^n+1$). Let ψ be a form of dimension $\geq 2^n+1$ such that $\phi_{F(\psi)}$ is isotropic. Then*

- *ψ has maximal splitting and $2^n < \dim\psi \leq 2^{n+1}$,*
- *$\psi_{F(\phi)}$ is isotropic (and hence $\phi \overset{st}{\sim} \psi$),*
- *if ψ is a Pfister neighbor, then ϕ is also a Pfister neighbor.*

The first and third statements of this theorem are due to D. Hoffmann [H1]. The second statement is proved in [Izh4]. Setting $n=3$ in Theorems 1.10 and 1.11, we get the following corollary.

Corollary 1.12. *Let ϕ be a form $9 \leq \dim\phi \leq 16$. Suppose that ϕ has maximal splitting (e.g., $\dim\phi = 9$), and ϕ is not a Pfister neighbor. Let ψ be a form of dimension ≥ 9 such that $\phi_{F(\psi)}$ is isotropic. Then*

- *$9 \leq \dim\phi, \dim\psi \leq 10$,*
- *ψ is not a Pfister neighbor and $\psi \notin I^2(F)$,*
- *$\psi \overset{st}{\sim} \phi$.*

The following result concerning the isotropy of 9-dimensional forms over function fields of quadrics has recently been proved by Nikita Karpenko.

Theorem 1.13. (N. Karpenko [Kar3]). *Let ϕ be a 9-dimensional essential form (see Definition 0.2). Let ψ be a form of dimension 9 such that $\phi_{F(\psi)}$ is isotropic. Then ψ is similar to ϕ.*

It should be pointed out that Theorem 1.13 plays a key role in the proof of our main Theorem 0.1. We also need the following statement from Karpenko's paper [Kar3].

Lemma 1.14. *Let ϕ be a 9-dimensional form which is not a Pfister neighbor. Then ϕ_L is not a Pfister neighbor for all odd extensions L/F.*

The following theorem was proved by A. Vishik by using calculations in Voevodsky's motivic category [Vi2]. In [Kar4], N. Karpenko simplified Vishik's proofs by using arguments in the framework of the classical category of Grothendieck Chow-motives and presented the current formulation of the theorem:

Theorem 1.15. *Let ϕ and ψ be anisotropic forms such that $\phi \overset{st}{\sim} \psi$. Then $\dim\phi - i_1(\phi) = \dim\psi - i_1(\psi)$.*

The rest of this section contains several lemmas concerning quadratic forms. These results will be used in other sections.

Lemma 1.16. *Let F be a field which has no nontrivial odd extensions. Let ϕ be an F-form with $\operatorname{ind}\phi \geq 2^s$. Then there exists a finite field extension E/F such that*

- $\operatorname{ind}\phi_E = 2^s$,
- *if $\dim\phi$ is even, then $\phi_E \in I^2(E)$,*
- *the homomorphism $N_{E/F} : K(C_0(\phi_E)) \to K(C_0(\phi))$ is surjective.* [2]

Proof. Let D be the simple division algebra corresponding to the algebra $C_0'(\phi)$. Let L be the center of D (in the cases when $\dim\phi$ is odd or $\phi \in I^2(F)$ we obviously have $L = F$). Let M/L be a maximal subfield of D. Clearly, $[M : L] = 2^r$ with $r = \operatorname{ind}\phi$. By our assumption, $[M : L] = 2^r \geq 2^s$. Since F has no odd extensions, Galois theory shows that there exists a tower of fields

$$M = M_0 \supset M_1 \supset \cdots \supset M_r = L$$

such that $[M_i : M_{i+1}] = 2$ for all i. Setting $E = M_s$, we get the field with the required properties. $\square$

Lemma 1.17. *Let L/F be a field extension, A be a central simple F-algebra of exponent 2, and m be an integer. Suppose that one of the following conditions holds:*

(i) $[L : F] \leq 2$, $\operatorname{ind} A_L = 1$, *and* $m = 2$;
(ii) $L = F$, $\operatorname{ind} A \leq 2$, *and* $m = 3$;
(iii) $[L : F] \leq 2$, $\operatorname{ind} A_L \leq 2$, *and* $m = 4$;
(iv) $L = F$, $\operatorname{ind} A \leq 4$, *and* $m = 5$.

Then there exists an m-dimensional form μ over F such that the algebra $C_0'(\mu)$ is Brauer equivalent to A_L.

Proof. (i) If $L = F$, we set $\mu = \mathbb{H}$, if $L = F(\sqrt{d})$, we set $\mu = \langle\langle d\rangle\rangle$.

(ii) Since $\operatorname{ind} A \leq 2$, it follows that A is Brauer equivalent to a quaternion algebra (a, b). Now, it suffices to set $\mu = \langle 1, -a, -b\rangle$.

(iii) If $L = F$, then $A_L = A$ is Brauer equivalent to a biquaternion algebra (a, b). In this case we set $\mu = \langle\langle a, b\rangle\rangle$. If $[L : F] = 2$, the condition $\operatorname{ind} A_L \leq 2$ implies that $\operatorname{ind} A \leq 4$. Therefore A is Brauer equivalent to a biquaternion algebra. Let q be an Albert form corresponding to A. Since $\operatorname{ind} A_L \leq 2$, the form q_L is isotropic. Hence q can be written in the form $q \simeq c\,\langle\langle d\rangle\rangle \perp q_0$, where d is such that $L = F(\sqrt{d})$. Now, it suffices to set $\mu = q_0$.

(iv) Let q be an Albert form corresponding to A. We define μ as an arbitrary 5-dimensional subform of q. $\square$

Lemma 1.18. *Let ϕ and ψ be F-forms such that $\dim\phi \equiv \dim\psi \pmod{2\mathbb{Z}}$ and $C_0'(\phi)$ is Brauer equivalent to $C_0'(\psi)$. Then there exists $k \in F^*$ such that $\phi \equiv k\psi \pmod{I^3(F)}$.*

Proof. If $\dim\phi$ and $\dim\psi$ are odd, it suffices to set $k = d_\pm\phi/d_\pm\psi$. Then $\gamma = \phi \perp -k\psi \in I^2(F)$ and $c(\gamma) = c(\phi) + c(\psi) = 0$. Therefore, $\gamma \in I^3(F)$. In

[2] Here $K(A)$ denotes the Grothendieck group of the ring A.

the case when $\dim\phi$ and $\dim\psi$ are even, the lemma is proved in [Izh2, Lemma 4.3] □

Lemma 1.19. *Let ϕ be a form satisfying one of the following conditions:*
(i) $\dim\phi = 2n$, *and* $\operatorname{ind}\phi = 1$,
(ii) $\dim\phi = 2n-1$ *and* $\operatorname{ind}\phi \le 2$,
(iii) $\dim\phi = 2n-2$ *and* $\operatorname{ind}\phi \le 2$,
(iv) $\dim\phi = 2n-3$ *and* $\operatorname{ind}\phi \le 4$.

Then there exists a $(2n+1)$-dimensional form $\tilde\phi$ and a $(2n+2)$-dimensional form $\gamma \in I^3(F)$ such that $\phi \subset \tilde\phi \subset \gamma$ and $\operatorname{ind}\tilde\phi = 1$.

Proof. Let $m = 2n+2-\dim\phi$. By Lemma 1.17, there exists an m-dimensional form μ such that $C_0'(\phi)$ is Brauer equivalent to $C_0'(\mu)$. Then there exists $k \in F^*$ such that $\phi \equiv k\mu \pmod{I^3(F)}$ (Lemma 1.18). We set $\gamma = \phi \perp -k\mu$. Now it suffices to set $\tilde\phi = \phi \perp -k\mu_0$, where μ_0 is a subform of μ of codimension 1. □

Lemma 1.20. (cf. [H1, Lemma 3]). *Let ψ be a form over F, and let $\phi \subset \psi$. If $\dim\phi \ge \dim\psi - i_W(\psi_{F(\psi)}) + 1$, then $\phi \overset{st}{\sim} \psi$.*

Proof. It suffices to verify that $\phi_{F(\psi)}$ is isotropic, but this is an obvious consequence of [H1, Lemma 3]. □

Lemma 1.21. (1) *Let γ be a 12-dimensional form from $I^3(F)$, and let ψ be an 11-dimensional subform of γ. Then $\psi \overset{st}{\sim} \gamma$.*

(2). *Let ψ be an 11-dimensional form with* $\operatorname{ind}\psi = 1$. *Then there exists a 12-dimensional form $\gamma \in I^3(F)$ such that $\psi \overset{st}{\sim} \gamma$.*

Proof. (1). Obvious in view of Lemma 1.20 and Corollary 1.9.

(2). We set $\gamma = \psi \perp \langle\det\psi\rangle$. Clearly $\gamma \in I^2(F)$ and $\operatorname{ind}\gamma = \operatorname{ind}\psi = 1$. Hence, $\gamma \in I^3(F)$. Item (1) shows that $\psi \overset{st}{\sim} \gamma$. □

Lemma 1.22. (see, e.g., [Izh4, Lemma 2.11]). *Let μ and ν be F-forms of dimension ≥ 1, and let $\phi = \mu \perp -t\nu$ be a form over $\tilde F = F(t)$. Then the extension $\tilde F(\phi)/F$ is purely transcendental.*

Lemma 1.23. *Let ϕ be an F-form and E/F be a unirational field extension. Then the form ϕ_E is a Pfister neighbor if and only if the form ϕ is a Pfister neighbor.*

Proof. It suffices to consider the case where E/F is purely transcendental. In this case lemma is proved in [H1, Prop.7]. □

Lemma 1.24. *Let ϕ be a 9-dimensional essential form over F (see Definition 0.2). Let E/F be either an odd extension or a unirational extension. Then ϕ_E is an essential form.*

Proof. By Springer's theorems, ϕ_E is anisotropic. By Lemma 1.2, $\operatorname{ind}\phi_E = \operatorname{ind}\phi \ge 4$. By Lemmas 1.14 and 1.23, ϕ_E is not a Pfister neighbor. □

At the end of this section, we prove the implication (1)⇒(2) in Theorem 0.3 (this is the simplest part of the theorem).

Proof. Let E be as in Item (1) of Theorem 0.3. We must prove that ϕ_E is essential. By our assumption, ϕ_E is anisotropic. Since $u(E) = 9$, it follows that all 4-fold Pfister forms over E are isotropic. Hence all 4-fold Pfister neighbors are also isotropic. Hence, ϕ_E is not a 4-fold Pfister neighbor. Now, it suffices to verify that $\operatorname{ind} \phi_E \geq 4$. Since all 10-dimensional forms over E are isotropic, it follows that all 9-dimensional E-form are universal (i.e., $D_E(\phi) = E^*$). In particular, $d = \det \phi \in D_E(\phi)$. Hence, ϕ_E can be written in the form $\phi_E = \tau \perp \langle d \rangle$, where τ is an 8-dimensional form over E with trivial discriminant. Since $\operatorname{ind} \tau = \operatorname{ind} \phi_E$, it suffices to prove that $\operatorname{ind} \tau \geq 4$. Assume the converse, $\operatorname{ind} \tau \leq 2$. Then there exists a quaternion E-algebra (a, b) such that $c(\tau) = (a, b)$. We have $\tau \equiv \langle\langle a, b \rangle\rangle \pmod{I^3(F)}$. Let $\gamma = d\tau \perp \langle\langle a, b \rangle\rangle$. Since $\dim \gamma = 12 > 9 = u(E)$, the form γ is isotropic. Therefore, the forms $(-d)\tau$ and $\langle\langle a, b \rangle\rangle$ have a nontrivial common value. Let $k \in E^*$ be such that $k \in D_E(-d\tau)$ and $k \in D_E(\langle\langle a, b \rangle\rangle)$. Set $\pi = \langle\langle k \rangle\rangle \otimes \tau = \tau \perp -k\tau$. Since $k \in D_E(-d\tau)$, it follows that that $d \in D_E(-k\tau)$, and hence $\phi_E = \tau \perp \langle d \rangle \subset \tau \perp -k\tau = \pi$.

Since $k \in D_E(\langle\langle a, b \rangle\rangle)$, it follows that $\langle\langle a, b, k \rangle\rangle = 0$. Therefore $\pi = \langle\langle k \rangle\rangle \otimes \tau \equiv \langle\langle k \rangle\rangle \otimes \langle\langle a, b \rangle\rangle = 0 \pmod{I^4(F)}$. Since $\dim \pi = 16$, we have $\pi \in GP_4(E)$. Since $\phi_E \subset \pi$, we see that ϕ_E is a Pfister neighbor. However, we have proved earlier that ϕ_E is not a Pfister neighbor, a contradiction. □

2. Graded Grothendieck groups of quadrics

For a smooth variety X we will denote by $K(X)$ the Grothendieck ring of X. This ring is supplied with the "topological" filtration $K(X)^{(i)}$ (which respects multiplication). The factor group $K(X)^{(i)}/K(X)^{(i+1)}$ is denoted by $G^i K(X)$. Thus, we get the adjoint graded ring $G^* K(X) = \oplus_i G^i K(X)$. There exists a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $G^* K(X)$.

Now, let $X = X_\phi$ be the projective quadric corresponding to the n-dimensional form ϕ. We consider X as a subvariety of codimension 1 in $\mathbb{P}^{n-1}$ (in particular, $\dim X_\phi = n - 2$). The embedding $X_\phi \subset \mathbb{P}^{n-1}$ determines the local free sheaf $\mathcal{O}_X(-1)$ on X. The group $K(X)^{(1)}$ contains the element $h := 1 - [\mathcal{O}_X(-1)]$. By $\bar{h}$, we denote the class of the element h in the group $G^1 K(X)$. It is well known that $\bar{h} \in G^1 K(X)$ corresponds the "hyperplane" class in $\mathrm{CH}^1(X)$ via the natural isomorphism $\mathrm{CH}^1(X) \simeq G^1 K(X)$. For all $i = 0, \ldots, \dim X = \dim \phi - 2$ the homomorphism $\mathbb{Z} \to G^i K(X)$, $1 \to \bar{h}^i$ is injective. The image of this homomorphism will be denoted by $\bar{h}^i \mathbb{Z}$.

Let us recall some basic properties of the group $K(X_\phi)$ (see [Kar1]). Let $s = i_S(\phi)$ (i.e., $C_0'(\phi)$ has the form $M_{2^s}(D)$, where D is a division algebra). Clearly, $s \leq \frac{1}{2} \dim X$. If $s = 0$, then the group $K(X)$ is generated (as a free Abelian group) by the elements h^i, where $i = 0, \ldots, \dim X$. If $s > 0$, the group $K(X)$ contains elements $l_0, \ldots, l_{s-1}$ such that $2^{i+1} l_i = h^d + 2h^{d-1} + \cdots + 2^i h^{d-i}$, where $d = \dim X$. Since the group $K(X_\phi)$ is torsion free, the elements $l_0, \ldots, l_{s-1}$

are uniquely defined. There exists a convenient geometric characterization of the elements l_i. Namely, l_i coincides with the "class of an i-dimensional line". More formally, this means that the image of the element l_i under the push-forward homomorphism $K(X) \to K(\mathbb{P}^{n-1})$ coincides with the image of the unit $1 = [\mathcal{O}_{\mathbb{P}^i}]$ under the push-forward homomorphism $K(\mathbb{P}^i) \to K(\mathbb{P}^{n-1})$ induced by the natural embedding $\mathbb{P}^i \subset \mathbb{P}^{n-1}$.

Theorem 2.1. (N. Karpenko, [Kar1, Th. 3.8 and 3.10] [3]). *Let ϕ be a quadratic form over F. Let X be the quadric corresponding to ϕ. Let i be an arbitrary integer such that $0 \le i < \dim X = \dim\phi - 2$.*

(1) *if $i < \frac{1}{2}\dim X = \frac{1}{2}\dim\phi - 1$, then $G^iK(X) = \operatorname{Tors} G^iK(X) \oplus \bar{h}^i\mathbb{Z}$,*
(2) *if $\phi \notin I^2(F)$ and $l_{i_S(\phi)-1} \in K(X)^{(i+1)}$, then*
$$\operatorname{Tors} G^0K(X) = \cdots = \operatorname{Tors} G^iK(X) = 0.$$
(3) *if $i < \frac{1}{2}\dim\phi - 1$ and $\operatorname{Tors} G^0K(X) = \cdots = \operatorname{Tors} G^iK(X) = 0$, then $l_{i_S(\phi)-1} \in K(X)^{(i+1)}$.*

Lemma 2.2. *Let $\phi \subset \tilde{\phi}$ be quadratic forms such that $\phi, \tilde{\phi} \notin I^2(F)$. Let $m = \dim\tilde{\phi} - \dim\phi$. Let p be an integer such that $0 \le p < \frac{1}{2}\dim\phi - 1$.*

(1) *if $i_S(\tilde{\phi}) = i_S(\phi)$ and $\operatorname{Tors} G^iK(X_\phi) = 0$ for all $i \le p$, then $\operatorname{Tors} G^iK(X_{\tilde{\phi}}) = 0$ for all $i \le p + m$.*
(2) *if $i_S(\tilde{\phi}) = i_S(\phi) + m$ and $\operatorname{Tors} G^iK(X_{\tilde{\phi}}) = 0$ for all $i \le p$, then $\operatorname{Tors} G^iK(X_\phi) = 0$ for all $i \le p$.*

Proof. Let $s = i_S(\phi)$ and $\tilde{s} = i_S(\tilde{\phi})$.

(1) Theorem 2.1(3) shows that $l_{s-1} \in K(X_\phi)^{(p+1)}$. Let us consider the push-forward homomorphism $K(X_\phi) \to K(X_{\tilde{\phi}})$. This homomorphism maps $K(X_\phi)^{(i)}$ to $K(X_{\tilde{\phi}})^{(i+m)}$ for all i, and maps l_j to l_j for all $j \le s-1$. Taking the image of l_{s-1} under this homomorphism, we get $l_{s-1} \in K(X_{\tilde{\phi}})^{(p+1+m)}$. Since $\tilde{s} = s$, Theorem 2.1(2) shows that $\operatorname{Tors} G^iK(X_{\tilde{\phi}}) = 0$ for all $i \le p + m$.

(2) Theorem 2.1(3) shows that $l_{\tilde{s}-1} \in K(X_{\tilde{\phi}})^{(p+1)}$. Let us consider the pull-back homomorphism $K(X_{\tilde{\phi}}) \to K(X_\phi)$. This homomorphism maps $K(X_{\tilde{\phi}})^{(i)}$ to $K(X_\phi)^{(i)}$ for all i, and maps l_j to l_{j-m} for all $j = m, \ldots, \tilde{s}-1$. Taking the image of $l_{\tilde{s}-1}$ under this homomorphism, we get $l_{\tilde{s}-1-m} \in K(X_\phi)^{(p+1)}$. Since $\tilde{s} - m = s$, Theorem 2.1(2) shows that $\operatorname{Tors} G^iK(X_\phi) = 0$ for all $i \le p$. ☐

Corollary 2.3. *Let $\phi \subset \tilde{\phi}$ be odd-dimensional forms. Let k be an integer such that $\dim\tilde{\phi} = \dim\phi + 2k$ and $\operatorname{ind}\tilde{\phi} = \frac{1}{2^k}\operatorname{ind}\phi$. Let p be an integer such that $0 \le p < \frac{1}{2}\dim\phi - 1$. Suppose also that $\operatorname{Tors} G^iK(X_{\tilde{\phi}}) = 0$ for all $i \le p$. Then $\operatorname{Tors} G^iK(X_\phi) = 0$ for all $i \le p$.*

[3]Actually, Theorems 3.8 and 3.10 from [Kar1] are proved only for anisotropic quadrics. However, the proof of three statements included in the formulation of Theorem 2.1 does not use anything specific from the anisotropic case.

Proof. By Lemma 2.2(2), it suffices to prove that $i_S(\tilde{\phi}) = i_S(\phi) + 2k$. Applying Lemma 1.1, we get $2^{(\dim\tilde{\phi}-\dim\phi)/2} = 2^{i_S(\tilde{\phi})-i_S(\phi)} \cdot \operatorname{ind}\tilde{\phi}/\operatorname{ind}\phi$. Since $\dim\tilde{\phi} - \dim\phi = 2k$, and $2^k \operatorname{ind}\tilde{\phi} = \operatorname{ind}\phi$, we have $i_S(\tilde{\phi}) - i_S(\phi) = 2k$. □

Corollary 2.4. *Let ϕ be an even-dimensional form with nontrivial discriminant and $\tilde{\phi}$ be an odd-dimensional form such that $\phi \subset \tilde{\phi}$. Let k be an integer such that $\dim\tilde{\phi} = \dim\phi + (2k+1)$ and $\operatorname{ind}\tilde{\phi} = \frac{1}{2^k}\operatorname{ind}\phi$. Let p be an integer such that $0 \le p < \frac{1}{2}\dim\phi - 1$. Suppose that $\operatorname{Tors} G^i K(X_{\tilde{\phi}}) = 0$ for all $i \le p$. Then $\operatorname{Tors} G^i K(X_\phi) = 0$ for all $i \le p$.*

Proof. By Lemma 2.2(2), it suffices to prove that $i_S(\tilde{\phi}) = i_S(\phi) + 2k + 1$. Let $\dim\phi = 2n$ and $\dim\tilde{\phi} = 2\tilde{n} + 1$. Since $\dim\tilde{\phi} - \dim\phi = 2k+1$, we have $\tilde{n} - n = k$. Applying Lemma 1.1, we get $2^{\tilde{n}-(n-1)} = 2^{i_S(\tilde{\phi})-i_S(\phi)} \cdot \operatorname{ind}\tilde{\phi}/\operatorname{ind}\phi$. Since $\tilde{n} - n = k$ and $2^k \operatorname{ind}\tilde{\phi} = \operatorname{ind}\phi$, we have $i_S(\tilde{\phi}) - i_S(\phi) = 2k+1$. □

Corollary 2.5. *Let $\phi = \phi_0 \perp \langle a\rangle$ be an even-dimensional form with nontrivial discriminant such that that $\operatorname{ind}\phi = \operatorname{ind}\phi_0$. Let p be an integer such that $1 \le p < \frac{1}{2}\dim\phi - 1$. Suppose that $\operatorname{Tors} G^i K(X_{\phi_0}) = 0$ for $i \le p-1$. Then $\operatorname{Tors} G^i K(X_\phi) = 0$ for $i \le p$.*

Proof. Since $\operatorname{ind}\phi = \operatorname{ind}\phi_0$, we have $i_S(\phi) = i_S(\phi_0)$. Clearly, $0 \le p-1 < \frac{1}{2}\dim\phi_0 - 1$. Now, Lemma 2.2(1) completes the proof. □

Proposition 2.6 (Karpenko)**.** *Let ϕ be an arbitrary quadratic form over F, and let E/F be a finite extension such that the norm map*

$$N_{E/F} : K(C_0(\phi_E)) \to K(C_0(\phi))$$

is surjective (e.g., E is a subfield of the division algebra corresponding to the simple algebra $C_0'(\phi)$). Let $p \ge 0$ be such that $G^i K(X_{\phi_E}) = \bar{h}^i\mathbb{Z}$ for all $i < p$. Then

- *$G^i K(X_\phi) = \bar{h}^i\mathbb{Z}$ for all $i < p$,*
- *the norm map $N_{E/F} : G^p K(C_0(\phi_E))/\bar{h}^p_E\mathbb{Z} \to G^p K(C_0(\phi))/\bar{h}^p\mathbb{Z}$ is surjective.*

Proof. The first statement coincides with Corollary 4.9 in [Kar2]. The proof of the second statement is the same as that of Corollary 4.9 in [Kar2]. □

Corollary 2.7. *Let m, s, and p be integers such that $p < m/2 - 1$. Suppose that for any field F and any F-form ρ satisfying the following conditions:*

- *ρ has dimension m,*
- $\operatorname{ind}\rho = s$,
- *if m is even, then $\rho \in I^2(F)$,*

we necessarily have $\operatorname{Tors} G^i K(X_\rho) = 0$ for all $i \le p$.

Then for any field F and any F-form ϕ satisfying two conditions:

- *ϕ has dimension m,*
- $\operatorname{ind}\phi \ge s$,

we necessarily have $\operatorname{Tors} G^i K(X_\phi) = 0$ for all $i \le p$.

Proof. Standard transfer arguments reduce the general case to the case where F has no nontrivial odd extensions. Let E/F be the extension constructed in Lemma 1.16. Applying Theorem 2.1(1) to the form ϕ_E, we have $G^iK(X_{\phi_E}) = \operatorname{Tors} G^iK(X_{\phi_E}) \oplus \bar{h}^i\mathbb{Z}$ for all $i \le p$. Applying the hypothesis of the corollary to the form $\rho = \phi_E$, we have $\operatorname{Tors} G^iK(X_{\phi_E}) = 0$ for $i \le p$. Hence, $G^iK(X_{\phi_E}) = \bar{h}^i\mathbb{Z}$ for all $i \le p$. By Proposition 2.6, we have $G^iK(X_\phi) = \bar{h}^i\mathbb{Z}$ for all $i \le p$. Therefore, $\operatorname{Tors} G^iK(X_\phi) = 0$ for $i \le p$. □

Proposition 2.8. *Let $n \ge 3$ and $p \ge 0$ be integers satisfying the following condition: for any $(2n+1)$-dimensional form τ over an arbitrary field F, the group $\operatorname{Tors} G^iK(X_\tau)$ is zero for all $i \le p$.*

Now, let ϕ be a form such that $p < \frac{1}{2}\dim\phi - 1$. Suppose also that ϕ satisfies one of the following conditions:

(i) $\dim\phi = 2n$, $d_\pm\phi \notin F^{*2}$, *and* $\operatorname{ind}\phi = 1$,
(ii) $\dim\phi = 2n-1$ *and* $\operatorname{ind}\phi \ge 2$,
(iii) $\dim\phi = 2n-2$, $d_\pm\phi \notin F^{*2}$, *and* $\operatorname{ind}\phi = 2$,
(vi) $\dim\phi = 2n-3$ *and* $\operatorname{ind}\phi \ge 4$.

Then $\operatorname{Tors} G^iK(X_\phi)$ *is zero for all* $i \le p$.

Proof. Corollary 2.7 shows that, instead of the cases (ii) and (iv), it suffices to consider their subcases in which we have

(ii') $\dim\phi = 2n-1$ and $\operatorname{ind}\phi = 2$,
(iv') $\dim\phi = 2n-3$ and $\operatorname{ind}\phi = 4$.

After this, Lemma 1.19 and Corollaries 2.3 and 2.4 complete the proof. □

3. Chow groups of quadrics

Our computation of the third Chow group of quadrics is based on the following assertion.

Theorem 3.1 (Karpenko, [Kar1, Kar2])**.** *Let ϕ be a quadratic form and X_ϕ be the projective quadric corresponding to ϕ. Then*

- *the homomorphism $\mathrm{CH}^i(X_\phi) \to G^iK(X_\phi)$ is an isomorphism for $i \le 3$.*
- $\operatorname{Tors}\mathrm{CH}^i(X_\phi) = \operatorname{Tors} G^iK(X_\phi) = 0$ *for* $i \le 1$.
- $\operatorname{Tors}\mathrm{CH}^2(X_\phi) = \operatorname{Tors} G^2K(X_\phi) = 0$ *except for the case when ϕ is an anisotropic 3-fold Pfister neighbor. If ϕ is an anisotropic 3-fold Pfister neighbor, then* $\operatorname{Tors}\mathrm{CH}^2(X_\phi) = \operatorname{Tors} G^2K(X_\phi) = \mathbb{Z}/2\mathbb{Z}$.
- $\operatorname{Tors}\mathrm{CH}^3(X_\phi)$ *is either* 0 *or* $\mathbb{Z}/2\mathbb{Z}$. *If* $\dim\phi > 12$, *then* $\operatorname{Tors}\mathrm{CH}^3(X_\phi) = 0$.

We will also use the following statement concerning the Chow groups of isotropic quadrics.

Lemma 3.2 ([Kar1])**.** *If $\phi = \psi \perp \mathbb{H}$, then $\operatorname{Tors}\mathrm{CH}^i(X_\phi) = \operatorname{Tors}\mathrm{CH}^{i-1}(X_\psi)$ for all i. If ϕ splits (i.e., $\dim\phi_{an} \le 1$), then $\operatorname{Tors}\mathrm{CH}^i(X_\phi) = 0$ for all i.*

Corollary 3.3. *Let ϕ be an isotropic form. Then the group $\operatorname{Tors}\mathrm{CH}^3(X_\phi)$ is nonzero only in the case when*

$$\phi \simeq (\textit{anisotropic 3-fold neighbor}) \perp \mathbb{H}.$$

In particular, this implies that $7 \leq \dim \phi \leq 10$*.*

Proof. Since ϕ is isotropic, we can write ϕ in the form $\phi \simeq \mu \perp \mathbb{H}$. By Lemma 3.2, $\operatorname{Tors} \operatorname{CH}^3(X_\phi) \simeq \operatorname{Tors} \operatorname{CH}^2(X_\mu)$. Theorem 3.1 completes the proof. ☐

Corollary 3.4. *Let* ϕ *be an isotropic form of dimension* ≥ 9*. Then the group* $\operatorname{Tors} \operatorname{CH}^3(X_\phi)$ *is nonzero only in the following two cases:*

- $\dim \phi = 10$ *and* $\phi \simeq \pi \perp \mathbb{H}$*, where* π *is similar to an anisotropic 3-fold Pfister form,*
- $\dim \phi = 9$ *and* $\phi \simeq \mu \perp \mathbb{H}$*, where* μ *is an anisotropic 7-dimensional Pfister neighbor.*

Corollary 3.5. *Let* ϕ *be a quadratic form of dimension* ≥ 9*. Then*

- $\operatorname{Tors} G^i K(X_\phi) = \operatorname{Tors} \operatorname{CH}^i(X_\phi) = 0$ *for* $i \leq 2$*,*
- $\operatorname{Tors} G^3 K(X_\phi) = \operatorname{Tors} \operatorname{CH}^3(X_\phi)$*;*
- *If* $\dim \phi > 12$*, then* $\operatorname{Tors} G^3 K(X_\phi) = 0$*;*

Remark 3.6. In the remaining part of this section we mostly work with forms of dimension ≥ 9. On the other hand, we are interested in the group $\operatorname{Tors} \operatorname{CH}^i(X_\phi)$ only in the case when $i \leq 3$. In this case, the condition $i < \frac{1}{2}\dim \phi - 1$ obviously holds. This shows that we can use all results of the previous section.

Now, we can prove our first result concerning the third Chow group of quadrics.

Proposition 3.7. *Let* ϕ *be a form satisfying one of the following conditions:*

(i) $\dim \phi = 12$*,* $d_\pm \phi \notin F^{*2}$*, and* $\operatorname{ind} \phi = 1$
(ii) $\dim \phi = 11$ *and* $\operatorname{ind} \phi \geq 2$*,*
(iii) $\dim \phi = 10$*,* $d_\pm \phi \notin F^{*2}$*, and* $\operatorname{ind} \phi = 2$*,*
(vi) $\dim \phi = 9$ *and* $\operatorname{ind} \phi \geq 4$*.*

Then $\operatorname{Tors} \operatorname{CH}^3(X_\phi) = 0$*.*

Proof. Let $n = 6$ and $p = 3$. By Corollary 3.5, we have $\operatorname{Tors} G^i K(X_\tau) = 0$ for $i \leq p = 3$ and all forms τ of dimension $13 = 2n + 1$. Applying Proposition 2.8, we see that $\operatorname{Tors} G^3 K(X_\phi) = 0$ for all quadratic forms satisfying conditions (i)–(iv). Since $\operatorname{CH}^3(X_\phi) \simeq G^3 K(X_\phi)$, the proof is complete. ☐

Lemma 3.8 (Karpenko)**.** *Let* ϕ *be an arbitrary quadratic form over* F *of dimension* ≥ 9*, and let* E/F *be a finite extension such that the norm map*

$$N_{E/F} : K(C_0(\phi_E)) \to K(C_0(\phi))$$

is surjective (e.g., E *may be any subfield of the division algebra corresponding to* $C_0'(\phi)$*). Then the homomorphism* $N_{E/F} : \operatorname{Tors} \operatorname{CH}^3(X_{\phi_E}) \to \operatorname{Tors} \operatorname{CH}^3(X_\phi)$ *is surjective.*

Proof. Theorem 2.1 shows that $\operatorname{Tors} G^i K(X_\phi) \simeq G^i K(X_\phi)/\bar{h}^i \mathbb{Z}$ for all $i \leq 3$. By Corollary 3.5, we have $\operatorname{Tors} G^i K(X_\phi) = 0$ for $i \leq 2$. Finally, Proposition 2.6 together with the second item of Corollary 3.5 complete the proof. ☐

Corollary 3.9. *Let ϕ be an even-dimensional form of dimension > 8. Suppose that $d = d_\pm\phi \notin F^{*2}$ and put $L = F(\sqrt{d})$. Then the norm homomorphism $N_{L/F} : \operatorname{Tors}\operatorname{CH}^3(X_{\phi_L}) \to \operatorname{Tors}\operatorname{CH}^3(X_\phi)$ is surjective.*

Corollary 3.10. *Let ϕ be a form of even dimension > 8. Suppose that $d = d_\pm\phi \notin F^{*2}$ and $\phi_{F(\sqrt{d})}$ is hyperbolic. Then $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$.*

Proof. Let $L = F(\sqrt{d})$. By Lemma 3.2, we have $\operatorname{Tors}\operatorname{CH}^3(X_{\phi_L}) = 0$. Corollary 3.9 implies that $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$. □

Lemma 3.11. *Let $d \notin F^{*2}$ and ϕ be a 10-dimensional form with discriminant d. Suppose that ϕ has the form $\phi = \tau \perp c\,\langle\langle d\rangle\rangle$, where $\tau \in I^2(F)$ (i.e., $\phi_{F(\sqrt{d})}$ is isotropic). Then $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$ except possibly when*

$$\operatorname{ind}\phi = \operatorname{ind}\tau_{F(\sqrt{d})} = 1, \qquad \operatorname{ind}\tau = 2, \qquad \phi_{F(\sqrt{d})} \text{ is not hyperbolic.}$$

Proof. Suppose that $\operatorname{Tors}\operatorname{CH}^3(X_\phi) \neq 0$. Let $L = F(\sqrt{d})$. By Lemma 3.9, we have $\operatorname{Tors}\operatorname{CH}^3(X_{\phi_L}) \neq 0$. Since $\phi_L = \tau_L \perp \mathbb{H}$, Corollary 3.3 implies that τ_L is an anisotropic 8-dimensional Pfister neighbor. In particular, this means that ϕ_L is not hyperbolic and $\operatorname{ind}\phi = \operatorname{ind}\tau_L = 1$.

Now, it suffices to verify that $\operatorname{ind}\tau = 2$. Since $\operatorname{ind}\tau_L = 1$, we obviously have $\operatorname{ind}\tau \leq 2$. Assume that $\operatorname{ind}\tau = 1$ (i.e., $\tau \in GP_3(F)$). Let $\phi_0 = \tau \perp \langle c\rangle \subset \phi$. Clearly, $\operatorname{ind}\phi_0 = \operatorname{ind}\tau = 1 = \operatorname{ind}\phi$. Since $\dim\phi_0 = 9 > 8$, it follows that $\operatorname{Tors} G^iK(X_{\phi_0}) = 0$ for $i \leq 2$. By Corollary 2.5, we get $\operatorname{Tors} G^iK(X_\phi) = 0$ for $i \leq 3$. Hence, $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = \operatorname{Tors} G^3K(X_\phi) = 0$. We get a contradiction to our assumption. Hence $\operatorname{ind}\tau = 2$. □

Lemma 3.12. *Let ϕ be a 9-dimensional form such that $\operatorname{ind}\phi \neq 1$. Suppose that ϕ has one of the following forms:*

(i) *$\phi = \gamma \perp \langle u, v\rangle$, where γ is a 7-dimensional Pfister neighbor,*
(ii) *$\phi = \tau \perp \langle d\rangle$, where $\tau \subset I^2(F)$.*

Then $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$.

Proof. The case $\operatorname{ind}\phi \geq 4$ was considered in Proposition 3.7. Thus, we can assume that $\operatorname{ind}\phi = 2$.

(i) Since $\operatorname{ind}\phi = 2$ and $\dim\phi = 9$, Lemma 1.1 shows that $i_S(\phi) = 3$. Since γ is a 7-dimensional Pfister neighbor, we have $\operatorname{ind}\gamma = 1$. By Lemma 1.1, we get $i_S(\gamma) = 3$. By Theorem 3.1, we have $\operatorname{Tors} G^iK(X_\gamma) = 0$ for $i \leq 1$. Lemma 2.2(1) shows that $\operatorname{Tors} G^iK(X_\phi) = 0$ for $i \leq 3$. Thus, $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = \operatorname{Tors} G^3K(X_\phi) = 0$.

(ii) Since $\tau \in I^2(F)$ and $\operatorname{ind}\tau = \operatorname{ind}\phi = 2$, we can write τ in the form $\phi = \langle\langle a\rangle\rangle \otimes \rho$ with $\dim\rho = 4$. Let $L = F(\sqrt{a})$. Since ϕ_L splits, we have $\operatorname{Tors}\operatorname{CH}^i(X_{\phi_L}) = 0$ for all i. By Lemma 3.8, the homomorphism $N_{L/F} : \operatorname{Tors}\operatorname{CH}^3(X_{\phi_L}) \to \operatorname{Tors}\operatorname{CH}^3(X_\phi)$ is surjective. Hence $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$. □

Lemma 3.13. *Let F be a field such that all 14-dimensional forms from $I^3(F)$ are isotropic. Let ϕ be a form over F satisfying one of the following conditions:*

- *$\dim\phi = 10$, $\phi \in I^2(F)$ and $\operatorname{ind}\phi = 4$,*

- $\dim\phi = 12$, $\phi \in I^2(F)$ *and* $\operatorname{ind}\phi = 2$.

Then $\operatorname{Tors}\mathrm{CH}^3(X_\phi) = 0$

Proof. Let us define the form τ as follows: if $\dim\phi = 10$, then τ is an Albert form corresponding to $c(\phi)$; if $\dim\phi = 12$, then τ is the 2-fold Pfister form corresponding to $c(\phi)$. Let $k \in F^*$ be such that the form $\gamma := \phi \perp -k\tau$ is isotropic. By definition, we have $\dim\gamma = 16$ and $\gamma \in I^3(F)$. Since γ is isotropic, we obtain $\dim\gamma_{an} \leq 14$. By the hypothesis of the lemma, we have $\dim\gamma_{an} \leq 12$. Therefore ϕ and $k\tau$ contain a common subform of dimension 2. Hence, there exists a quadratic extension L/F such that ϕ_L and τ_L are isotropic. By Corollary 3.3, we have $\operatorname{Tors}\mathrm{CH}^3(X_{\phi_L}) = 0$. Since τ_L is isotropic, it follows that L is a subfield of the division algebra corresponding to $C_0'(\phi)$. Then Lemma 3.8 shows that the homomorphism $N_{L/F} : \operatorname{Tors}\mathrm{CH}^3(X_{\phi_L}) \to \operatorname{Tors}\mathrm{CH}^3(X_\phi)$ is surjective. Hence $\operatorname{Tors}\mathrm{CH}^3(X_\phi) = 0$. $\square$

4. Galois cohomology

Throughout the paper we use the notation $H^n(F)$ for the Galois cohomology

$$H^n(\mathrm{Gal}(F^{sep}/F), \mathbb{Z}/2\mathbb{Z})$$

of the field F. It is well known that $H^1(F) \simeq F^*/F^{*2}$. Thus, any element $a \in F^*$ determines an element of $H^1(F)$, which is denoted by (a). The cup-product $(a_1) \cup \cdots \cup (a_n) \in H^n(F)$ is denoted by $(a_1, \ldots, a_n)$. It is well known that there exists a well-defined map

$$P_n(F) \to H^n(F), \qquad \langle\langle a_1, \ldots, a_n\rangle\rangle \mapsto (a_1, \ldots, a_n).$$

If $n \leq 4$, this map yields the homomorphism ([Ara, JR, Szy]):

$$e^n : I^n(F)/I^{n+1}(F) \to H^n(F).$$

Recently, Orlov-Vishik-Voevodsky have announced deep results concerning existence and bijectivity of e^n for arbitrary n, but in our paper, we need only old (already published) results concerning e^n. We list these results in the following theorem ([Ara, AEJ2, M1, MS, R1, Szy, JR]).

Theorem 4.1. *Let F be a field, and $\pi, \pi_1, \pi_2 \in P_n(F)$, where $n \leq 4$.*

- *The form π is isotropic (and hence hyperbolic) if and only if $e^n(\pi) = 0$.*
- *If $e^n(\pi_1) = e^n(\pi_2)$, then $\pi_1 \simeq \pi_2$.*
- *If $n \leq 3$, then the homomorphism $e^n : I^n(F)/I^{n+1}(F) \to H^n(F)$ is an isomorphism.*

Corollary 4.2. *Let $n \leq 3$, and let E/F be a field extension such that the homomorphism $H^i(F) \to H^i(E)$ is injective for $i = 0, \ldots, n$. Then the homomorphism*

$$W(F)/I^{n+1}(F) \to W(E)/I^{n+1}(E)$$

is injective.

Definition 4.3. Let E/F be a field extension. By $H^n(E/F)$ we denote the kernel of the homomorphism $H^n(F) \to H^n(E)$.

The following theorem was proved by J. K. Arason in the case when $n \leq 3$ and proved by B. Kahn, M. Rost, and R. Sujatha, in the case $n = 4$.

Theorem 4.4 ([Ara, KRS1]). *Let ϕ be a form of dimension ≥ 9. Then*

- *for $n \leq 3$, the homomorphism $H^n(F) \to H^n(F(\phi))$ is injective (i.e., $H^n(F(\phi)/F) = 0$),*
- *if ϕ is not an anisotropic 4-fold Pfister neighbor, then the homomorphism $H^4(F) \to H^4 F(\phi))$ is injective (i.e., $H^4(F(\phi)/F) = 0$).*
- *If ϕ is a Pfister neighbor of a Pfister form $\pi = \langle\langle a, b, c, d\rangle\rangle$, then the group $H^4(F(\phi)/F)$ is generated by the element $e^4(\pi) = (a, b, c, d)$.*

Corollary 4.5. (cf. [KRS1, Cor.10]). *Let ϕ be a form of dimension ≥ 9. Then the homomorphism*

$$W(F)/I^n(F) \to W(F(\phi))/I^n(F(\phi))$$

is injective for all $n \leq 4$.

Proof. Obvious in view of Corollary 4.2 and Theorem 4.4. □

Proposition 4.6. ([AEJ1, AEJ2]). *Let F be a field satisfying two conditions: F has no nontrivial odd extensions and $I^4(F) = 0$. Then $H^4(F) = 0$ and $cd_2(F) \leq 3$.*

Lemma 4.7. *Let E/F be an arbitrary field extension and ψ be a 4-fold Pfister neighbor over F. Then $H^4(E(\psi)/F) = H^4(E/F) + H^4(F(\psi)/F)$.*

Proof. Clearly, $H^4(E(\psi)/F) \supset H^4(E/F) + H^4(F(\psi)/F)$. It suffices to verify that any element $u \in H^4(E(\psi)/F)$ belongs to $H^4(E/F) + H^4(F(\psi)/F)$. Let $\pi = \langle\langle a, b, c, d\rangle\rangle$ be the Pfister form associated with ψ. Since $u \in H^4(E(\psi)/F)$, we have $u_E \in H^4(E(\psi)/E)$. By Theorem 4.4, the group $H^4(E(\psi)/E)$ is generated by the element $(a, b, c, d)_E \in H^4(E)$. Let $m \in \mathbb{Z}$ be such that $u_E = m \cdot (a, b, c, d)_E$. We have $u - m \cdot (a, b, c, d) \in H^4(E/F)$. Since $m \cdot (a, b, c, d) \in H^4(F(\psi)/F)$, we get $u \in H^4(E/F) + H^n(F(\psi)/F)$. □

Corollary 4.8. *Let ψ be a 4-fold Pfister neighbor over F. Then for any form ϕ, we have $H^4(F(\phi, \psi)/F) = H^4(F(\phi)/F) + H^4(F(\psi)/F)$.*

Proof. It suffices to set $E = F(\phi)$ in the Lemma 4.7. □

Corollary 4.9. *Let ϕ and ψ be forms of dimension ≥ 9. Suppose that ϕ is not a 4-fold neighbor and ψ is a 4-fold Pfister neighbor. Then $H^4(F(\phi, \psi)/F) = H^4(F(\psi)/F)$.*

Proof. By Theorem 4.4, the group $H^4(F(\phi)/F)$ is zero. Hence, $H^4(F(\phi, \psi)/F) = H^4(F(\phi)/F) + H^4(F(\psi)/F) = H^4(F(\psi)/F)$. □

5. Unramified cohomology of quadrics

In this section we use the terminology and notation of [CT1] and [KRS1, KS2]. For a smooth variety X we define the *unramified cohomology* $H^n_{nr}(F(X)/F)$ and the unramified Witt ring $W_{nr}(F(X)/F)$ as follows:

$$H^n_{nr}(F(X)/F) := \ker(H^n(F(X)) \to \coprod_{x\in X^{(1)}} H^{n-1}(F(x))),$$
$$W_{nr}(F(X)/F) := \ker(W(F(X)) \to \coprod_{x\in X^{(1)}} W(F(x))).$$

Besides that, we set

$$I^n_{nr}(F(X)/F) := \ker(I^n(F(X)) \to \coprod_{x\in X^{(1)}} I^{n-1}(F(x))).$$

Clearly, $I^n_{nr}(F(X)/F) = I^n(F(X)) \cap W_{nr}(F(X)/F)$. For $n \leq 4$, the homomorphism $e^n : I^n(F(X)) \to H^n(F(X))$ determines the homomorphism $e^n : I^n_{nr}(F(X)/F) \to H^n_{nr}(F(X)/F)$.

Since the image of the homomorphism $H^n(F) \to H^n(F(X))$ belongs to $H^n_{nr}(F(X)/F)$, we get the homomorphism $\eta^n_{2,X} : H^n(F) \to H^n_{nr}(F(X)/F)$. The cokernel of this homomorphism will be denoted by $\tilde{H}^n_{nr}(F(X)/F)$:

$$\tilde{H}^n_{nr}(F(X)/F) := \operatorname{coker}(\eta^n_{2,X} : H^n(F) \to H^n_{nr}(F(X)/F)).$$

For a form ϕ, we set $\tilde{H}^n_{nr}(F(\phi)/F) := \tilde{H}^n_{nr}(F(X_\phi)/F)$, where X_ϕ is the projective quadric corresponding to the form ϕ.

The following statement is well known (see, e.g., [KRS1, Prop.2.5]).

Lemma 5.1. *If ϕ is an isotropic form over F, then $\tilde{H}^n_{nr}(F(\phi)/F) = 0$. If two forms ϕ_1 and ϕ_2 are stable rational equivalent (i.e., $\phi_1 \overset{st}{\sim} \phi_2$), then*

$$\tilde{H}^n_{nr}(F(\phi_1)/F) \simeq \tilde{H}^n_{nr}(F(\phi_2)/F).$$

To state the following theorem, we need one more notation:

$$H^n(F(X)/F)_0 = \{\alpha \in H^n(F(X)/F) \mid (-1) \cup \alpha = 0 \in H^{n+1}(F)\}.$$

We note, that $H^n(F(X)/F)_0$ is a subgroup of $H^n(F)$.

The essential part of the following theorem is contained in the paper of B. Kahn, M. Rost, and R. Sujatha [KRS1].

Theorem 5.2. *Let $X = X_\phi$ be the projective quadric corresponding to the quadratic form ϕ of dimension ≥ 9. Then*

(1) *There exists a natural exact sequence*

$$0 \longrightarrow H^4(F(X)/F)_0 \xrightarrow{\delta} \tilde{H}^4_{nr}(F(X)/F) \xrightarrow{\epsilon} \operatorname{Tors}\operatorname{CH}^3 X.$$

(2) *The orders* [4] *of the groups $H^4(F(X)/F)_0$, $\tilde{H}^4_{nr}(F(X)/F)$, and* $\operatorname{Tors}\operatorname{CH}^3 X$ *are at most 2.*

(3) *If ϕ is not a 4-fold Pfister neighbor, then $H^4(F(X)/F)_0 = 0$. This (in particular) means that the homomorphism*

$$\epsilon : \tilde{H}^4_{nr}(F(X)/F) \to \operatorname{Tors}\operatorname{CH}^3 X$$

is injective and $|\tilde{H}^4_{nr}(F(X)/F)| \leq |\operatorname{Tors}\operatorname{CH}^3 X| \leq 2$.

[4]Below we will use the notation $|A|$ for the order of a set A.

(4) *If ϕ is a 4-fold neighbor, then the homomorphism ϵ is zero and the homomorphism δ is an isomorphism*

$$\delta : H^4(F(X)/F)_0 \simeq \tilde{H}^4_{nr}(F(X)/F).$$

Proof. (1) This is a formal consequence of [KRS1, Prop.3 and Th.6(1)].

(3) The group $H^4(F(\phi)/F)$ is zero in view of Theorem 4.4. Hence, its subgroup $H^4(F(\phi)/F)_0$ is also zero. The rest of the statement is a formal consequence of the first item of this theorem and Theorem 3.1.

(4) By Theorem 4.4, the group $H^4(F(X)/F)_0$ is finite. By Item (1), it suffices to prove that $|\tilde{H}^4_{nr}(F(X)/F)| = |H^4(F(X)/F)_0|$. Since the groups $H^4(F(X)/F)_0$ and $\tilde{H}^4_{nr}(F(X)/F)$ are stably birational invariants of X, we can change the 4-fold Pfister neighbor ϕ by its associated 4-fold Pfister form. Thus, we can assume that $\dim \phi = 16$. In this case, we have $\operatorname{Tors} \mathrm{CH}^3(X_\phi) = 0$ in view of Theorem 3.1. By Item (1) of the theorem, we obtain that δ is an isomorphism, and hence $|\tilde{H}^4_{nr}(F(X)/F)| = |H^4(F(X)/F)_0|$.

(2) The groups $H^4(F(X)/F)_0$ and $\operatorname{Tors} \mathrm{CH}^4(X_\phi)$ have orders at most 2 in view of Theorems 4.4 and 3.1. After this, the statement concerning the group $\tilde{H}^4_{nr}(F(X)/F)$ follows readily from Items (3) and (4). □

Corollary 5.3. *Let ϕ be a form of dimension ≥ 9. Suppose that ϕ is not a 4-fold Pfister neighbor and $\tilde{H}^4_{nr}(F(\phi)/F) \neq 0$. Then*

$$\tilde{H}^4_{nr}(F(\phi)/F) \simeq \operatorname{Tors} \mathrm{CH}^3 X_\phi \simeq \mathbb{Z}/2\mathbb{Z}$$

and the homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\phi)/F) \to \operatorname{Tors} \mathrm{CH}^3 X_\phi$ is an isomorphism.

Proof. Obvious in view of Item (3) of Theorem 5.2. □

Corollary 5.4. ([KRS1, Cor. 8(3)(a)]). *If $\dim \phi > 12$ and ϕ is not a 4-fold neighbor, then $\tilde{H}^4_{nr}(F(\phi)/F) = 0$.*

Proof. Follows from Theorems 5.2(3) and Theorem 3.1. □

Corollary 5.5. *Let ψ be a 4-fold Pfister neighbor. Let E/F be an extension such that $H^4(E/F) = 0$ (for example, $E = F(\phi)$, where ϕ is a form of dimension ≥ 9 which is not a 4-fold neighbor).*

Then the homomorphism $\tilde{H}^4_{nr}(F(X_\psi)/F) \to \tilde{H}^4_{nr}(E(X_\psi)/E)$ is injective.

Proof. Since $H^4(F) \to H^4(E)$ is injective, it follows that the homomorphism $H^4(F(X_\psi)/F)_0 \to H^4(E(X_\psi)/E)_0$ is also injective (because $H^4(F(X_\psi)/F)_0$ is a subgroup of $H^4(F)$). Now the corollary follows from Theorem 5.2(4). □

6. Pfister neighbors over function fields

Definition 6.1. Let ϕ be a quadratic form over F. By $\mathrm{Pf}(\phi)$ we denote the form defined as follows:

- if ϕ is not a Pfister neighbor, then $\mathrm{Pf}(\phi) = 0$,
- if ϕ is a Pfister neighbor of a Pfister form π, we set $\mathrm{Pf}(\phi) = \pi$.

Lemma 6.2. *Let X be a smooth F-variety, and let ϕ be a quadratic form over F. Then* $\mathrm{Pf}(\phi_{F(X)}) \in W_{nr}(F(X)/F)$.

Proof. We can assume that $1 \in D_F(\phi)$. Let $\pi = \mathrm{Pf}(\phi_{F(X)})$. If $\pi = 0$, the statement is trivial. Hence we can assume that $\phi_{F(X)}$ is a Pfister neighbor of a Pfister form $\pi \in P_n(F(X))$. We must prove that $\delta_x(\pi) = 0$ for any $x \in X^{(1)}$. Since $\pi \in I^n(F(X))$, it follows that $\delta_x(\pi) \in I^{n-1}(F(x))$. Since $1 \in D_F(\phi)$, we have $\phi_{F(X)} \subset \pi$. Let ξ be an $F(X)$-form such that $\phi_{F(X)} \perp \xi = \pi$. Clearly, $\dim \xi < 2^{n-1}$. Since ϕ is defined over F, it follows that $\delta_x(\phi_{F(X)}) = 0$. Therefore, $\delta_x(\pi) = \delta_x(\xi)$. Since $\dim \xi < 2^{n-1}$ and $\delta_x(\xi) \in I^{n-1}(F(x))$, the Arason–Pfister Hauptsatz shows that $\delta_x(\pi) = 0$. Therefore, $\pi \in W_{nr}(F(X)/F)$. □

Corollary 6.3. *Let X be a smooth F-variety and ϕ be a quadratic form over F. Suppose that $\phi_{F(X)}$ is a Pfister neighbor of $\pi \in P_n(F(X))$, where $n \leq 4$. Then $e^n(\pi) \in H^n_{nr}(F(X)/F)$.*

Lemma 6.4. *Let ϕ be a quadratic form over F, and let $E = F(\phi)$. Let X be a smooth F-variety. Suppose that $\phi_{F(X)}$ is an anisotropic n-fold Pfister neighbor ($n \leq 4$). Then one of the following conditions holds:*

- *the kernel of the natural homomorphism*
$$i_{L/F} : \tilde{H}^n_{nr}(F(X)/F) \to \tilde{H}^n_{nr}(E(X)/E)$$
contains the nonzero element $\tilde{e}^n(\mathrm{Pf}(\phi_{F(X)}))$. In particular, $i_{L/F}$ is not injective and $\tilde{H}^n_{nr}(F(X)/F) \neq 0$.
- *$H^n(F(\phi, X)/F) \supsetneq H^n(F(X)/F)$. In particular, $H^n(F(\phi, X)/F) \neq 0$ and $H^n(F) \neq 0$.*

Proof. Let $\pi = \mathrm{Pf}(\phi_{F(X)})$. By Corollary 6.3, we have $e^n(\pi) \in H^n_{nr}(F(X)/F)$.

Case 1: the element $e^n(\pi)$ is not defined over F. In this case, the element $e^n(\pi)$ determines the nonzero element $\tilde{e}^n(\pi) \in \tilde{H}^n_{nr}(F(X)/F)$. In particular, $\tilde{H}^n_{nr}(F(X)/F) \neq 0$. Since $\phi_{F(X)}$ is a Pfister neighbor of π, we conclude that π is hyperbolic over the function field of $\phi_{F(X)}$. Since the function field of $\phi_{F(X)}$ coincides with $E(X)$, we see that the element $\tilde{e}^n(\pi)$ maps to zero under the homomorphism $i_{L/F} : \tilde{H}^n_{nr}(F(X)/F) \to \tilde{H}^n_{nr}(E(X)/E)$.

Case 2: the element $e^n(\pi) \in H^n(F(X))$ is defined over F. Let $\lambda \in H^n(F)$ be an element such that $e^n(\pi) = \lambda_{F(X)}$. Since π is anisotropic, it follows that $e^n(\pi) \neq 0$ (see Theorem 4.1). Hence $\lambda \notin H^n(F(X)/F)$. Since $\phi_{F(X)}$ is a subform of π, it follows that $\pi_{F(\phi,X)}$ is hyperbolic. Hence, $\lambda_{F(\phi,X)} = e^n(\pi_{F(\phi,X)}) = 0$. Therefore, $\lambda \in H^n(F(\phi, X)/F)$. Thus, we have proved that λ belongs to the group $H^n(F(\phi, X)/F)$ but does not belong to $H^n(F(X)/F)$. Since $H^n(F(\phi, X)/F) \supset H^n(F(X)/F)$, we obtain that $H^n(F(\phi, X)/F) \supsetneq H^n(F(X)/F)$. □

Corollary 6.5. *Let F be a field such that $H^4(F) = 0$. Let ψ be a form over F such that $\tilde{H}^4_{nr}(F(\psi)/F) = 0$. Then for any form ϕ over F, the form $\phi_{F(\psi)}$ is not an anisotropic 4-fold neighbor.* □

Corollary 6.6. *Let F be a field such that $H^4(F) = 0$, and let ψ be a form of dimension ≥ 9 over F. Suppose that* $\operatorname{Tors}\operatorname{CH}^3(X_\psi) = 0$ *(for example,* $\dim\psi > 12$*). Then for any form ϕ over F, the form $\phi_{F(\psi)}$ is not an anisotropic 4-fold neighbor.*

Proof. Since $H^4(F) = 0$, the form ψ is not an anisotropic 4-fold neighbor. By Theorem 5.2, we have $|\tilde{H}^4_{nr}(F(\psi)/F)| \leq |\operatorname{Tors}\operatorname{CH}^3(X_\psi)| = 0$. Hence, $\tilde{H}^4_{nr}(F(\psi)/F) = 0$. Now, the required result follows from Corollary 6.5. □

Lemma 6.7. *Let F be a field such that $H^4(F) = 0$. Let ϕ_1, ϕ_2, and ψ be forms of dimension ≥ 9 such that $(\phi_1)_{F(\psi)}$ and $(\phi_2)_{F(\psi)}$ are anisotropic 4-fold Pfister neighbors. Then $(\phi_1)_{F(\psi)} \overset{st}{\sim} (\phi_2)_{F(\psi)}$.*

Proof. Let $\pi_i = \operatorname{Pf}((\phi_i)_{F(\psi)})$ for $i = 1, 2$. By our assumption, the forms π_1 and π_2 are anisotropic. Hence, $e^4(\pi_1)$ and $e^4(\pi_2)$ are nonzero elements of the group $H^4_{nr}(F(\psi)/F)$. Since $|\tilde{H}^4_{nr}(F(\psi)/F)| \leq 2$ and $H^4(F) = 0$, we have $|H^4_{nr}(F(\psi)/F)| \leq 2$. Therefore, $e^4(\pi_1) = e^4(\pi_2)$. Thus $\pi_1 = \pi_2$, and so $(\phi_1)_{F(\psi)}$ and $(\phi_2)_{F(\psi)}$ are Pfister neighbors in the same Pfister form. Hence $(\phi_1)_{F(\psi)} \overset{st}{\sim} (\phi_2)_{F(\psi)}$. □

Lemma 6.8. *Let ϕ be an anisotropic form of dimension 9 which is not a Pfister neighbor. Let ψ be a 4-fold Pfister neighbor. Then $\phi_{F(\psi)}$ is anisotropic form which is not a Pfister neighbor.*

Proof. The form $\phi_{F(\psi)}$ is anisotropic in view of Corollary 1.12. Suppose that $\phi_{F(\psi)}$ is a Pfister neighbor. Let $X = X_\psi$ and $E = F(\phi)$. By Corollary 4.9, we have $H^4(F(\phi, X)/F) = H^4(F(X)/F)$. By Corollary 5.5, the homomorphism $\tilde{H}^4_{nr}(F(X)/F) \to \tilde{H}^4_{nr}(E(X)/E)$ is injective. We get a contradiction to the statement of Lemma 6.4. □

Corollary 6.9. *Let ϕ be an essential 9-dimensional form, and ψ be a 4-fold Pfister neighbor. Then $\phi_{F(\psi)}$ is essential.*

Proof. Follows from Lemma 6.8 and Lemma 1.5. □

Proposition 6.10. *For any field F there exists a field extension E/F with the following properties:*

(i) *E has no nontrivial odd extensions, $I^4(F) = 0$, and $H^4(F) = 0$ (in particular, $cd_2(F) \leq 3$),*
(ii) *for any F-form τ, we have* $\operatorname{ind}\tau_E = \operatorname{ind}\tau$,
(iii) *for any anisotropic 9-dimensional form ϕ which is not a Pfister neighbor, the form ϕ_E is also anisotropic and is not a Pfister neighbor.*
(iv) *for any essential 9-dimensional form ϕ over F, the form ϕ_E is also essential,*
(v) *the homomorphism $W(F)/I^n(F) \to W(E)/I^n(E)$ is injective for all $n \leq 4$,*
(vi) *for any 10-dimensional form ϕ with nontrivial discriminant d the form $\phi_{E(\sqrt{d})}$ is hyperbolic only if $\phi_{F(\sqrt{d})}$ is hyperbolic.*

Proof. Let us construct the fields

$$F_0 \subset F_1 \subset \cdots \subset F_{2i} \subset F_{2i+1} \subset \dots$$

as follows. First, we set $F_{-1} = F$.

If $n = 2i$, we define F_n as the maximal odd extension of F_{n-1}.

If $n = 2i+1$, we define F_n as the free composite of all fields $F_{n-1}(\psi)$, where ψ runs over all 4-fold Pfister forms over F_{n-1}.

Now, we set $E = \cup_{n\geq 0} F_n$. We claim that E satisfies all needed properties.

(i) By definition, E has no odd extensions. Clearly all 4-fold Pfister forms over E are isotropic. Hence $I^4(E) = 0$. Therefore $H^4(E) = 0$ and $cd_2(E) \leq 3$ (see Proposition 4.6.)

(ii) Follows from Lemmas 1.2 and 1.5;

(iii) Follows from Lemmas 1.14 and 6.8;

(iv) Follows from Lemma 1.24 and Corollary 6.9;

(v) Follows from Lemma 1.2 and Corollary 4.5;

(vi) Follows from Corollary 1.7. □

Remark 6.11. It is possible to include many additional properties of the field extension E/F in the formulation of Proposition 6.10. Here we point out only the following modification of property (iii) (which has the same proof): *Let ϕ be a 9-dimensional form, and let X be an F-variety such that $\phi_{F(X)}$ is anisotropic and is not a Pfister neighbor. Then $\phi_{E(X)}$ is also anisotropic and is not a Pfister neighbor.*

Lemma 6.12. *Let E/F be a field extension constructed [5] in Theorem 6.10. Let ϕ be a form with maximal splitting satisfying the condition $9 \leq \dim \phi \leq 16$. Then*

1. *If ϕ is anisotropic and is not a Pfister neighbor, then ϕ_E is also anisotropic and is not a Pfister neighbor.*
2. *If ψ is an F-form such that $\phi_{F(\psi)}$ is anisotropic and is not a Pfister neighbor, then $\phi_{E(\psi)}$ is also anisotropic and is not a Pfister neighbor.*

Proof. (1) Let ϕ_0 be a 9-dimensional subform of ϕ. By Theorem 1.11, $\phi_0 \overset{st}{\sim} \phi$ and hence $(\phi_0)_E \overset{st}{\sim} \phi_E$. Replacing ϕ by ϕ_0, we can assume that $\dim \phi = 9$. In this case the required statement coincides with Item (iii) of Theorem 6.10.

(2). Taking into account Remark 6.11, one can can give the same proof as for Item (1). □

Proposition 6.13. *Let ϕ be an anisotropic form of dimension 9 that is not a Pfister neighbor. Let ψ be a form of dimension > 12. Then $\phi_{F(\psi)}$ is an anisotropic form that is not a Pfister neighbor.*

Proof. In view of Proposition 6.10, we can assume that $H^4(F) = 0$. By Corollary 1.12, the form $\phi_{F(\psi)}$ is anisotropic. Now, the required result follows immediately from Corollary 6.6. □

[5]Here we mean not only the formulation of Theorem 6.10, but also the proof of this theorem. In all other statements we will refer the reader only to the formulation of Theorem 6.10.

Corollary 6.14. *Let ϕ be an anisotropic form and ψ be a form of dimension > 12. Suppose that $\phi_{F(\psi)}$ is an anisotropic 4-fold Pfister neighbor. Then ϕ is a 4-fold Pfister neighbor.*

Proof. Since $\phi_{F(\psi)}$ is an anisotropic 4-fold Pfister neighbor, it follows that $\phi_{F(\psi)}$ has maximal splitting and $9 \leq \dim\phi \leq 16$. By [H1, Lemma 5], the form ϕ also has maximal splitting. Let ϕ_0 be an arbitrary 9-dimensional subform of ϕ. Since $\phi_{F(\psi)}$ is a 4-fold Pfister neighbor, it follows that $(\phi_0)_{F(\psi)}$ is a 4-fold Pfister neighbor. By Proposition 6.13, ϕ_0 is a 4-fold neighbor. Since ϕ has maximal splitting, it follows that $(\phi_0)_{F(\phi)}$ is isotropic. Hence ϕ is a 4-fold neighbor. □

Remark 6.15. As it will be shown in the following sections, Proposition 6.13 and Corollary 6.14 cannot be generalized to the case of 12-dimensional forms ψ.

7. Construction of a field with u-invariant 9

In this section, we prove Theorems 0.1 and 0.3, and Conjecture 0.10. We start the proofs with two easy lemmas.

Lemma 7.1. *Let ϕ, ψ, and ψ_0 be forms over F such that $\psi_{F(\psi_0)}$ is isotropic (for example, $\psi_0 \subset \psi$). Suppose that $\phi_{F(\psi_0)}$ is an essential 9-dimensional form. Then $\phi_{F(\psi)}$ is also essential.*

Proof. Let $E = F(\psi_0)$. Since $\psi_E = \psi_{F(\psi_0)}$ is isotropic, it follows that the extension $E(\psi)/E$ is purely transcendental. Since $\phi_E = \phi_{F(\psi_0)}$ is essential and $E(\psi)/E$ is purely transcendental, Lemma 1.24 implies that $\phi_{E(\psi)}$ is essential. Since $F(\psi) \subset E(\psi)$, it follows that $\phi_{F(\psi)}$ is also essential. □

Lemma 7.2. *Let F be a field such that $H^4(F) = 0$, and let ϕ be an essential 9-dimensional F-form. Let ψ be a form of dimension ≥ 9 such that $\psi \not\sim \phi$ and* $\operatorname{Tors}\operatorname{CH}^3(X_\psi) = 0$. *Then $\phi_{F(\psi)}$ is an essential form.*

Proof. By Theorem 1.13, the form $\phi_{F(\psi)}$ is anisotropic. Lemma 1.5 shows that $\operatorname{ind}\phi_{F(\psi)} \geq 4$. By Corollary 6.6, the form $\phi_{F(\psi)}$ is not a Pfister neighbor. Therefore, $\phi_{F(\psi)}$ is an essential form. □

The following theorem is a basic tool in the proof of our main results concerning the u-invariant.

Theorem 7.3. *Let ϕ be an essential* 9*-dimensional form over F. Then for an F-form ψ of dimension ≥ 9, the following conditions are equivalent:*

(1) $\dim\psi = 9$ *and there exists $k \in F^*$ such that* $\psi \equiv k\phi \pmod{I^4(F)}$,
(2) *the form $\phi_{F(\psi)}$ is not essential.*

In particular, $\phi_{F(\psi)}$ is always essential if $\dim\psi \geq 10$.

Proof. (1)⇒(2). Let ψ and k be as in (1). Let ψ_0 be a 7-dimensional $F(\psi)$-form such that $\psi_{F(\psi)} = \psi_0 \perp \mathbb{H}$. Put $\pi = \phi_{F(\psi)} \perp -k\psi_0$. Clearly, $\dim\pi = 9+7 = 16$.

In the Witt ring $W(F(\psi))$, we have $\pi = \phi_{F(\psi)} - k\psi_0 = (\phi - k\psi)_{F(\psi)} \in I^4(F(\psi))$. By the Arason-Pfister Hauptsatz, $\pi \in GP_4(F(\psi))$. Since $\phi_{F(\psi)} \subset \pi$, the form $\phi_{F(\psi)}$ is a Pfister neighbor. Therefore, $\phi_{F(\psi)}$ is not an essential form.

(2)$\Rightarrow$(1). First of all, we introduce the following notation:

Definition 7.4. Let ϕ and ψ be forms over F. We will write $\phi \sim \psi \pmod{I^n(F)}$ if there exists $k \in F^*$ such that $\phi \equiv k\psi \pmod{I^n(F)}$. Otherwise, we write $\phi \not\sim \phi \pmod{I^n(F)}$.

Remark 7.5. In Definition 7.4 we do not assume that $\dim\phi = \dim\psi$.

Lemma 7.6. *Let ϕ be an odd-dimensional form and $n \geq 2$. Then for a form ψ the following conditions are equivalent:*
(i) $\psi \sim \phi \pmod{I^n(F)}$,
(ii) $\psi \equiv k\phi \pmod{I^n(F)}$, *where* $k = d_\pm\psi/d_\pm\phi$.

Proof. (i)$\Rightarrow$(ii). Let $x \in F^*$ be such that $\psi \equiv x\phi \pmod{I^n(F)}$. It suffices to verify that $x \equiv d_\pm\psi/d_\pm\phi \in F^*/F^{*2}$. Since $n \geq 1$ and $\dim\phi$ is odd, it follows that $\dim\psi$ is odd. Since $n \geq 2$, it follows that $d_\pm\psi \equiv d_\pm(x\phi) \in F^*/F^{*2}$. Hence, $x \equiv d_\pm\psi/d_\pm\phi \in F^*/F^{*2}$.

(ii)$\Rightarrow$(i). Obvious. □

Corollary 7.7. *Let ϕ be an odd-dimensional form and let $n \geq 2$. Let E/F be an extension such that the homomorphism $W(F)/I^n(F) \to W(E)/I^n(E)$ is injective. Then for any F-form ψ, the condition $\psi \not\sim \phi \pmod{I^n(F)}$ implies that $\psi_E \not\sim \phi_E \pmod{I^n(E)}$.*

Proof. Suppose that $\psi_E \sim \phi_E \pmod{I^n(E)}$. By Lemma 7.6, $\psi_E \equiv k\phi_E \pmod{I^n(E)}$, where $k = d_\pm\psi/d_\pm\phi$. Since the homomorphism $W(F)/I^n(F) \to W(E)/I^n(E)$ is injective, it follows that $\psi \equiv k\phi \pmod{I^n(F)}$. Hence, $\psi \sim \phi \pmod{I^n(F)}$, a contradiction. □

Corollary 7.8. *Let ϕ and ψ be odd-dimensional forms such that $\phi \not\sim \psi \pmod{I^4(F)}$. Then*
(a) *if E/F is an extension such that $W(F)/I^4(F) \to W(E)/I^4(E)$ is injective, then $\phi_E \not\sim \psi_E \pmod{I^4(E)}$,*
(b) *if $\tau_1, \dots, \tau_m$ are forms of dimension ≥ 9, then $\phi_{F(\tau_1,\dots,\tau_m)} \not\sim \psi_{F(\tau_1,\dots,\tau_m)} \pmod{I^4(F(\tau))}$.*

Proof. Statement (a) follows from Corollary 7.7; Statement (b) follows from Statement (a) and Corollary 4.5. □

Lemma 7.9. *Let ψ be an anisotropic form of dimension 10. Then there exists an extension E/F and a 9-dimensional form $\psi_0 \subset \psi$ such that:*
(1) *for any 9-dimensional form ϕ over F we have $\phi_E \not\sim \psi_0 \pmod{I^4(E)}$.*
(2) *E/F is a purely transcendental field extension.*
(3) $\operatorname{ind}\psi_0 \geq 4$ *except for the cases where*
  – *either $\psi \in I^2(F)$ and* $\operatorname{ind}\psi \leq 2$,
  – *or $\psi \notin I^2(F)$ and* $\operatorname{ind}\psi = 1$.

Proof. We define ψ_0 as the "*generic subform of ψ of codimension 1*". Let us give the explicit definition of ψ_0 and E. Let $\tilde{F} = F(t)$ and $\tilde{\psi} = \psi_{\tilde{F}} \perp \langle -t \rangle$. We define E as $\tilde{F}(\tilde{\psi})$. Since $\tilde{\psi}_E$ is isotropic, there exists a E-form ψ_0 such that $\tilde{\psi}_E = \psi_0 \perp \mathbb{H}$. We have

$$\psi_0 \perp \langle t \rangle \perp \mathbb{H} = \tilde{\psi}_E \perp \langle t \rangle = \psi_E \perp \langle -t \rangle \perp \langle t \rangle = \psi_E \perp \mathbb{H}.$$

Hence, $\psi_0 \perp \langle t \rangle = \psi_E$. Therefore, ψ_0 is a 9-dimensional subform of ψ_E.

(1) Let ϕ be an arbitrary 9-dimensional form over F. We set $\tilde{\phi} = \phi_{\tilde{F}}$ and $\tilde{k} = d_{\pm}\tilde{\psi}/d_{\pm}\tilde{\phi}$. Clearly, $\tilde{k} = kt$, where $k = d_{\pm}\psi/d_{\pm}\phi$.

We claim that $\tilde{\psi} \not\sim \tilde{\phi} \pmod{I^4(\tilde{F})}$. Indeed, assuming the contrary, we have $\tilde{\psi} \equiv \tilde{k}\tilde{\phi} \pmod{I^4(\tilde{F})}$. Then $\psi_{\tilde{F}} + \langle -t \rangle \equiv tk\phi_{\tilde{F}} \pmod{I^4(\tilde{F})}$. Computing the homomorphism δ^1_t (see e.g., [Lam1, Ch.6. Cor.1.6]), we get $\psi \equiv 0 \pmod{I^3(F)}$. Since $\dim\psi = 10$, Pfister's theorem shows that ψ is isotropic, a contradiction.

Since $E = \tilde{F}(\tilde{\psi})$ and $\dim\tilde{\psi} = 11 \geq 9$, Corollary 7.8(2), shows that $\phi_E \not\sim \tilde{\psi}_E$ $\pmod{I^4(E)}$. Since $\tilde{\psi}_E$ coincides with ψ_0 in the Witt ring $W(E)$, we have $\phi_E \not\sim \psi_0 \pmod{I^4(E)}$.

(2) Obvious in view of Lemma 1.22.

(3) Suppose that ψ is not "exceptional". By Lemma 1.4, $\operatorname{ind}\tilde{\psi} \geq 4$. By Lemma 1.5, we have $\operatorname{ind}\psi_0 = \operatorname{ind}(\tilde{\psi}_{\tilde{F}(\tilde{\psi})}) \geq 4$. $\square$

Now, we return to the proof of the implication (2)⇒(1) in Theorem 7.3.

We start with the following case:

Case 1. $\dim\psi = 10$, $\psi \in I^2(F)$, and $\operatorname{ind}\psi \geq 4$.

Since $\dim\psi \neq 9$, we must prove that $\phi_{F(\psi)}$ is an essential form. Let E/F and ψ_0 be as in Lemma 7.9. Since E/F is purely transcendental, it follows that ϕ_E is essential and $\operatorname{ind}\psi_E = \operatorname{ind}\psi \geq 4$ (see Lemmas 1.2 and 1.24). Replacing the field F by E, we can assume that the 9-dimensional form ψ_0 is defined over the ground field and $\phi \not\sim \psi_0 \pmod{I^4(F)}$. After this, applying Proposition 6.10 together with Corollary 7.8(a), we can assume that $H^4(F) = 0$. Since $\operatorname{ind}\psi_0 = \operatorname{ind}\psi \geq 4$, Proposition 3.7(iv) shows that $\operatorname{Tors}\operatorname{CH}^3(X_{\psi_0}) = 0$. Since $\phi \not\sim \psi_0 \pmod{I^4(F)}$, we have $\phi \not\sim \psi_0$. By Lemma 7.2, the form $\phi_{F(\psi_0)}$ is essential. By Lemma 7.1, the form $\phi_{F(\psi)}$ is also essential.

Case 2. $\dim\psi = 10$, $\psi \in I^2(F)$, and $\operatorname{ind}\psi \leq 2$.

Since $\dim\psi \neq 9$, we must prove that $\phi_{F(\psi)}$ is an essential form. By Proposition 6.10, we can assume that $H^4(F) = 0$. Our assumption concerning the form ψ shows that ψ (up to similarity) can be written in the form $\pi' \perp -\langle\langle u, v\rangle\rangle'$, where π' is the pure subform of a 3-fold Pfister form π and $\langle\langle u, v\rangle\rangle'$ is the pure subform of $\langle\langle u, v\rangle\rangle$ (see e.g, [H2, Th. 5.1]). Consider the subform $\psi_0 = \pi' \perp \langle u, v\rangle \subset \psi$. Since $\operatorname{ind}\psi_0 = \operatorname{ind}\psi \leq 2 < \operatorname{ind}\phi$, we have $\psi_0 \not\sim \phi$. By Lemma 3.12(i), we have $\operatorname{Tors}\operatorname{CH}^3(X_{\psi_0}) = 0$. By Lemma 7.2, the form $\phi_{F(\psi_0)}$ is essential. Lemma 7.1 shows the form $\phi_{F(\psi)}$ is also essential.

Case 3. $\dim\psi = 9$. Suppose that $\phi \not\sim \psi \pmod{I^4(F)}$. We must verify that $\phi_{F(\psi)}$ is essential.

Let $\tau = \psi \perp \langle -d \rangle$, where $d = \det \psi$. Since $\tau \in I^2(F)$ and $\dim \tau = 10$, the results of Cases 1 and 2 show that $\phi_{F(\tau)}$ is an essential form. By Corollary 7.8, we have $\phi_{F(\tau)} \not\sim \psi_{F(\tau)} \pmod{I^4(F(\tau))}$. Hence, replacing F by $F(\tau)$, we can assume that τ is isotropic. Then ψ has the form $\psi = \psi_0 \perp \langle d \rangle$, where ψ_0 is a 8-dimensional form from $I^2(F)$. By Proposition 6.10 and Corollary 7.8(a), we can assume that $H^4(F) = 0$. All needed properties of ϕ and ψ are preserved. In particular, we still have $\phi \not\sim \psi \pmod{I^4(F)}$. Since $H^4(F) = 0$, all 4-fold Pfister neighbors are isotropic. Hence, we can assume that ψ is not a Pfister neighbor. Then $\psi_0 \notin GP_3(F)$ and therefore, $\operatorname{ind} \psi = \operatorname{ind} \psi_0 \geq 2$. By Lemma 3.12(ii), we have $\operatorname{Tors} \mathrm{CH}^3(X_\psi) = 0$. By Lemma 7.2, the form $\phi_{F(\psi)}$ is essential.

Case 4. $\dim \psi \geq 10$. Changing ψ by a 10-dimensional subform, we can assume that $\dim \psi = 10$ (Lemma 7.1). As in Case 1, we can assume that there exists a 9-dimensional subform $\psi_0 \subset \psi$ such that $\psi_0 \not\sim \phi \pmod{I^4(F)}$. By Case 3, the form $\phi_{F(\psi_0)}$ is essential. By Lemma 7.1, the form $\phi_{F(\psi)}$ is also essential.

The proof of Theorem 7.3 is complete. □

Corollary 7.10. *Let ϕ be an essential 9-dimensional form, and let $\psi_1, \dots, \psi_m$ be forms of dimension ≥ 9. Then the following conditions are equivalent:*

(1) *there exists i such that $\dim \psi_i = 9$ and $\psi_i \sim \phi \pmod{I^4(F)}$,*

(2) *the form $\phi_{F(\psi_1, \dots \psi_m)}$ is not essential.*

In particular, the form $\phi_{F(\psi_1, \dots, \psi_m)}$ is always essential if $\dim \psi_i \geq 10$ for all $i = 1, \dots, m$.

Proof. (1)⇒(2). Obvious in view of Theorem 7.3.

(2)⇒(1). Suppose that $\phi_{F(\psi_1, \dots \psi_m)}$ is not an essential form. Set $F_0 = F$, $F_1 = F(\psi_1)$, $F_2 = F(\psi_1, \psi_2)$, $\dots$, $F_m = F(\psi_1, \dots \psi_m)$. By our assumption, ϕ_{F_0} is essential and ϕ_{F_m} is not. Hence, there exists $i \geq 1$ such that $\phi_{F_{i-1}}$ is essential and ϕ_{F_i} is not essential. Since $F_i = F_{i-1}(\psi_i)$, Theorem 7.3 shows that $\dim \psi_i = 9$ and $(\psi_i)_{F_{i-1}} \sim \phi_{F_{i-1}} \pmod{I^4(F_{i-1})}$. Finally, Corollary 7.8(b) shows that $\psi_i \sim \phi \pmod{I^4(F)}$. □

Proof of Theorem 0.3. Implication (1)⇒(2) was proved in Section 1.

(2)⇒(1). Let us construct the fields

$$F_0 \subset F_1 \subset \dots \subset F_{3i} \subset F_{3i+1} \subset F_{3i+1} \subset \dots$$

as follows. First, we set $F_{-1} = F$.

If $n = 3i$, we define F_n as the maximal odd extension of F_{n-1}.

If $n = 3i + 1$, we define F_n as the free composite of the fields $F_{n-1}(\psi)$, where ψ runs over all F_{n-1}-forms of dimension ≥ 10.

If $n = 3i + 2$, we define F_n as the free composite of the fields $F_{n-1}(\psi)$, where ψ runs over all 9-dimensional F_{n-1}-forms satisfying the condition $\psi \not\sim \phi_{F_{n-1}} \pmod{I^4(F_{n-1})}$.

Induction on n, Lemma 1.14, and Corollary 7.10 show that ϕ_{F_n} is an essential form for all n. Now, we set $E = \cup_{n \geq 0} F_n$. Clearly, ϕ_E is an essential form. In particular, ϕ_E is anisotropic. Hence $u(E) \geq \dim \phi = 9$. By the definition of F_{3i+1}, all 10-dimensional forms over E are isotropic. Hence, $u(E) = 9$.

Therefore $I^4(E) = 0$. By the definition of F_{3i+2}, each anisotropic 9-dimensional form ψ over E satisfies the condition $\psi \sim \phi_E \pmod{I^4(E)}$. Since $I^4(E) = 0$, we have $\psi \sim \phi_E$. Clearly, E has no odd extensions. Taking into account the equation $I^4(E) = 0$, we conclude that $cd_2(E) \le 3$ (see Proposition 4.6). Now, it suffices to verify that $cd_2(E) \not\le 2$. Indeed, assuming the contrary, we obtain $I^3(E) = 0$. In this case, [Lam1, Ch. 11, Lemma 4.9] claims that $u(E)$ is even. We get a contradiction. $\square$

Proof of Theorem 0.1. By Theorem 0.3, it suffices to construct at least one example of a 9-dimension essential form. We present here the following example: Let F_0 be an arbitrary field and $F = F_0((t_1))\dots((t_9))$. We define ϕ as $\langle t_1, \dots, t_9\rangle$. An easy computation using Tignol's Theorem (see Corollary 1.4) shows that $\operatorname{ind}\phi = 16$. By Springer's theorem, ϕ is anisotropic. Now, it suffices to prove that ϕ is not a Pfister neighbor. Assume the contrary. Let μ be the complementary 7-dimensional form. Since $\phi \perp \mu \in GP_4(F)$, it follows that $\phi \equiv -\mu \pmod{I^3(F)}$ and hence $\operatorname{ind}\phi = \operatorname{ind}\mu$. Since $\dim\mu = 7$, we have $\operatorname{ind}\mu \le 8$, a contradiction. $\square$

At the end of this section we present a proof of Conjecture 0.10. [6]

Proof of Conjecture 0.10. Let ϕ be an anisotropic 10-dimensional form with maximal splitting. We can assume that ϕ is not a Pfister neighbor. We must prove that $\phi \simeq \langle\langle d\rangle\rangle \otimes \tau$ for a suitable $d \in F^*$ and a 5-dimensional form τ. Let us consider the following three cases.

Case 1: $\phi \in I^2(F)$. Since all 10-dimensional forms from $I^2(F)$ with maximal splitting are necessarily Pfister neighbors (see the last Item of Theorem 1.10), we get a contradiction to our assumption.

Case 2: $\phi \notin I^2(F)$ and $\operatorname{ind}\phi = 1$. Let $d = \det\phi$ and $L = F(\sqrt{d})$. Since $\phi_L \in I^2(L)$ and $\operatorname{ind}\phi_L = \operatorname{ind}\phi = 1$, it follows that $\phi_L \in I^3(L)$. By Pfister's theorem the form ϕ_L is isotropic. Since ϕ has maximal splitting, we have $\dim(\phi_L)_{an} \le 6$. Then the Arason–Pfister Hauptsatz implies that ϕ_L is hyperbolic. Therefore, ϕ is divisible by $\langle\langle d\rangle\rangle$. Hence ϕ has the form $\langle\langle d\rangle\rangle \otimes \tau$ with $\dim\tau = 5$.

Case 3: $\phi \notin I^2(F)$ and $\operatorname{ind}\phi \ge 2$. By Lemma 7.9, there exists a purely transcendental extension E/F and a 9-dimensional subform $\phi_0 \subset \phi_E$ with $\operatorname{ind}\phi_0 \ge 4$. Since ϕ has maximal splitting, the form $(\phi_0)_{E(\phi)}$ is isotropic. Assume that ϕ_0 is a Pfister neighbor. Since $(\phi_0)_{E(\phi)}$ is isotropic, it follows that ϕ_E is a Pfister neighbor. Since E/F is unirational, ϕ is also a Pfister neighbor (Lemma 1.23). We get a contradiction. Hence ϕ_0 is not a Pfister neighbor. Then ϕ_0 is an essential E-form. Since $\dim\phi_E = 10$, it follows from Theorem 7.3 that $(\phi_0)_{E(\phi)}$ is anisotropic. We get a contradiction, and the proof is complete. $\square$

8. Special pair of forms: definition and basic properties

The main goal of this section is to study the properties of some specific class of pairs of forms. We will call these pairs *special* (see Definition 8.3 below). Since

[6] This conjecture was proved in [Izh3] in the case when $\operatorname{char} F = 0$ and $\sqrt{-1} \in F^*$.

many basic properties of special pairs are closely related to linkage properties of Pfister forms, we recall some results of R. Elman and T. Y. Lam on this subject.

Theorem 8.1. ([EL3, §4]) [7] *Let $\tau_1 \in P_{n_1}(F)$ and $\tau_2 \in P_{n_2}(F)$.*

(1) *Let $k \in F^*$ be such that the form $\tau_1 \perp -k\tau_2$ is isotropic. Then the forms $(\tau_1 \perp -k\tau_2)_{an}$ and $(\tau_1 \perp -\tau_2)_{an}$ are similar. In particular,*
$$\dim(\tau_1 \perp -k\tau_2)_{an} = \dim(\tau_1 \perp -\tau_2)_{an}.$$
(2) *Let $\rho \in P_n(F)$, $\mu \in P_m(F)$, and $\nu \in P_k(F)$ be such that $\tau_1 \simeq \rho \otimes \mu$, $\tau_2 \simeq \rho \otimes \nu$, and $m, k > 0$. Let μ' and ν' be the pure subforms of μ and ν. Then*
   - *either $(\tau_1 \perp -\tau_2)_{an} \simeq \rho \otimes (\mu' \perp -\nu')$ (in particular, this means that the form $\rho \otimes (\mu' \perp -\nu')$ is anisotropic),*
   - *or there exist $\mu_0 \in P_{m-1}(F)$, $\nu_0 \in P_{k-1}(F)$, and $d \in F^*$ such that $\tau_1 \simeq \rho \otimes \langle\langle d \rangle\rangle \otimes \mu_0$ and $\tau_2 \simeq \rho \otimes \langle\langle d \rangle\rangle \otimes \nu_0$.*

Lemma 8.2. *Let $\rho = \tau \otimes q$, where q is an Albert form and $\tau \in P_n(F)$. Suppose that there exists a form $\tilde{\rho}$ such that $\dim \tilde{\rho} < \dim \rho$ and $\rho \equiv \tilde{\rho} \pmod{I^{n+3}(F)}$. Then the form ρ is isotropic.*

Proof. Assume that ρ is anisotropic. Then τ is anisotropic. We obviously have $\rho \in I^{n+2}(F)$. Since $\tilde{\rho} \equiv \rho \pmod{I^{n+3}(F)}$, we also have $\tilde{\rho} \in I^{n+2}(F)$. Clearly, $\dim \tilde{\rho} < \dim \rho = 6 \cdot 2^n < 2^{n+3}$.

If we assume that $\tilde{\rho}$ is hyperbolic, we get $\rho \equiv \tilde{\rho} \equiv 0 \pmod{I^{n+3}(F)}$. Since $\dim \rho < 2^{n+3}$, the Arason–Pfister Hauptsatz shows that ρ is hyperbolic. We get a contradiction. Hence, $\tilde{\rho}$ is not hyperbolic. Changing $\tilde{\rho}$ by its anisotropic part, we can assume that $\tilde{\rho}$ is non-zero and anisotropic.

We have $\tilde{\rho}_{F(\tau)} \equiv \rho_{F(\tau)} \equiv 0 \pmod{I^{n+3}(F)}$. Since $\dim \tilde{\rho} < 2^{n+3}$, the Arason–Pfister Hauptsatz shows that $\tilde{\rho}_{F(\tau)}$ is hyperbolic. Hence there exists a form λ such that $\tilde{\rho} = \tau \otimes \lambda$. Since $\dim \tau = 2^n$ and $\dim \tilde{\rho} < \dim \rho = 6 \cdot 2^n$, it follows that $\dim \lambda < 6$. First, consider the case when $\dim \lambda$ is odd. Then $\langle 1 \rangle \equiv \lambda \pmod{I(F)}$ and we have $\tau \equiv \tau \otimes \lambda \equiv \tilde{\rho} = 0 \pmod{I^{n+1}(F)}$. Since $\dim \tau = 2^n$, the Arason-Pfister Hauptsatz shows that τ is hyperbolic, a contradiction. Now, we can assume that $\dim \lambda$ is even. Since $\dim \lambda < 6$, we have $\dim \lambda \leq 4$. Hence, $\dim \tilde{\rho} = \dim(\tau \otimes \lambda) \leq 2^n \cdot 4 = 2^{n+2}$. Since $\tilde{\rho} \in I^{n+2}(F)$, the Arason-Pfister Hauptsatz implies that $\tilde{\rho} \in GP_{n+2}(F)$. Therefore, $\dim \tilde{\rho} = 2^{n+2}$ and the form $\tilde{\rho}_{F(\tilde{\rho})}$ is hyperbolic. Hence $\rho_{F(\tilde{\rho})} \equiv \tilde{\rho}_{F(\tilde{\rho})} = 0 \pmod{I^{n+3}(F(\tilde{\rho}))}$. Since $\dim \rho < 2^{n+3}$, the Arason–Pfister Hauptsatz shows that the form $\rho_{F(\tilde{\rho})}$ is hyperbolic. Hence, the form ρ is divisible by $\tilde{\rho}$. On the other hand, $\dim \rho = 6 \cdot 2^n$ is not divisible by $\dim \tilde{\rho} = 2^{n+2}$, a contradiction. □

[7] For the proof of the first statement, see the proof of Theorem 4.5 in [EL3] or [H4, Lemma 3.2]. The proof of the second statement is the same as *Step 2* of the proof of Proposition 4.4 in [EL3].

Definition 8.3. Let n and m be integers such that $n \geq 0$ and $m \geq 2$. We say that (ρ, ρ_0) is an *(n,m)-special pair of forms* if there exist $u, v \in F^*$ and $\tau \in P_n(F)$, $\mu \in P_m(F)$ such that

$$\rho \simeq \tau \otimes (\mu' \perp -\langle\langle u, v\rangle\rangle') \qquad \text{and} \qquad \rho_0 \simeq \tau \otimes (\mu' \perp \langle u, v\rangle),$$

where μ' and $\langle\langle u, v\rangle\rangle'$ are the pure subforms of μ and $\langle\langle u, v\rangle\rangle$.

Our interest in special pairs is motivated by the following conjecture.

Conjecture 8.4. *Let s be a positive integer and ϕ be an F-form such that:*
- *ϕ is not an s-fold Pfister neighbor,*
- $\dim \phi > 2^{s-1}$ *and* $\tilde{H}^s_{nr}(F(\phi)/F) \neq 0$.

Then $\tilde{H}^s_{nr}(F(\phi)/F) \simeq \mathbb{Z}/2\mathbb{Z}$ and there exists an (n,m)-special pair (ρ, ρ_0) with the following properties:

(i) *$n \geq 0$, $m \geq 2$, and $m + n + 1 = s$,*
(ii) *$\rho_{F(\phi)}$ is isotropic and $(\rho_0)_{F(\phi)}$ is anisotropic,*
(iii) *the group $\tilde{H}^s_{nr}(F(\phi)/F)$ is generated by $\tilde{e}^s(\mathrm{Pf}((\rho_0)_{F(\phi)}))$.*

Remark 8.5. (1) If $s \leq 2$, then the group $\tilde{H}^s_{nr}(F(\phi)/F)$ is zero for all non Pfister neighbors ϕ (see [KRS1, Th. 4 and Prop. 3]). Hence, in this case the conjecture is obvious. We also note that for $s \leq 2$ there are no integers n and m satisfying condition (i).
(2) In the case $s = 3$, the conjecture follows easily from the results of [Kah3] and [KRS1]. See also the following section: Example 9.2 and the proof of Lemma 9.5.
(3) In the case $s = 4$, the conjecture will be proved in Section 12.
(4) In this section, we show (Theorem 8.6(2)) that condition (ii) of Conjecture 8.4 implies that $(\rho_0)_{F(\phi)}$ is an anisotropic s-fold Pfister neighbor. Hence $\mathrm{Pf}((\rho_0)_{F(\phi)})$ is an anisotropic s-fold Pfister form.

In this section, we study the properties of (n,m)-special pairs. Let us start with the following obvious observation: if (ρ, ρ_0) is an (n,m)-special pair over F, then $(\rho_L, (\rho_0)_L)$ is an (n,m)-special pair over L for any field extension L/F.

All basic properties of (n,m)-special pair of forms are collected in the following theorem.

Theorem 8.6. *Let n and m be integers such that $n \geq 0$ and $m \geq 2$. Let (ρ, ρ_0) be an (n,m)-special pair of forms. Then*

(1) $\dim \rho_0 = 2^n(2^m + 1)$, $\dim \rho = 2^n(2^m + 2)$, *and ρ_0 is a subform of ρ. In particular,*
$$2^{n+m} < \dim \rho_0 < \dim \rho < 2^{n+m+1}.$$
(2) *The following conditions are equivalent:*
   (a) *the form ρ_0 is a Pfister neighbor,*
   (b) *the form ρ contains an $(n+m+1)$-fold Pfister neighbor,*
   (c) *there exists a form $\tilde{\rho}$ such that $\dim \tilde{\rho} < \dim \rho$ and $\tilde{\rho} \equiv \rho$ $(\mathrm{mod}\ I^{n+m+1}(F))$,*

(d) *the form* ρ *is isotropic.*

(3) *If* ρ_0 *is an anisotropic form, then*
- ρ_0 *has maximal splitting,*
- ρ *is not a Pfister neighbor,*
- $(\rho_0)_{F(\rho)}$ *is an anisotropic Pfister neighbor.*

(4) *If* ρ *is anisotropic, then* $i_1(\rho) = 2^n$ *and* $\dim(\rho_{F(\rho)})_{an} = 2^{n+m}$.

Proof. Item (1) is obvious.

Let us prove Statement (2).

(a)⇒(b). Obvious.

(b)⇒(c). Suppose that ρ contains an $(n+m+1)$-fold Pfister neighbor. Then we can write ρ in the form $\rho = \lambda \perp \mu$, where λ is an $(n+m+1)$-fold Pfister neighbor. Then there exists a form $\tilde{\lambda}$ such that $\lambda \perp -\tilde{\lambda} \in GP_{n+m+1}(F)$. In particular, $\tilde{\lambda} \equiv \lambda \pmod{I^{n+m+1}(F)}$. By the definition of Pfister neighbors, we have $\dim \tilde{\lambda} < \dim \lambda$. To complete the proof, it suffices to set $\tilde{\rho} = \tilde{\lambda} \perp \mu$.

(c)⇒(d). Let $\tilde{\rho}$ be such that $\dim \tilde{\rho} < \dim \rho$ and $\rho \equiv \tilde{\rho} \pmod{I^{n+m+1}(F)}$.

Set $\pi = (\rho \perp -\tilde{\rho})_{an}$ and $\lambda = \tau \otimes \mu$. Clearly, $\pi \in I^{n+m+1}(F)$ and $\lambda \in P_{n+m}(F)$. Since $m \geq 2$, we have

$$\dim \pi \leq \dim \rho + \dim \tilde{\rho} < 2 \dim \rho = 2^{n+1}(2^m + 2) \leq 2^{n+1}(2^m + 2^{m-1}) = 3 \cdot 2^{n+m}.$$

We can assume that π is a nonhyperbolic form (otherwise, $\rho_{an} = \tilde{\rho}_{an}$ and the proof is obvious). Since $\pi \in I^{n+m+1}(F)$, it follows that $\dim \pi \geq 2^{n+m+1}$. We have proved $2 \cdot 2^{n+m} \leq \dim \pi < 3 \cdot 2^{n+m}$.

We consider the cases $m > 2$ and $m = 2$ separately.

Case 1: $m > 2$. In the Witt ring, we have $\rho = \tau(\mu - \langle\langle u, v\rangle\rangle) = \lambda - \tau \langle\langle u, v\rangle\rangle$. Hence $\rho_{F(\lambda)} = -\tau \langle\langle u, v\rangle\rangle_{F(\lambda)}$. Therefore, $\dim(\rho_{F(\lambda)})_{an} \leq 2^{n+2}$. Taking into account the inequality $6 < 2^m$, we have

$$\begin{aligned}\dim(\pi_{F(\lambda)})_{an} &\leq \dim(\rho_{F(\lambda)})_{an} + \dim \tilde{\rho} < 2^{n+2} + \dim \rho \\ &\leq 2^{n+2} + 2^n(2^m + 2) = 2^n(6 + 2^m) < 2^{n+m+1}.\end{aligned}$$

Since $\pi \in I^{n+m+1}(F)$, the Arason–Pfister Hauptsatz shows that $\pi_{F(\lambda)}$ is hyperbolic. Hence, there exists a form ξ such that $\pi = \lambda \otimes \xi$. Since $\dim \lambda = 2^{n+m}$ and $2 \cdot 2^{n+m} \leq \dim \pi < 3 \cdot 2^{n+m}$, we conclude that $\dim \xi = 2$. Let us write ξ in the form $\xi = k \langle\langle d\rangle\rangle$. Then $\pi = \lambda \otimes \xi = k\tau \otimes \mu \otimes \langle\langle d\rangle\rangle$. In the Witt ring $W(F)$ we have

$$\begin{aligned}\tilde{\rho} = \rho - \pi &= \tau(\mu - \langle\langle u, v\rangle\rangle) - k\tau\mu \langle\langle d\rangle\rangle = \\ &= \tau\mu \langle 1, -k, kd\rangle - \tau \langle\langle u, v\rangle\rangle = \\ &= \tau\mu(\langle\langle k, d\rangle\rangle + \langle d\rangle) - \tau \langle\langle u, v\rangle\rangle = \\ &= \tau(d\mu - \langle\langle u, v\rangle\rangle) + \tau\mu \langle\langle k, d\rangle\rangle .\end{aligned}$$

Let $\hat{\rho} = \tau \otimes (d\mu \perp - \langle\langle u, v\rangle\rangle)$. Since $\tau\mu \langle\langle k, d\rangle\rangle \in GP_{n+m+2}(F)$, we have $\tilde{\rho} \equiv \hat{\rho} \pmod{I^{n+m+2}(F)}$. On the other hand, $\dim \tilde{\rho} + \dim \hat{\rho} < \dim \rho + \dim \hat{\rho} \leq 2^n(2^m + 2) + 2^n(2^m + 4) = 2^n(2^{m+1} + 6) < 2^{n+m+2}$. The Arason–Pfister Hauptsatz shows that $\tilde{\rho}_{an} \simeq \hat{\rho}_{an}$. Since $\dim \tilde{\rho} < \dim \rho < \dim \hat{\rho}$, it follows that $\hat{\rho} = \tau \otimes (d\mu \perp$

$- \langle\langle u, v\rangle\rangle)$ is isotropic. Applying Theorem 8.1(1) to the forms $\tau_1 = \tau \otimes \mu$ and $\tau_2 = \tau \otimes \langle\langle u, v\rangle\rangle$, we see that $\hat{\rho}_{an}$ is similar to ρ_{an}. Hence, $\rho_{an} \sim \hat{\rho}_{an} \simeq \tilde{\rho}_{an}$. Since $\dim \tilde{\rho} < \dim \rho$, the form ρ is isotropic.

Case 2: $m = 2$. Since $\mu \in P_m(F) = P_2(F)$, there exist $a, b \in F^*$ such that $\mu = \langle\langle a, b\rangle\rangle$. Then $\rho = \tau \otimes (\mu' \perp - \langle\langle u, v\rangle\rangle') = \tau \otimes q$, where $q = \langle -a, -b, ab, u, v, -uv\rangle$. Since q is an Albert form, the required statement follows readily from Lemma 8.2.

(d)$\Rightarrow$(a). Let us assume that ρ is isotropic. Applying Theorem 8.1(2) to the forms $\tau_1 = \tau \otimes \mu$ and $\tau_2 = \tau \otimes \langle\langle u, v\rangle\rangle$, we see that there exists $d \in F^*$ such that $\tau \otimes \langle\langle d\rangle\rangle$ divides the both forms $\tau \otimes \mu$ and $\tau \otimes \langle\langle u, v\rangle\rangle$. Thus, there exists $\mu_0 \in P_{m-1}(F)$ and $k \in F^*$ such that $\tau \otimes \langle\langle d\rangle\rangle \otimes \mu_0 = \tau \otimes \mu$ and $\tau \otimes \langle\langle d, k\rangle\rangle = \tau \otimes \langle\langle u, v\rangle\rangle$.

We claim that ρ_0 is a subform of the Pfister form

$$\pi := \tau \otimes \mu \otimes \langle\langle -uvk\rangle\rangle \,.$$

Besides, we claim that the complementary form is equal to

$$\rho_0' := uv\tau \otimes (k\mu_0' \otimes \langle\langle d\rangle\rangle \perp \langle -d\rangle).$$

Since $\dim \rho_0 = 2^n(2^m + 1)$, $\dim \rho_0' = 2^n(2^m - 1)$, and $\pi \in P_{n+m+1}(F)$, it suffices to verify the equation $\rho_0 + \rho_0' = \pi$ in the Witt ring. We have

$$\begin{aligned}
\pi - \rho_0 &= \tau(\mu + uvk\mu) - \tau(\mu + \langle u, v, -1\rangle)\\
&= \tau(uvk\mu + \langle 1, -u, -v\rangle)\\
&= uv\tau(k\mu + \langle uv, -v, -u\rangle)\\
&= uv\tau(k\mu + \langle\langle u, v\rangle\rangle - \langle 1\rangle)\\
&= uv\tau(k\mu_0 \langle\langle d\rangle\rangle + \langle\langle d, k\rangle\rangle - \langle 1\rangle)\\
&= uv\tau(k\mu_0' \langle\langle d\rangle\rangle + \langle -d\rangle)\\
&= \rho_0'
\end{aligned}$$

The proof of Item (2) is complete. To prove Item (3), we need the following lemma.

Lemma 8.7. *Let $\phi = \tau \otimes \nu$, where τ is an n-fold Pfister form. Let L/F be an arbitrary extension such that τ_L is anisotropic. Then $i_W(\phi_L)$ is divisible by 2^n.*

Proof. Since $\phi_{L(\tau)}$ is hyperbolic, there exists an L-form γ such that $(\phi_L)_{an} = \tau_L \otimes \gamma$. In the Witt ring $W(L)$, we have $\tau_L \cdot (\nu_L - \gamma) = 0$. Since $\tau_L \neq 0 \in W(L)$, $(\nu_L - \gamma)$ is a zero-divisor in $W(L)$. By [Lam1, Ch. VIII, Cor. 6.7], $\dim \nu - \dim \gamma$ is even. Hence $\dim \phi - \dim(\phi_L)_{an} = 2^n(\dim \nu - \dim \gamma)$ is divisible by 2^{n+1}. Therefore, $i_W(\phi_L)$ is divisible by 2^n. □

Corollary 8.8. *Let $\phi = \tau \otimes \nu$ be an anisotropic form, where τ is an n-fold Pfister form and $\dim \nu \geq 2$. Then $i_1(\phi)$ is divisible by 2^n.*

Proof. Since ϕ is anisotropic, τ is also anisotropic. Since $\dim \nu \geq 2$, we have $\dim \phi > \dim \tau$. The Cassels–Pfister subform theorem shows that $\tau_{F(\phi)}$ is anisotropic. Now, Lemma 8.7 shows that $i_1(\phi) = i_W(\phi_{F(\phi)})$ is divisible by 2^n. $\square$

Corollary 8.9. *Let $\phi = \tau \otimes \nu$ be an anisotropic form, where τ is an n-fold Pfister form and* $\dim \nu = 2^m + 1$. *Then ϕ has maximal splitting (i.e., $i_1(\phi) = 2^n$).*

Proof. Since $\dim \phi = 2^{n+m} + 2^n$, Theorem 1.10 shows that $1 \le i_1(\phi) \le 2^n$. From Corollary 8.8 it follows that $i_1(\phi)$ is divisible by 2^n. Hence $i_1(\phi) = 2^n$. $\square$

Corollary 8.10. *Let $\phi = \tau \otimes \nu$ be an anisotropic form, where τ is an n-fold Pfister form and* $\dim \nu = 2^m + 2$ *with $m \ge 1$. Then*

- *either $i_1(\phi) = 2^{n+1}$ (in this case ϕ has maximal splitting),*
- *or $i_1(\phi) = 2^n$ (in this case* $\dim(\phi_{F(\phi)})_{an} = 2^{n+m}$*).*

Proof. Since $\dim \phi = 2^{n+m} + 2^{n+1}$, Theorem 1.10 shows that $i_1(\phi) \le 2^{n+1}$. From Corollary 8.8 it follows that $i_1(\phi)$ is divisible by 2^n. Hence $i_1(\phi) = 2^{n+1}$ or 2^n. The rest of the proof is obvious. $\square$

Now, we return to the proof of Item (3) of Theorem 8.6. Here we can assume that ρ_0 is anisotropic. Corollary 8.9 shows that ρ_0 has maximal splitting. Now, we must verify that ρ is not a Pfister neighbor. Suppose at the moment that ρ is a Pfister neighbor of π. Then ρ_0 is also a Pfister neighbor of π. Item (2) of Theorem 8.6 shows that ρ is isotropic. Then π is isotropic. Since ρ_0 is a neighbor of π, the form ρ_0 is also isotropic. This contradicts our assumption. Hence ρ is not a Pfister neighbor.

To complete the proof of Item (3) it suffices to verify that $(\rho_0)_{F(\rho)}$ is an anisotropic Pfister neighbor. Suppose that $(\rho_0)_{F(\rho)}$ is isotropic. Since ρ_0 has maximal splitting, ρ also has maximal splitting (Theorem 1.11). By [H1, Prop. 6] there exists an extension K/F such that ρ_K is an anisotropic Pfister neighbor. Changing the field F by K, we can assume that ρ is an anisotropic Pfister neighbor. However, we have proved above that ρ is not a Pfister neighbor, a contradiction. Hence $(\rho_0)_{F(\rho)}$ is anisotropic. To prove that $(\rho_0)_{F(\rho)}$ is a Pfister neighbor, we consider the special pair $(\rho_{F(\rho)}, (\rho_0)_{F(\rho)})$. Since $\rho_{F(\rho)}$ is isotropic, Item (2) of the theorem shows that $(\rho_0)_{F(\rho)}$ is a Pfister neighbor. The proof of Item (3) is complete.

In the proof of Item (4) of Theorem 8.6, we can assume that ρ is anisotropic. By Corollary 8.10, the form ρ has maximal splitting or $i_1(\rho) = 2^n$. If ρ has maximal splitting, then there exists an extension K/F such that ρ_K is an anisotropic Pfister neighbor ([H1, Prop. 6]). This contradicts Item (3) of the theorem. By Corollary 8.10, we have $i_1(\rho) = 2^n$ and $\dim(\rho_{F(\rho)})_{an} = 2^{n+m}$. The proof of Theorem 8.6 is complete. $\square$

Definition 8.11. We say that a special pair (ρ, ρ_0) is *anisotropic*, if ρ (and so ρ_0) is anisotropic.

Lemma 8.12. *Let (ρ, ρ_0) be an anisotropic (n, m)-special pair. Then*

- *ρ contains no $(n + m + 1)$-fold Pfister neighbors,*

- *ρ_0 and ρ are not Pfister neighbors,*
- *ρ_0 is a form with maximal splitting,*
- *$(\rho_0)_{F(\rho)}$ is an anisotropic Pfister neighbor.*

Proof. Obvious in view of Theorem 8.6. □

Proposition 8.13. *Let (ρ, ρ_0) be an anisotropic (n, m)-special pair, and $(\tilde{\rho}, \tilde{\rho}_0)$ be an anisotropic $(\tilde{n}, \tilde{m})$-special pair with $n + m = \tilde{n} + \tilde{m}$. Suppose that ρ_0 is isotropic over the function field of $\tilde{\rho}_0$. Then $n = \tilde{n}$, $m = \tilde{m}$, $\dim \rho = \dim \tilde{\rho}$, $\dim \rho_0 = \dim \tilde{\rho}_0$, $\rho \overset{st}{\sim} \tilde{\rho}$, and $\rho_0 \overset{st}{\sim} \tilde{\rho}_0$.*

Proof. Since $n + m = \tilde{n} + \tilde{m}$, we have $2^{n+m} < \dim \rho_0, \dim \tilde{\rho}_0 < 2^{n+m+1}$. Since ρ_0 and $\tilde{\rho}_0$ have maximal splitting and ρ_0 is isotropic over the function field of $\tilde{\rho}_0$, Theorem 1.11 shows that $\rho_0 \overset{st}{\sim} \tilde{\rho}_0$.

By Lemma 8.12, the form $(\rho_0)_{F(\rho)}$ is a Pfister neighbor. Since $\rho_0 \overset{st}{\sim} \tilde{\rho}_0$, it follows that $(\tilde{\rho}_0)_{F(\rho)}$ is also a Pfister neighbor. Applying Theorem 8.6(2) to the special pair $(\tilde{\rho}_{F(\rho)}, (\tilde{\rho}_0)_{F(\rho)})$, we see that the form $\tilde{\rho}_{F(\rho)}$ is isotropic. Analogously, $\rho_{F(\tilde{\rho})}$ is isotropic. Hence $\rho \overset{st}{\sim} \tilde{\rho}$. By Theorem 1.15, $\dim \rho - i_1(\rho) = \dim \tilde{\rho} - i_1(\tilde{\rho})$. Taking into account Items (1) and (4) of Theorem 8.6, we have $2^n(2^m+2)-2^n = 2^{\tilde{n}}(2^{\tilde{m}}+2)-2^{\tilde{n}}$. Since $n+m = \tilde{n}+\tilde{m}$, we obviously get $n = \tilde{n}$ and $m = \tilde{m}$. □

9. Special pairs of degree 4 and unramified cohomology

We recall that if (ρ, ρ_0) is an (n, m)-special pair, then $2^{n+m} < \dim \rho_0 < \dim \rho < 2^{n+m+1}$. When we say that a pair is "*(n, m)-special*", we always assume that $n \geq 0$ and $m \geq 2$.

Definition 9.1. Let (ρ, ρ_0) be an (n, m)-special pair. We define the degree of (ρ, ρ_0) by the formula $\deg(\rho, \rho_0) = n + m + 1$.

Since $n \geq 0$ and $m \geq 0$, the degree d of any special pair satisfies the condition $d = n + m + 1 \geq 3$.

Example 9.2. Let us consider special pairs of degree 3. Since $n \geq 0$, $m \geq 2$, and $n+m+1 = 3$, we obviously have $n = 0$ and $m = 2$. In this case, $\tau \in P_0(F)$ and $\mu \in P_2(F)$. Hence $\tau = \langle 1 \rangle$ and μ is of the form $\mu = \langle\langle s, r \rangle\rangle$. Then we get

$$\rho = \langle\langle s, r \rangle\rangle' \perp - \langle\langle u, v \rangle\rangle' = \langle -s, -r, sr, u, v, -uv \rangle \, ,$$
$$\rho_0 = \langle\langle s, r \rangle\rangle' \perp \langle u, v \rangle \, . = \langle -s, -r, sr, u, v \rangle$$

Thus, ρ is an Albert form, and ρ_0 is a 5-dimensional subform of ρ.

In what follows, we are interested in the case when the degree is equal to 4. Since $m \geq 2$ and $n \geq 0$, we have exactly two possibilities for special pairs of degree 4, namely:

(i) $n = 1$ and $m = 2$,
(ii) $n = 0$ and $m = 3$.

Let us consider consider these cases separately.

Example 9.3. Let $n = 1$ and $m = 2$. In this case, $\tau \in P_1(F)$ and $\mu \in P_2(F)$. Then we can write τ and μ in the forms $\tau = \langle\langle a\rangle\rangle$ and $\mu = \langle\langle s, r\rangle\rangle$. We obtain

$$\rho = \langle\langle a\rangle\rangle \otimes (\langle\langle s, r\rangle\rangle' \perp -\langle\langle u, v\rangle\rangle') = \langle\langle a\rangle\rangle \otimes \langle -s, -r, sr, u, v, -uv\rangle\,,$$
$$\rho_0 = \langle\langle a\rangle\rangle \otimes (\langle\langle s, r\rangle\rangle' \perp \langle u, v\rangle) = \langle\langle a\rangle\rangle \otimes \langle -s, -r, sr, u, v\rangle\,.$$

We will say that ρ is a *special* 12-dimensional form and ρ_0 is a *special subform* of ρ.

Example 9.4. Let $n = 0$ and $m = 3$. In this case, $\tau \in P_0(F)$ and $\mu \in P_3(F)$. Then $\tau = \langle 1\rangle$ and μ can be written in the form $\mu = \langle\langle a, b, c\rangle\rangle$. We get

$$\rho = \langle\langle a, b, c\rangle\rangle' \perp -\langle\langle u, v\rangle\rangle'\,,$$
$$\rho_0 = \langle\langle a, b, c\rangle\rangle' \perp \langle u, v\rangle\,.$$

We will say that ρ is a *special* 10-dimensional form and ρ_0 is a *special 9-dimensional subform* of ρ.

Lemma 9.5. *Let (ρ, ρ_0) be an anisotropic special pair of degree* 4 *(in particular, $9 \le \dim \rho_0 < \dim \rho < 16$). Then $\tilde{H}^4_{nr}(F(\rho)/F) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. By Lemma 8.12, the form $(\rho_0)_{F(\rho)}$ is an anisotropic 4-fold neighbor. Now, we apply Lemma 6.4. This lemma (in particular) shows that either $\tilde{H}^4_{nr}(F(\rho)/F) \neq 0$ or $H^4(F(\rho_0, \rho)/F) \neq 0$.

We claim that $H^4(F(\rho_0, \rho)/F) = 0$. To prove this, we note that $\rho_0 \subset \rho$, and hence $H^4(F(\rho_0, \rho)/F) = H^4(F(\rho_0)/F)$. Since ρ is anisotropic, it follows that ρ_0 is not a Pfister neighbor (Lemma 8.12). Hence $H^4(F(\rho_0)/F) = 0$ and therefore $H^4(F(\rho_0, \rho)/F) = 0$.

Since $H^4(F(\rho_0, \rho)/F) = 0$, it follows that $\tilde{H}^4_{nr}(F(\rho)/F) \neq 0$. Now, the lemma follows from Theorem 5.2. □

Now, we need the following well-known statement.

Lemma 9.6.
- (Pfister, [Pf2, Satz 14]). *Let ρ be a 12-dimensional form from $I^3(F)$. Then there exist $a, s, r, u, v \in F^*$ such that ρ is similar to the form*
$$\langle\langle a\rangle\rangle \otimes \langle -s, -r, sr, u, v, -uv\rangle\,.$$
In other words, the form ρ up to similarity coincides with the "special 12-dimensional form" defined in Example 9.3.
- (see e.g., [H2, Th.5.1]). *Let ρ be a 10-dimensional form from $I^2(F)$ with* $\operatorname{ind} C(\phi) \le 2$. *Then there exist $a, b, c, u, v \in F^*$ such that ρ is similar to the form*
$$\langle\langle a, b, c\rangle\rangle' \perp -\langle\langle u, v\rangle\rangle'\,.$$
In other words, the form ρ up to similarity coincides with the "special 12-dimensional form" defined in Example 9.4.

Corollary 9.7. *Let $\rho \in I^2(F)$ be either an anisotropic 12-dimensional form with* $\operatorname{ind} \rho = 1$*, or an anisotropic 10-dimensional form with* $\operatorname{ind} \rho = 2$*. Then ρ does not contain 4-fold Pfister neighbors.*

Proof. By Lemma 9.6, we can assume that ρ is either a special 12-dimensional form, or a special 10-dimensional form. In any case, Lemma 8.12 shows that ρ contains no 4-fold Pfister neighbors. $\square$

Lemma 9.8. *Let ρ be an anisotropic form from $I^2(F)$. Suppose also that ρ is either a 10-dimensional form with $\operatorname{ind}\rho = 2$, or a 12-dimensional form with $\operatorname{ind}\rho = 1$. Then ρ is not a Pfister neighbor and $\tilde{H}^4_{nr}(F(\rho)/F) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Corollary 9.7 shows that ρ is not a Pfister neighbor. The isomorphism $\tilde{H}^4_{nr}(F(\rho)/F) = \mathbb{Z}/2\mathbb{Z}$ follows from Lemmas 9.6 and 9.5. $\square$

Corollary 9.9. *Let ρ be an anisotropic 11-dimensional form with $\operatorname{ind}\rho = 1$. Then ρ is not a Pfister neighbor and $\tilde{H}^4_{nr}(F(\rho)/F) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Since $c(\rho) = 1$, we obtain from Lemma 1.21(2) that there exists a 12-dimensional form $\gamma \in I^3(F)$ such that $\rho \overset{st}{\sim} \gamma$. Since ρ is anisotropic, γ is also anisotropic. By Lemma 9.8, γ is not a Pfister neighbor and $\tilde{H}^4_{nr}(F(\gamma)/F) = \mathbb{Z}/2\mathbb{Z}$. Since $\gamma \overset{st}{\sim} \rho$, the proof is complete (see Lemma 5.1). $\square$

Proposition 9.10. *Let ρ be an F-form satisfying one of the following conditions:*

(a) $\dim\rho = 12$ *and* $\rho \in I^3(F)$,
(b) $\dim\rho = 11$ *and* $c(\rho) = 1$,
(c) $\dim\rho = 10$, $\rho \in I^2(F)$, *and* $\operatorname{ind} C(\rho) = 2$.

Then

- *if ρ is isotropic, then $\tilde{H}^4_{nr}(F(\rho)/F) \simeq \operatorname{Tors}\operatorname{CH}^3(X_\rho) \simeq 0$,*
- *if ρ is anisotropic, then $\tilde{H}^4_{nr}(F(\rho)/F) \simeq \operatorname{Tors}\operatorname{CH}^3(X_\rho) \simeq \mathbb{Z}/2\mathbb{Z}$.*

In any case, the homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\rho)/F) \to \operatorname{Tors}\operatorname{CH}^3(X_\rho)$ is an isomorphism.

Proof. If ρ is isotropic, then the proposition is trivial in view of Corollary 3.3 and Lemma 5.1. Hence, we can assume that ρ is anisotropic. By Lemma 9.8 and Corollary 9.9, the form ρ is not a Pfister neighbor and $\tilde{H}^4_{nr}(F(\rho)/F) = \mathbb{Z}/2\mathbb{Z}$. The claim follows now from Corollary 5.3. $\square$

Corollary 9.11. *Let ϕ be a quadratic form of dimension ≥ 10. If $\dim\phi = 10$ we suppose in addition that $\operatorname{ind}\phi \neq 1$. Then*

(i) *the homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\phi)/F) \to \operatorname{Tors}\operatorname{CH}^3(X_\phi)$ is surjective,*
(ii) *if ϕ is not a Pfister neighbor, then the homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\phi)/F) \to \operatorname{Tors}\operatorname{CH}^3(X_\phi)$ is an isomorphism,*
(iii) *If ϕ is a Pfister neighbor, then $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$.*

Proof. (i) If $\dim\phi > 12$, then $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$ by Theorem 3.1. Hence we can assume that $\dim\phi = 10$, 11, or 12. The standard transfer arguments reduce the general case to the case where F has no nontrivial odd extensions. By Lemma 1.16, there exists an extension E/F such that the form $\rho = \phi_E$ satisfies one of the conditions (a)–(c) of Proposition 9.10, and the homomorphism $N_{E/F}$:

$K(C_0(\phi_E)) \to K(C_0(\phi))$ is surjective. By Proposition 9.10, the homomorphism $\epsilon : \tilde{H}^4_{nr}(E(\phi)/E) \to \operatorname{Tors} \mathrm{CH}^3(X_{\phi_E})$ is an isomorphism. By Theorem 3.8, the homomorphism $N_{E/F} : \operatorname{Tors} \mathrm{CH}^3(X_{\phi_E}) \to \operatorname{Tors} \mathrm{CH}^3(X_\phi)$ is surjective.

Clearly, the diagram

$$\begin{array}{ccc} \tilde{H}^4_{nr}(E(\phi)/E) & \xrightarrow[\text{isomorphism}]{\epsilon_E} & \operatorname{Tors} \mathrm{CH}^3(X_{\phi_E}) \\ {\scriptstyle N_{E/F}}\Big\downarrow & & {\scriptstyle N_{E/F}}\Big\downarrow \text{surjective} \\ \tilde{H}^4_{nr}(F(\phi)/F) & \xrightarrow{\epsilon} & \operatorname{Tors} \mathrm{CH}^3(X_\phi) \end{array}$$

is commutative. Hence, the homomorphism ϵ is surjective.

(ii) is obvious in view of Item (i) and Theorem 5.2.

(iii) Since ϕ is a Pfister neighbor, Theorem 5.2(4) shows that the homomorphism ϵ is zero. Since ϵ is surjective (Item (i)), we have $\operatorname{Tors} \mathrm{CH}^3(X_\phi) = 0$. □

10. Proof of the Conjectures 0.8 and 0.9

In this section we prove the conjectures 0.8 and 0.9 (see Theorems 10.5 and 10.6). We start with the following lemma.

Lemma 10.1. *Let (ρ, ρ_0) be an anisotropic special pair of degree 4. Let E/F be the extension constructed in Proposition 6.10 and ψ be an F-form. Then*

1. *the special pair $(\rho_E, (\rho_0)_E)$ is anisotropic,*
2. *if $\rho_{F(\psi)}$ is anisotropic, then $\rho_{E(\psi)}$ is also anisotropic,*
3. *if $\dim \psi \geq 11$, then $(\rho_0)_{E(\psi)}$ is anisotropic.*

Proof. (1) Since (ρ, ρ_0) is a special pair of degree 4, it follows that ρ_0 is a form with maximal splitting and $9 \leq \dim \rho_0 \leq 16$. By Lemma 8.12, ρ_0 is anisotropic and is not a Pfister neighbor. By Lemma 6.12, $(\rho_0)_E$ is also anisotropic and is not a Pfister neighbor. Then Theorem 8.6(2) shows that ρ_E is anisotropic.

(2) The proof is the same as for Item (1).

(3) The form $(\rho_0)_E$ is anisotropic and is not a Pfister neighbor (see the proof of Item (1)). Since ρ_0 has maximal splitting, it follows that $(\rho_0)_E$ has maximal splitting. Since $\dim \psi \geq 11$, Corollary 1.12 shows that $(\rho_0)_{E(\psi)}$ is anisotropic. □

Lemma 10.2. *Let (ρ, ρ_0) be an anisotropic special pair of degree 4, and let ψ be a form such that $\rho_{F(\psi)}$ is isotropic. Then $\dim \psi \leq 12$. Moreover,*

- *if $\dim \psi = 12$, then $\psi \in I^3(F)$,*
- *if $\dim \psi = 11$, then $c(\psi) = 1$.*

Proof. Since $\rho_{F(\psi)}$ is isotropic, Theorem 8.6 shows that $(\rho_0)_{F(\psi)}$ is a Pfister neighbor. Let E/F be the extension constructed in Proposition 6.10. Obviously, $(\rho_0)_{E(\psi)}$ is a Pfister neighbor. By Lemma 10.1(3), the form $(\rho_0)_{E(\psi)}$ is anisotropic. Hence $\phi_{E(\psi)}$ is an anisotropic Pfister neighbor, where ϕ is an arbitrary 9-dimensional subform of ρ_0. Since $H^4(E) = 0$, Corollary 6.6 shows that $\operatorname{Tors} \mathrm{CH}^3(X_{\psi_E}) \neq 0$. By Theorem 3.1, we have $\dim \psi \leq 12$.

Now we consider the case $\dim\psi = 11$. Since $\operatorname{Tors}\operatorname{CH}^3(X_{\psi_E}) \neq 0$, Proposition 3.7(ii) shows that $\operatorname{ind}\psi_E = 1$. Hence $\operatorname{ind}\psi = 1$ (see Theorem 6.10(ii)).

Now, it suffices to consider the case $\dim\psi = 12$. Let ψ_0 be an 11-dimensional subform of ψ. Since $\rho_{F(\psi)}$ is isotropic, it follows that $\rho_{F(\psi_0)}$ is also isotropic. We have proved above that $\operatorname{ind}\psi_0 = 1$. Hence $\operatorname{ind}\psi = 1$ and $\operatorname{ind}\psi_E = 1$. Since $\operatorname{Tors}\operatorname{CH}^3(X_{\psi_E}) \neq 0$, Proposition 3.7(i) shows that $d_\pm\psi_E = 1$. Since $\operatorname{ind}\psi_E = 1$, we have $\psi_E \in I^3(E)$. Now, Item (v) of Theorem 6.10 shows that $\psi \in I^3(F)$. The proof is complete. □

Lemma 10.3. *Let (ρ,ρ_0) be an anisotropic special pair of degree 4, and let ψ be a 12-dimensional form such that $\rho_{F(\psi)}$ is isotropic. Then $\psi \in I^3(F)$, $\dim\rho = \dim\psi = 12$, and $\rho \overset{st}{\sim} \psi$.*

Proof. By Lemma 10.2, we have $\psi \in I^3(F)$. Since $\dim\psi = 12$, we can assume that ψ is a special 12-dimensional form containing a special 10-dimensional subform ψ_0 (see Lemma 9.6 and Example 9.3). Clearly, the special pair (ψ,ψ_0) is anisotropic (since the form $\rho_{F(\psi)}$ is isotropic). Now, let E/F be the field extension constructed in Proposition 6.10. By Lemma 10.1(1), the pairs $(\rho_E,(\rho_0)_E)$ and $(\psi_E,(\psi_0)_E)$ are anisotropic.

Since $\rho_{F(\psi)}$ is isotropic, $(\rho_0)_{F(\psi)}$ is a 4-fold neighbor (Theorem 8.6). By Lemma 10.1(3), the Pfister neighbor $(\rho_0)_{E(\psi)}$ is anisotropic. By Theorem 8.6(3), the form $(\psi_0)_{E(\psi)}$ is also an anisotropic 4-fold Pfister neighbor. By Lemma 6.7, we have $(\psi_0)_{E(\psi)} \overset{st}{\sim} (\rho_0)_{E(\psi)}$. Hence, $(\rho_0)_{E(\psi,\psi_0)}$ is isotropic. Since $\psi_0 \subset \psi$, the form $(\rho_0)_{E(\psi_0)}$ is also isotropic. By Lemma 8.13, $\rho_E \overset{st}{\sim} \psi_E$ and $\dim\rho = \dim\psi$. Hence the form $\psi_{E(\rho)}$ is isotropic. By Lemma 10.1(2) [8], the form $\psi_{F(\rho)}$ is also isotropic. Since $\rho_{F(\psi)}$ and $\psi_{F(\rho)}$ are both isotropic, we conclude that $\rho \overset{st}{\sim} \psi$. □

Corollary 10.4. *Let (ρ,ρ_0) be an anisotropic special pair of degree 4, and let ψ be an 11-dimensional form such that $\rho_{F(\psi)}$ is isotropic. Then $\operatorname{ind}\psi = 1$, $\dim\rho = 12$, and $\rho \overset{st}{\sim} \psi$.*

Proof. By Lemma 10.2, we have $\operatorname{ind}\psi = 1$. By Lemma 1.21(2), there is a 12-dimensional form $\gamma \in I^3(F)$ such that $\psi \overset{st}{\sim} \gamma$. Since $\rho_{F(\psi)}$ is isotropic and $\psi \overset{st}{\sim} \gamma$, it follows that $\rho_{F(\gamma)}$ is isotropic. Lemma 10.3 shows that $\dim\rho = 12$ and $\rho \overset{st}{\sim} \gamma$. Therefore $\rho \overset{st}{\sim} \gamma \overset{st}{\sim} \psi$. □

Theorem 10.5. *Let $\rho \in I^3(F)$ be an anisotropic 12-dimensional form. Let ψ be a form such that $\phi_{F(\psi)}$ is isotropic. Then $\dim\psi \leq 12$. Moreover,*

(1) *if $\dim\psi = 12$, then $\psi \in I^3(F)$ and $\psi \overset{st}{\sim} \rho$,*
(2) *if $\dim\psi = 11$, then $\operatorname{ind}\psi = 1$ and $\psi \overset{st}{\sim} \rho$.*

Proof. We can assume that ρ is a special 12-dimensional form containing a special 10-dimensional subform ρ_0. By our assumption, the special pair (ρ,ρ_0)

[8]Here we apply Lemma 10.1(2) for the special pair (ψ,ψ_0) over the function field of the form ρ.

is anisotropic. Now, the results follow readily from Lemmas 10.2 and 10.3 and Corollary 10.4 □

Theorem 10.6. *Let $\rho \in I^2(F)$ be an anisotropic 10-dimensional form with $\operatorname{ind} C(\phi) = 2$. Let ψ be a form of dimension > 10. Then the form $\rho_{F(\psi)}$ is anisotropic.*

Proof. We can assume that ρ is a special 10-dimensional form containing a special 9-dimensional subform ρ_0. By Lemma 10.2, $\dim \psi \leq 12$. Hence $\dim \psi =$ 11 or 12. Then Lemma 10.3 and Corollary 10.4 show that $\dim \rho = 12$, a contradiction. □

11. The group $\operatorname{Tors} \mathrm{CH}^3(X_\phi)$ for forms $\phi \in I^2(F)$

The main goal of this section is the computation of the group $\operatorname{Tors} \mathrm{CH}^3(X_\phi)$ for all forms $\phi \in I^2(F)$. We start the proof with the following easy lemma.

Lemma 11.1. *Let $\psi \in I^2(F)$ be a 14-dimensional Pfister neighbor. Then ψ is hyperbolic.*

Proof. By our assumption, there exists a 2-dimensional form μ such that $\psi + \mu \in GP_4(F) \subset I^2(F)$. Since $\psi \in I^2(F)$, it follows that $\mu \in I^2(F)$. Since $\dim \mu = 2$, the form μ is hyperbolic. Hence π is hyperbolic. Therefore $\psi = \pi - \mu$ is also hyperbolic. □

Corollary 11.2. *For any 14-dimensional form $\phi \in I^2(F)$, we have $\tilde{H}^4_{nr}(F(\psi)/F) = 0$.*

Proof. The case when ψ is isotropic is obvious (Lemma 5.1). Now we assume that ψ is anisotropic. Then Lemma 11.1 shows that ψ is not a Pfister neighbor. In this case the statement is proved in Corollary 5.4 □

Lemma 11.3. *Let ψ be a 14-dimensional form from $I^2(F)$ and let $E = F(\psi)$. Then for any form ϕ of dimension ≥ 9, the homomorphism $\tilde{H}^4_{nr}(F(\phi)/F) \to \tilde{H}^4_{nr}(E(\phi)/E)$ is injective.*

Proof. Let $X = X_\phi$ and $Y = Y_\psi$ be the projective quadrics corresponding to the forms ϕ and ψ. Let us consider the commutative diagram

$$\begin{tikzcd}
H^4(F) \arrow[r] \arrow[d] & H^4(F(X)) \arrow[r] \arrow[d, "\alpha"'] & \coprod\limits_{x \in X^{(1)}} H^3(F(x)) \arrow[d] \\
H^4(F(Y)) \arrow[r] \arrow[d] & H^4(F(X \times Y)) \arrow[r] \arrow[d] & \coprod\limits_{x \in X_{F(Y)}{}^{(1)}} H^3(F(Y)(x)) \\
\coprod\limits_{y \in Y^{(1)}} H^3(F(y)) \arrow[r, "\beta"] & \coprod\limits_{y \in Y_{F(X)}{}^{(1)}} H^3(F(X)(y)) &
\end{tikzcd}$$

Clearly, all columns and rows of this diagram are zero sequences. The homology group of the first and the second rows coincide with $\tilde{H}^4_{nr}(F(\phi)/F)$ and $\tilde{H}^4_{nr}(E(\phi)/E)$ respectively. Thus, we must verify that the homology group of the first row maps injectively to the homology group of the second row. This is a formal consequence of the following three properties of the diagram:

(i) the homomorphism α is injective,
(ii) the homomorphism β is injective,
(iii) the first column of the diagram is an exact sequence.

Let us verify these properties.

(i) By Theorem 4.4, the homomorphism α is not injective only in the case when $\psi_{F(X_\phi)}$ is an anisotropic 4-fold Pfister neighbor, which is impossible in view of Lemma 11.1.

(ii) The homomorphism β is injective in view of Theorem 4.4.

(iii) The homology group of the first column equals $\tilde{H}^4_{nr}(F(\psi)/F)$. By Corollary 11.2, this group is zero. Hence, the first column is exact. □

Corollary 11.4. *For any field F there exists an extension E/F with the following properties:*

(i) *all 14-dimensional forms from $I^3(E)$ are isotropic;*
(ii) *for all quadratic forms ϕ over F, we have $\operatorname{ind}\phi_E = \operatorname{ind}\phi$,*
(iii) *for all F-forms ϕ of dimension ≥ 9, the homomorphism $\tilde{H}^4_{nr}(F(\phi)/F) \to H^4_{nr}(E(\phi)/E)$ is injective.*
(iv) *for any F-form ϕ of dimension ≥ 9 which is not a 4-fold Pfister neighbor, the form ϕ_E is not an anisotropic 4-fold Pfister neighbor.*

Proof. Let us construct the fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_n \subset \dots$$

as follows. First, we set $F_0 = F$. For $n \geq 1$, we define F_n as the free composite of all fields $F_{n-1}(\psi)$, where ψ runs over all 14-dimensional forms from $I^3(F)$. We set $E = \cup_{n\geq 0} F_n$. We claim that the field E satisfies all required properties. Indeed, Item (i) follows from the definition of E; Item (ii) follows from Lemma 1.5; Item (iii) follows from Lemma 11.3 and Item (iv) follows from Corollary 6.14. □

Lemma 11.5. *Let ϕ be an F-form which is not an anisotropic 4-fold Pfister neighbor. Suppose also that ϕ satisfies one of the following conditions:*

- $\dim\phi = 10$, $\phi \in I^2(F)$, *and* $\operatorname{ind}\phi = 4$,
- $\dim\phi = 12$, $\phi \in I^2(F)$, *and* $\operatorname{ind}\phi = 2$.

Then $\tilde{H}^4_{nr}(F(\phi)/F) = 0$.

Proof. Corollary 11.4 reduces the proof to the case where all 14-dimensional forms from $I^3(F)$ are isotropic. Then Lemma 3.13 shows that $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$. By Theorem 5.2(3), we have $\tilde{H}^4_{nr}(F(\phi)/F) = 0$. □

Corollary 11.6. *Let ϕ be an F-form satisfying one of the following conditions:*

- $\dim \phi = 10$, $\phi \in I^2(F)$, *and* $\operatorname{ind} \phi = 4$,
- $\dim \phi = 12$, $\phi \in I^2(F)$, *and* $\operatorname{ind} \phi = 2$.

Then $\operatorname{Tors} \mathrm{CH}^3(X_\phi) = 0$. *Moreover,* $\operatorname{Tors} G^i K(X_\phi) = 0$ *for all* $i \leq 3$.

Proof. The first statement follows from Lemma 11.5 and items (ii) and (iii) of Corollary 9.11. The second statement is obvious in view of Theorem 3.1. □

Corollary 11.7. *Let ϕ be a form satisfying one of the following conditions:*

- $\dim \phi = 10$ *and* $\operatorname{ind} \phi \geq 4$,
- $\dim \phi = 12$ *and* $\operatorname{ind} \phi \geq 2$.

Then $\operatorname{Tors} \mathrm{CH}^3(X_\phi) = 0$.

Proof of Corollary 11.7. Corollaries 2.7 and 11.6 show that $\operatorname{Tors} G^i K(X_\phi) = 0$ for all $i \leq 3$. Hence $\operatorname{Tors} \mathrm{CH}^3(X_\phi) = \operatorname{Tors} G^3 K(X_\phi) = 0$. □

Proposition 11.8. *Let ϕ be an anisotropic quadratic form from $I^2(F)$. Then the group* $\operatorname{Tors} \mathrm{CH}^3(X_\phi)$ *is nonzero only in the following cases:*

- $\dim \phi = 8$ *and ϕ is similar to a 3-fold Pfister form,*
- $\dim \phi = 10$ *and* $\operatorname{ind} \phi = 2$,
- $\dim \phi = 12$ *and* $\operatorname{ind} \phi = 1$ *(i.e., $\phi \in I^3(F)$).*

In all cases listed above, the group $\operatorname{Tors} \mathrm{CH}^3(X_\phi)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

Proof. In case $\dim \phi \leq 8$, the proposition is proved in [Kar1]. Suppose that $\dim \phi > 8$. By Theorem 3.1, we can assume that $\dim \phi = 10$ or 12. If $\dim \phi = 10$, we necessarily have $\operatorname{ind} \phi \geq 2$ (otherwise, the form ϕ is isotropic by Pfister's theorem). Now, the required result follows readily from Corollary 11.7 and Proposition 9.10. □

12. Proof of Theorems 0.5 and 0.6

In this section we complete the computation of the third Chow group of quadrics X_ϕ for all forms of dimension ≥ 9 (Theorem 0.5). Besides, we prove our main results concerning unramified cohomology (Theorem 0.6 and Corollary 0.7). In the proofs we will use the following terminology: *we say that ϕ is of type (9-a), or (10-a), etc., if the form ϕ satisfies the corresponding conditions given in the formulation of Theorem 0.5.*

Lemma 12.1. *Let ϕ be a form of dimension ≥ 9 such that* $\operatorname{Tors} \mathrm{CH}^3(X_\phi) \neq 0$. *Then ϕ belongs to the list of forms given in Theorem 0.6.*

Proof. First, we consider the case when ϕ is isotropic. Corollary 3.4 shows that

- either $\dim \phi = 10$ and $\phi = \pi \perp \mathbb{H}$, where π is similar to an anisotropic 3-fold Pfister form,
- or $\dim \phi = 9$ and $\phi = \mu \perp \mathbb{H}$, where μ is an anisotropic 7-dimensional Pfister neighbor.

Obviously, in the case $\dim \phi = 10$, the form ϕ has type (10-a). Let us consider the case $\dim \phi = 9$. Let $d \in F^*$ be such that $\pi = \mu \perp \langle -d \rangle \in GP_3(F)$. Then we have $\phi = \mu \perp \mathbb{H} = \mu \perp \langle -d, d \rangle = \pi \perp \langle d \rangle$. Hence ϕ is of type (9-a).

In the following, we can assume that ϕ is an anisotropic form. By Theorem 3.1, it suffices to consider the following four cases separately: $\dim\phi = 9$, 10, 11, or 12.

$\underline{\dim\phi = 12}$. Since $\operatorname{Tors}\mathrm{CH}^3(X_\phi) \neq 0$, Corollary 11.7 shows that $\operatorname{ind}\phi = 1$. From Proposition 3.7(i) it follows that $\phi \in I^2(F)$. Since $\operatorname{ind}\phi = 1$ and $\phi \in I^2(F)$, we have $\phi \in I^3(F)$. Therefore ϕ is of type (12-a).

$\underline{\dim\phi = 11}$. Since $\operatorname{Tors}\mathrm{CH}^3(X_\phi) \neq 0$, Proposition 3.7(ii) shows that $\operatorname{ind}\phi = 1$ and hence ϕ has type (11-a).

$\underline{\dim\phi = 10}$. If $\phi \in I^2(F)$, then Proposition 11.8 shows that $\operatorname{ind}\phi = 2$. We get a form of type (10-b). Now, we can assume that $\phi \notin I^2(F)$. Then $d = d_\pm\phi \notin F^{*2}$. By Corollary 3.10, $\phi_{F(\sqrt{d})}$ is not hyperbolic. Proposition 3.7(iii) and Corollary 11.7 show that $\operatorname{ind}\phi = 1$. Hence, there exists a $c \in F^*$ such that $\phi \perp -c\langle\langle d\rangle\rangle \in I^3(F)$ (see, e.g., Lemma 1.19(i)). Now, it suffices to verify that the 12-dimensional form $\gamma = \phi \perp -c\langle\langle d\rangle\rangle$ is anisotropic (see Item (10-c) of Theorem 0.5). Assume the contrary. Then $\dim\gamma_{an} \leq 10$ and Corollary 1.9 shows that γ is Witt equivalent to some $\tau \in GP_3(F)$. In the Witt ring $W(F)$ we have $\phi = \gamma + c\langle\langle d\rangle\rangle = \tau + c\langle\langle d\rangle\rangle$. Since $\dim\phi = 10 = 8 + 2 = \dim(\tau \perp c\langle\langle d\rangle\rangle)$, it follows that $\phi \simeq \tau \perp c\langle\langle d\rangle\rangle$. Since $\operatorname{ind}\tau = 1$, Lemma 3.11 shows that $\operatorname{Tors}\mathrm{CH}^3(X_\phi) = 0$. We get a contradiction.

$\underline{\dim\phi = 9}$. Proposition 3.7(iv) shows that $\operatorname{ind}\phi \leq 2$. First, we suppose that $d = \det\phi \in D_F(\phi)$. Then ϕ has the form $\phi = \tau \perp \langle d\rangle$, where τ is an 8-dimensional form from $I^2(F)$. Then Lemma 3.12(ii) shows that $\operatorname{ind}\tau = \operatorname{ind}\phi = 1$. Then $\tau \in GP_3(F)$. We get a form of type (9-a).

Now, we can assume that $d = \det\phi \notin D_F(\phi)$. In this case, the 10-dimensional form $\mu = \phi \perp \langle -d\rangle \in I^2(F)$ is anisotropic. Pfister's theorem shows that $\operatorname{ind}\phi = \operatorname{ind}\mu \neq 1$. Since $\operatorname{ind}\phi \leq 2$, we have $\operatorname{ind}\phi = 2$. We claim that ϕ has the type (9-b). We have already proved that $\operatorname{ind}\phi = 2$ and $d \notin D_F(\phi)$. Thus, it suffices to verify that ϕ contains no 7-dimensional Pfister neighbors. This has been proved in Lemma 3.12. □

Lemma 12.2. *Let ϕ be one of the forms listed in Theorem 0.5. Suppose also that the form ϕ is not of type (9-a) or (10-a). Then the homomorphism $\epsilon : \tilde{H}^4_{nr}(F(\phi)/F) \to \operatorname{Tors}\mathrm{CH}^3(X_\phi)$ is an isomorphism and $\tilde{H}^4_{nr}(F(\phi)/F) \simeq \operatorname{Tors}\mathrm{CH}^3(X_\phi) \simeq \mathbb{Z}/2\mathbb{Z}$*

Proof. If $\dim\phi = 11$ or 12 the statement is obvious in view of Proposition 9.10. If $\dim\phi = 9$ or 10, it suffices to verify the following two properties of the form ϕ:

(i) ϕ is not a Pfister neighbor,
(ii) $\tilde{H}^4_{nr}(F(\phi)/F) \neq 0$.

After this, the lemma will be obvious in view of Corollary 5.3.

Now, let us study the forms case by case.

(9-b). Since $d = \det\phi \notin D_F(\phi)$, the form $\rho := \phi \perp \langle -d\rangle$ is anisotropic. Clearly, $\dim\rho = 10$, $\rho \in I^2(F)$ and $\operatorname{ind}\rho = \operatorname{ind}\phi = 2$. In view of Lemma 9.6, we can assume that ρ is a "special 10-dimensional form" containing a special 9-dimensional subform ρ_0 (Example 9.4).

Since $\operatorname{ind}\phi = 2$, there exists a 3-dimensional form μ such that $\phi \perp \mu \in I^3(F)$ (see e.g, Lemma 1.19(ii)). Set $\tilde{\rho} = \phi \perp \mu$. Clearly, $\dim\tilde{\rho} = 12$. First, we suppose that $\tilde{\rho}$ is isotropic. Then $\tilde{\rho}$ is Witt equivalent to some form $\pi \in GP_3(F)$ (Corollary 1.9). In the Witt ring, we have $\phi + \mu = \tilde{\rho} = \pi$. Then $\pi - \phi = \mu$. Hence, ϕ and π contain a common subform of dimension $\frac{1}{2}(\dim\pi + \dim\phi - \dim\mu) = \frac{1}{2}(9 + 8 - 3) = 7$. Therefore, ϕ contains a 7-dimensional Pfister neighbor of π. This contradicts condition (9-b). Thus, we have proved that $\tilde{\rho}$ is anisotropic. In view of Lemma 9.6, we can assume that $\tilde{\rho}$ is a "special 12-dimensional form" which contains a special 10-dimensional subform $\tilde{\rho}_0$ (Example 9.3).

We have constructed two anisotropic special pairs (ρ, ρ_0) and $(\tilde{\rho}, \tilde{\rho}_0)$ such that $\phi \subset \rho$ and $\phi \subset \tilde{\rho}$. Corollary 9.7 shows that ϕ is not a Pfister neighbor. We have realized Item (i) of our plan. Since $\phi \subset \rho$ and $\phi \subset \tilde{\rho}$, the forms $\rho_{F(\phi)}$ and $\tilde{\rho}_{F(\phi)}$ are isotropic. Theorem 8.6 shows that $(\rho_0)_{F(\phi)}$ and $(\tilde{\rho}_0)_{F(\phi)}$ are Pfister neighbors. We get two elements $\tilde{e}^4(\operatorname{Pf}((\rho_0)_{F(\phi)}))$ and $\tilde{e}^4(\operatorname{Pf}((\tilde{\rho}_0)_{F(\phi)}))$ of the group $\tilde{H}^4_{nr}(F(\phi)/F)$. If at least one of these elements is nonzero, the proof is complete (see item (ii) of our plan of the proof).

Thus, we can assume that $\tilde{e}^4(\operatorname{Pf}((\rho_0)_{F(\phi)})) = \tilde{e}^4(\operatorname{Pf}((\tilde{\rho}_0)_{F(\phi)})) = 0$.

Let E/F be the field extension constructed in Proposition 6.10. By Lemma 10.1(1), the forms ρ_E and $\tilde{\rho}_E$ are anisotropic. Hence, $(\rho_0)_E$ and $(\tilde{\rho}_0)_E$ are also anisotropic. Since $H^4(E) = 0$, we have $e^4(\operatorname{Pf}((\rho_0)_{E(\phi)})) = e^4(\operatorname{Pf}((\tilde{\rho}_0)_{E(\phi)})) = 0$ in the group $H^4_{nr}(E(\phi)/E) \subset H^4(E(\phi))$ (without "tilde" !). By Theorem 4.1, $\operatorname{Pf}((\rho_0)_{E(\phi)})) = \operatorname{Pf}((\tilde{\rho}_0)_{E(\phi)}) = 0$. This means that both Pfister neighbors $(\rho_0)_{E(\phi)}$ and $(\tilde{\rho}_0)_{E(\phi)}$ are isotropic. Since $(\rho_0)_E$ and $(\tilde{\rho}_0)_E$ are anisotropic forms with maximal splitting (Theorem 8.6), it follows that $(\rho_0)_E \overset{st}{\sim} \phi_E$ and $(\tilde{\rho}_0)_E \overset{st}{\sim} \phi_E$ (see Theorem 1.11). Hence, $(\rho_0)_E \overset{st}{\sim} (\tilde{\rho}_0)_E$. By Proposition 8.13, we get $\dim\rho = \dim\tilde{\rho}$, which contradicts the equations $\dim\rho = 10$ and $\dim\tilde{\rho} = 12$. This completes the proof in case (9-b).

(10-b). In this case the result of the lemma is covered by Proposition 9.10.

(10-c). Let ϕ, d, and τ be as in (10-c). Since $\dim\tau = 12$ and $\tau \in I^3(F)$, we can assume that τ is a "special 12-dimensional form" containing a 10-dimensional special subform τ_0 (Example 9.3). Since $\phi \subset \tau$, Corollary 9.7 shows that ϕ is not a Pfister neighbor. This completes the proof of Item (i) of our plan. To prove Item (ii), it suffices to verify that the element $\tilde{e}^4(\operatorname{Pf}((\tau_0)_{F(\phi)}))$ is non-zero in the group $\tilde{H}^4_{nr}(F(\phi)/F)$. Assume the contrary, $\tilde{e}^4(\operatorname{Pf}((\tau_0)_{F(\phi)})) = 0$. Let E/F be the extension constructed in Proposition 6.10. By Lemma 10.1(1), the form τ_E is anisotropic. Hence $(\tau_0)_E$ is an anisotropic form which is not a Pfister neighbor. On the other hand, the form $(\tau_0)_{E(\psi)}$ is a Pfister neighbor because the form $\tau_{E(\psi)}$ is isotropic. Since $H^4(E) = 0$, we get $e^4(\operatorname{Pf}((\tau_0)_{E(\phi)})) = 0 \in H^4(E(\phi))$. Hence, the Pfister neighbor $(\tau_0)_{E(\phi)}$

is isotropic. Since $(\tau_0)_E$ is an anisotropic form with maximal splitting, Theorem 1.11 shows that ϕ_E also has maximal splitting. Since $\operatorname{ind}\phi = 1$, it follows that $\phi_{E(\sqrt{d})} \in I^3(F)$. Since $\dim\phi = 10$, Pfister's theorem shows that $\phi_{E(\sqrt{d})}$ is isotropic. Taking into account that ϕ_E has maximal splitting, we have $\dim(\phi_{E(\sqrt{d})})_{an} \leq 6$. Now, the Arason–Pfister Hauptsatz implies that $\phi_{E(\sqrt{d})}$ is hyperbolic. By Item (vi) of Proposition 6.10, we conclude that $\phi_{F(\sqrt{d})}$ is hyperbolic. This contradicts condition (10-c). The proof is complete. □

Proof of Theorem 0.6. If $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = 0$, then ϵ is an isomorphism by Theorem 5.2(3). Now, we can assume that $\operatorname{Tors}\operatorname{CH}^3(X_\phi) \neq 0$. By Lemma 12.1, the form ϕ belongs to the list of forms given in Theorem 0.5. Since ϕ is anisotropic and is not a Pfister neighbor, it follows that ϕ is not of type (9-a) or (10-a). Then Lemma 12.2 completes the proof. □

Lemma 12.3. *If ϕ has type (9-a) or (10-a), then* $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = \mathbb{Z}/2\mathbb{Z}$.

Proof. (9-a). If $\phi = \pi \perp \langle d\rangle$ is isotropic, then π can be written as $\mu \perp \langle -d\rangle$, where μ is a 7-dimensional anisotropic Pfister neighbor. Then $\phi = \mu \perp \langle -d, d\rangle = \mu \perp \mathbb{H}$. By Corollary 3.3, we have $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = \mathbb{Z}/2\mathbb{Z}$. Now we can assume that ϕ is anisotropic. Since $\operatorname{ind}\phi = 1$, we have $s = i_S(\phi) = 4$ (Lemma 1.1). By [Kar1, Th. 3.8], the set

$$U = \{i \mid \operatorname{Tors} G^i K(X_\phi) \neq 0\}$$

consists exactly of $s = 4$ elements. Since $\dim X_\phi = \dim\phi - 2 = 7$, we have $U \subset \{0, 1, \dots, 7\}$. By Theorem 3.1, $\operatorname{Tors} G^i K(X_\phi) = 0$ for $i \leq 2$. Hence, $U \subset \{3, 4, 5, 6, 7\}$. By [Sw], we have $\operatorname{Tors} G^7 K(X_\phi) = \operatorname{Tors} G_0 K(X_\phi) = 0$. Hence, $U \subset \{3, 4, 5, 6\}$. Since U consists of 4 elements, we get $U = \{3, 4, 5, 6\}$. In particular, $3 \in U$. Hence, $\operatorname{Tors}\operatorname{CH}^3 X_\phi \simeq \operatorname{Tors} G^3 K(X_\phi) = \mathbb{Z}/2\mathbb{Z}$.

(10-a). Let $\phi = \pi \perp \mathbb{H}$, where π is similar to an anisotropic Pfister form. In this case $\operatorname{Tors}\operatorname{CH}^3(X_\phi) = \mathbb{Z}/2\mathbb{Z}$ in view of Corollary 3.3. □

Proof of Theorem 0.5. The theorem is a formal consequence of Lemmas 12.1 12.2, and 12.3. □

Proof of Corollary 0.7. Let us consider the homomorphisms

$$\tilde{H}^4_{nr}(F(\phi)/F) \xrightarrow{\alpha} \tilde{H}^4_{nr}(F(\phi)/F, \mathbb{Q}/\mathbb{Z}(3)) \xrightarrow{\beta} \operatorname{Tors}\operatorname{CH}^3(X_\phi),$$

where α is the homomorphism induced by the natural homomorphism

$$H^4(F(\phi)) \to H^4(F(\phi), \mathbb{Q}/\mathbb{Z}(3))$$

and β is the homomorphism defined in [KRS1, Th.6(1)]. We must prove that α is surjective. If ϕ is a 4-fold Pfister neighbor or isotropic, then the group $\tilde{H}^4_{nr}(F(\phi)/F, \mathbb{Q}/\mathbb{Z}(3))$ is zero by [KRS1, Th.3(1) and Prop.2.5(a)]. Therefore, α is surjective. If ϕ is anisotropic and is not a 4-fold Pfister neighbor, then the composition $\beta \circ \alpha$ is surjective by Theorem 0.6. By [KRS1, Th.6(1)], the homomorphism β is injective. Hence α is surjective. □

13. Forms of height 2 and degree 3

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The main goal of this section is to prove Conjecture 0.11 in the case when $n = 3$ and $\operatorname{char} F = 0$. We will use the following theorem of A. Vishik.

Theorem 13.1. (A. Vishik, [Vi3, Vi4]). *Let F be a field of characteristic zero. Let ϕ be an even-dimensional F-form of height 2 and degree d. Then*

$$\dim\phi = 3\cdot 2^{d-1}, \qquad \dim\phi = 2^{d+1}, \text{ or } \dim\phi = 2^N - 2^d \text{ for some } N \geq d+1.$$

The proof of this theorem depends on Voevodsky's announced theorem [9] that Milnor's conjecture is valid ([Vo]). We need below only the following special case of Theorem 13.1: *If ϕ is a form of height 2 and degree 3, then* $\dim\phi \neq 14$. This special case was proved by Vishik in his thesis [Vi1, Statement 1.2.1] under the additional hypothesis $\sqrt{-1} \in F^*$.

Corollary 13.2. *Let F be a field of characteristic 0, and let $\phi \in I^3(F)$ be an anisotropic 14-dimensional form. Then $\dim(\phi_{F(\phi)})_{an} = 12$.*

Proof. The form $\phi_{F(\phi)}$ is nonhyperbolic because ϕ is not similar to a Pfister form. If $\dim(\phi_{F(\phi)})_{an} \neq 12$, then $\dim(\phi_{F(\phi)})_{an} \leq 8$ by Pfister's theorem. Since $\phi \in I^3(F)$, we get $\dim(\phi_{F(\phi)})_{an} \in GP_3(F(\phi))$. Hence $ht(\phi) = 2$ and $\deg\phi = 3$. However, Theorem 13.1 implies that $\dim\phi \neq 14$ for all forms of height 2 and degree 3, a contradiction. ☐

Corollary 13.3. *Let F be a field of characteristic zero, and $\tau \in I^3(F)$ be an anisotropic form of dimension 12 or 14. Let ψ be a form of dimension > 12. Then $\dim(\tau_{F(\psi)})_{an} \geq 12$.*

Proof. If $\dim\tau = 12$, the statement follows from Theorem 10.5. Hence we can assume that $\dim\tau = 14$. By Corollary 13.2, we have $\dim(\tau_{F(\tau)})_{an} = 12$. Applying Theorem 10.5 once again, we see that $(\tau_{F(\tau)})_{an}$ is anisotropic over the function field of the form $\psi_{F(\tau)}$. Hence $\dim(\tau_{F(\tau,\psi)})_{an} = 12$. Therefore $\dim(\tau_{F(\psi)})_{an} \geq \dim(\tau_{F(\tau,\psi)})_{an} = 12$. ☐

Below we will use the following terminology: we say that an element $u \in H^n(F)$ is a symbol, if there exist $a_1, \dots, a_n \in F^*$ such that $u = (a_1, \dots, a_n)$.

The following lemma is well known (it is an easy consequence of the isomorphism $H^3(F) \simeq I^3(F)/I^4(F)$ and the "linkage properties" of Pfister forms [EL3]).

Lemma 13.4. *Let $u = (a_1, a_2, a_3) + (b_1, b_2, b_3) \in H^3(F)$. The the following conditions are equivalent:*

- *the element u is a symbol,*
- *the form $(\langle\langle a_1, a_2, a_3\rangle\rangle \perp -\langle\langle b_1, b_2, b_3\rangle\rangle)_{an}$ is zero or belongs to $GP_3(F)$.*

[9] More precisely, the proof of Theorem 13.1 is based on the whole technique developed by Voevodsky in his proof of Milnor conjecture. We also note that, for good forms over a field of characteristic not 2, this result was proved earlier (see [H5] and [HR, §3]).

Proposition 13.5. (cf. [Kah1, Prop.5]). *Let F be a field of characteristic zero. Let ψ be an F-form of dimension > 12. Let $u \in H^3(F)$ be an element such that $u_{F(\psi)}$ is a symbol. Then u is also a symbol.*

Proof. Suppose that u is not a symbol. By [Kah1, Prop.5], u is a sum of at most two symbols. Thus, $u = (a_1, a_2, a_3) + (b_1, b_2, b_3)$ for suitable $a_i, b_i \in F^*$. Let $q = (\langle\langle a_1, a_2, a_3\rangle\rangle \perp -\langle\langle b_1, b_2, b_3\rangle\rangle)_{an}$. Clearly, $\dim q \leq 14$. Since u is not a symbol, Lemma 13.4 shows that $\dim q > 8$. By Pfister's theorem, $\dim q \geq 12$. Hence $\dim q = 12$ or 14. By Lemma 13.3, $\dim(q_{F(\psi)})_{an} \geq 12$. By Lemma 13.4, the element $u_{F(\psi)}$ is not a symbol, which contradicts the hypothesis of the proposition. □

Proposition 13.6. *Let F be a field of characteristic zero, and ψ be an F-form of dimension > 12. Let π be a 3-fold Pfister form over $F(\psi)$. Suppose also that $x\pi$ is unramified for some element $x \in F(\psi)^*$ (i.e., $x\pi \in W_{nr}(F(\psi)/F)$). Then π is defined over F by a 3-fold Pfister form (i.e., there exists $\tau \in P_3(F)$ such that $\pi \simeq \tau_{F(\psi)}$).*

Proof. Since $x\pi \in W_{nr}(F(\psi)/F)$, we have $e^3(\pi) = e^3(x\pi) \in H^3_{nr}(F(\psi)/F)$. Since $\dim \psi > 8$, the homomorphism $H^3(F) \to H^3_{nr}(F(\psi)/F)$ is surjective (see [KRS1, Cor.8(2a)]). Hence, there exists an element $u \in H^3(F)$ such that $u_{F(\psi)} = e^3(\pi)$. Since $e^3(\pi)$ is a symbol, Proposition 13.5 shows that u is also a symbol. Let $a_1, a_2, a_3 \in F^*$ be such that $u = (a_1, a_2, a_3)$, and let $\tau = \langle\langle a_1, a_2, a_3\rangle\rangle$. We have $e^3(\tau_{F(\psi)}) = u_{F(\psi)} = e^3(\pi)$. By Theorem 4.1, we have $\pi = \tau_{F(\psi)}$. The proof is complete. □

Corollary 13.7. *Let F be a field of characteristic zero and ϕ be a nongood F-form of degree 3 and height 2. Then $\dim \phi \leq 12$.*

Proof. Suppose that $\dim \phi > 12$. Since ϕ has degree 3 and height 2, it follows that $(\phi_{F(\phi)})_{an} \in GP_3(F(\phi))$. Hence, there exists $\pi \in P_3(F(\phi))$ and $x \in F(\phi)^*$ such that $(\phi_{F(\phi)})_{an} = x\pi$. By definition, $x\pi \in \operatorname{Im}(W(F) \to W(F(\phi)))$. Hence $x\pi \in W_{nr}(F(\phi)/F)$. By Proposition 13.6, π is defined over F by a Pfister form. This contradicts the definition of nongood forms. □

We recall the conjecture of Bruno Kahn concerning the classifications of forms of height 2.

Conjecture 13.8. (see [Kah2, Conjecture 7]). *Let ϕ be a quadratic form of height 2 and degree $n \geq 2$. Then at least one of the following conditions holds:*

- *(excellent forms) $\phi \simeq a\rho \otimes \tau'$, where $a \in F^*$, $\rho \in P_n(F)$ and τ' is the pure subform of $\tau \in P_m(F)$ with $m \geq 2$.*
- *(good nonexcellent forms) $\phi \simeq \rho \otimes \psi$, where $\rho \in P_{n-1}(F)$ and ψ is a 4-dimensional form.*
- *(nongood forms) $\phi \simeq \rho \otimes \gamma$, where $\rho \in P_{n-2}(F)$ and γ is an Albert form.*

Theorem 13.9. *Let F be a field of characteristic zero. Then Conjecture 13.8 is true for $n \leq 3$.*

Proof. M. Knebusch proved Conjecture 13.8 for all excellent forms. B. Kahn proved Conjecture 13.8 for $n = 2$ ([Kah2]). Moreover, he proved the second item of this conjecture (i.e., for good forms) for $n = 3$ (see [Kah2, Th.2.12]). To complete the proof of the theorem it suffices to classify nongood forms of height 2 and degree 3.

Since $\deg \phi = 3$ and $ht(\phi) = 2 > 1$, we have $\phi \in I^3(F)$ and $\dim \phi > 8$. By Pfister's theorem, we have $\dim \phi \geq 12$. On the other hand, Corollary 13.7 shows that $\dim \phi \leq 12$. Hence, ϕ is a 12-dimensional form from $I^3(F)$. Therefore, $\phi = \langle\langle a \rangle\rangle \otimes \gamma$, for suitable $a \in F^*$ and an Albert form γ (Lemma 9.6). □

Corollary 13.10. *Let F be a field of characteristic zero, and let ϕ be an F-form of degree 2 and height 3.* [10] *Then $\dim \phi \neq 14$. If $\dim \phi = 16$, then $\phi \in GP_{4,2}F$ (i.e. $\phi \simeq k \langle\langle a \rangle\rangle \otimes (\pi' \perp \langle b \rangle)$, where π' is the pure subform of some 3-fold Pfister form).*

Proof. This follows from Theorem 13.9 and [Lag, Th.6]. □

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