QUADRATIC FORMS OVER FRACTION FIELDS OF TWO-DIMENSIONAL HENSELIAN RINGS AND BRAUER GROUPS OF RELATED SCHEMES

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Introduction

Let A be an excellent henselian two-dimensional local domain (for the definition of excellent rings, see [EGA IV$_2$], 7.8.2). Let K be its field of fractions and k its residue field.

Assume that k is separably closed. If k of positive characteristic p, we show that the unramified Brauer group of K (with respect to all rank 1 discrete valuations of K) is a p-group. This group is trivial in each of the following cases: k is of characteristic zero, or A is complete, or A is a henselization of an R-algebra of finite type, where R is either a field or an excellent discrete valuation ring.

Under some more restrictive conditions such a result was obtained by Artin [Art$_2$] in 1987. We actually prove a generalization of Artin's result for the case of an arbitrary residue field k, following ideas of Artin and Grothendieck, as developed in Grothendieck's 1968 paper [GB III].

Assuming further that 2 is a unit in A, we prove that if k is separably closed or finite, then every quadratic form of rank 3 or 4 which is isotropic in all completions of K with respect to rank 1 discrete valuations is isotropic.

If k is separably closed of characteristic $p \geq 0$, we prove that any division algebra over K whose order in the Brauer group is n prime to p is cyclic of degree n. For A the henselization or the completion at a closed point of a normal surface over an algebraically closed field of characteristic zero, this result was first obtained by Ford and Saltman [FS]. For k the separable closure of a finite field, the result was obtained by Hoobler ([Ho], Thm. 13), who used higher class field theory à la Kato-Saito.

Let A, K, k be as in the beginning of this introduction, with k algebraically closed of characteristic different from the prime l. Gabber and Kato proved that the l-cohomological dimension of K is 2 (see Saito [Sai$_1$], Thm. 5.1). Combining this result and the above cyclicity statement, we prove that any quadratic form over K of rank at least 5 is isotropic.

The special case where K is the fraction field $\mathbb{C}((X,Y))$ of $A = \mathbb{C}[[X,Y]]$ had been considered earlier. In that case, the local-global principle for quadratic forms of rank 3 is easy. For rank 4 it was announced by Jaworski [Ja] after some special

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cases had been proved by Hatt-Arnold [HA]. That quadratic forms of rank at least 5 over $\mathbb{C}((X,Y))$ are isotropic was proved in [CDLR] using the Weierstraß preparation theorem.

Assume now that k is real closed. We show that every rank 4 quadratic form over K which is torsion in the Witt group of K and is isotropic in all completions of K with respect to rank 1 discrete valuations is isotropic. We also show that every quadratic form of even rank ≥ 6 which is torsion in the Witt group of K is isotropic. In particular, the u-invariant of K, as defined by Elman and Lam [EL], is 4.

In the whole paper, we shall only consider rank 1 discrete valuations, and we shall simply call them discrete valuations.

Given an integer $n > 0$ and an abelian group A, we let ${}_nA = \{x \in A, nx = 0\}$.

1. The unramified Brauer group.

We first recall a few definitions and theorems from Grothendieck's exposés on the Brauer group [GB I], [GB II], [GB III]. Given a scheme X, we denote its cohomological Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ by $Br(X)$. We let $Br_{Az}(X)$ denote the Azumaya Brauer group. This is a torsion group. There is a natural inclusion $Br_{Az}(X) \subset Br(X)$.

Given a discrete valuation ring R with field of fractions K, and a class $\alpha \in Br(K)$, one says that α is unramified with respect to R, if it is in the image of the natural embedding $Br(R) \to Br(K)$. This property can be checked by going over to the completion of R. Given a field K we denote by $Br_{nr}(K)$ the unramified Brauer group of K, consisting of all classes of $Br(K)$ which are unramified with respect to all discrete valuations of K.

We recall a result which was recorded in [OPS], although it was never used in that paper.

Lemma 1.1. *Let X be a noetherian reduced scheme and U an open subscheme containing all singular points and all generic points of X. Then the restriction map $Br(X) \to Br(U)$ is injective.*

Proof. See [OPS], Theorem 4.1.

Lemma 1.2. *(a) For a noetherian scheme X of dimension at most one, and for a regular noetherian scheme X of dimension at most two, the inclusion $Br_{Az}(X) \subset Br(X)$ is an equality.*

(b) For X a reduced, separated, excellent scheme of dimension at most two such that any finite set of closed points is contained in an affine open set, the natural inclusion $Br_{Az}(X) \subset Br(X)$ identifies $Br_{Az}(X)$ with the torsion subgroup of $Br(X)$.

(c) For any regular integral scheme X of dimension at most two, with field of fractions K, there are natural inclusions $Br_{nr}(K) \subset Br(X) \subset Br(K)$.

Proof. (a) This is Cor. 2.2 of [GB II].

(b) Since X is excellent and reduced, the singular locus is closed of dimension at most one. One may thus find two affine open sets U and V such that their union W contains the generic points of all components of X, and the complement of W in X consists of finitely many points whose local rings are regular of dimension 2. By Lemma 1.1, the restriction map $Br(X) \to Br(W)$ is injective. By a theorem of Gabber [Ga], the map $Br_{Az}(W) \to Br(W)$ identifies $Br_{Az}(W)$ with the torsion in $Br(W)$. Since all points of the complement of W are regular on X, the proof of Cor. 2.2 of [GB II] shows that the map $Br_{Az}(X) \to Br_{Az}(W)$ is surjective. Thus the map $Br_{Az}(X) \to Br_{Az}(W)$ is an isomorphism and (b) follows.

(c) This is Thm. 6.1.b of [GB III].

Let A be an excellent henselian two-dimensional local domain , let k be its residue field. A *model* of A is an integral scheme X equipped with a projective birational morphism $X \to Spec(A)$. According to Hironaka, Abhyankar and Lipman (see [Li$_1$], [Li$_2$]) there exist regular models of A. The fibre X_0 of $X \to Spec(A)$ at the closed point of $Spec(A)$ is a projective variety of dimension at most one over k. Given any one-dimensional reduced closed subscheme $C \subset X$, there exists a further projective birational morphism $\pi : X' \to X$, with X' regular integral, such that the support of the curve $\pi^{-1}(C)$ is a union of regular curves with normal crossings ([Sh], Theorem, page 38 and Remark 2, page 43; note that blow-ups of excellent schemes are excellent and so are closed subschemes of excellent schemes).

Theorem 1.3. *Let A be a henselian local ring. Let k be its residue field and $p \geq 0$ be the characteristic of k. Let $\pi : X \to Spec(A)$ be a proper morphism and assume that the fibre $X_0 \to Spec(k)$ of π over the closed point of $Spec(A)$ is of dimension at most one. For any prime l different from p, the restriction map $Br(X) \to Br(X_0)$ induces an isomorphism on l-primary torsion subgroups. If the scheme X is regular, the restriction map is an isomorphism up to p-primary torsion.*

Proof. Let n be an integer, prime to p if k is of characteristic p. The Kummer sequence of étale sheaves

$$1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1 ,$$

where $\mathbb{G}_m \to \mathbb{G}_m$ is given by $x \mapsto x^n$, induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Pic(X)/n & \longrightarrow & H^2_{\text{ét}}(X, \mu_n) & \longrightarrow & {}_nBr(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Pic(X_0)/n & \longrightarrow & H^2_{\text{ét}}(X_0, \mu_n) & \longrightarrow & {}_nBr(X_0) & \longrightarrow & 0 \end{array}$$

The vertical maps are induced by the inclusion $X_0 \to X$. Since X_0 is of dimension at most one, $H^2(X_0, \mathcal{O}_{X_0}) = 0$. Since A is henselian, this implies ([EGA IV$_4$], 21.9.12) that the map $Pic(X) \to Pic(X_0)$ is surjective. Since the morphism $\pi : X \to Spec(A)$ is projective and the ring A henselian, the proper base change theorem ([Mi], VI.2.7) implies that the restriction map $H^2_{\text{ét}}(X, \mu_n) \to H^2_{\text{ét}}(X_0, \mu_n)$ is an isomorphism. Thus the map ${}_nBr(X) \to {}_nBr(X_0)$ is an isomorphism.

If X is regular, then $Br(X)$ is torsion. The group $Br(X_0)$ is torsion because X_0 is a curve. The last statement of the theorem follows.

There are two cases where we may get hold of the p-part. The first one is the case where A is complete, which we now discuss. We start with a series of lemmas.

Let A be a local ring, let $\pi : X \to Spec(A)$ be a proper map such that the fibre $X_0 \to Spec(k)$ of π over the closed point of $Spec(A)$ is of of dimension at most one. Let m the maximal ideal of A and X_n the fibre of π over $Spec(A/m^{n+1})$.

Lemma 1.4. *The natural maps*

$$Pic(X_{n+1}) \to Pic(X_n)$$

are surjective.

Proof. We have the exact sequence of sheaves

$$0 \longrightarrow \frac{m^n\mathcal{O}_X}{m^{n+1}\mathcal{O}_X} \longrightarrow \Big(\frac{\mathcal{O}_X}{m^{n+1}\mathcal{O}_X}\Big)^* \longrightarrow \Big(\frac{\mathcal{O}_X}{m^n\mathcal{O}_X}\Big)^* \longrightarrow 1\ ,$$

where the left map is given by $x \mapsto 1 + x$. We have

$$H^2\Big(X, \frac{m^n\mathcal{O}_X}{m^{n+1}\mathcal{O}_X}\Big) = H^2\Big(X_0, \frac{m^n\mathcal{O}_X}{m^{n+1}\mathcal{O}_X}\Big) = 0$$

because X_0 is of dimension at most one. Hence the map

$$H^1(X_{n+1}, \mathcal{O}^*_{X_{n+1}}) \to H^1(X_n, \mathcal{O}^*_{X_n})$$

is surjective.

Lemma 1.5. *Assume that A is complete. Then the canonical homomorphism*

$$Br_{Az}(X) \to \lim_{\longleftarrow} Br_{Az}(X_n)$$

is an isomorphism.

Proof. Let $\mathcal{A}$ be an Azumaya algebra over X. Denote by $\mathcal{A}_n$ the algebra obtained from $\mathcal{A}$ under base change from X to X_n and suppose that it is trivial for each n. Let

$$u_n : \mathcal{A}_n \tilde{\to} \mathcal{E}nd(V_n)$$

be an isomorphism, where V_n is a locally free sheaf on X_n. The sheaf V_n is determined by $\mathcal{A}_n$ up to a line bundle. By Lemma 1.4 we can successively modify each V_{n+1} in such a way that V_n is isomorphic to $V_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}$ and the u_n's build up a projective system. By [EGA III_1], 5.1.4, the projective system $(V_n, n \in \mathbb{N})$ gives a locally free $\mathcal{O}_X$-module V and an isomorphism

$$u : \mathcal{A} \tilde{\to} \mathcal{E}nd(V)$$

of locally free sheaves such that u induces u_n on X_n. Since each u_n is an algebra homomorphism, so is u.

We now prove the surjectivity. We first show that there exists an open covering $X_0 = U_0 \cup V_0$ with U_0, V_0 and $U_0 \cap V_0 := W_0$ affine. Let $\overline{X_0}$ be the reduced scheme associated to X_0 and $f : Y \to \overline{X_0}$ its the normalization. By [EGA II], Corollaire 7.4.6 the morphism f is finite. We remark that, by the assumption that X is proper over $Spec(A)$, the scheme X_0 is separated and therefore Y is a projective regular curve (see [EGA II], Corollaire 7.4.10). Let $Y^\circ \subset Y$ be the open set on which f is an isomorphism. Choose two disjoint sets of closed points $\{P_1, \ldots, P_r\}$ and $\{Q_1, \ldots, Q_s\}$ on Y° such that $U^\circ = Y \setminus \{P_1, \ldots, P_r\}$ and $V^\circ = Y \setminus \{Q_1, \ldots, Q_s\}$ are affine. Then $U^\circ \cap V^\circ$ is affine too. The restriction of f to these three open sets is finite, hence, by Chevalley's theorem ([EGA II], Théorème 6.7.1) their images under f are affine open subsets of $\overline{X_0}$. Since a scheme is affine if and only if its associated reduced scheme is affine ([EGA I], Corollaire 5.1.10) the open sets $U_0 = X_0 \setminus f(\{P_1, \dots, P_s\})$, $V_0 = X_0 \setminus f(\{Q_1, \dots, Q_s\})$ and $U_0 \cap V_0$ are affine.

There are open sets U, V in X such that $U \cup V = X$, $U \cap X_0 = U_0$ and $V \cap X_0 = V_0$. Let U_n and V_n be the intersections of U and V with X_n. Since the maps $U_0 \to U_n$, $V_0 \to V_n$ and $W_0 \to W_n$ are finite, the sets U_n, V_n and $W_n := U_n \cap V_n$ are affine. We now show that any Azumaya algebra over X_n may be lifted to an Azumaya algebra over X_{n+1}. Let $\mathcal{A}_0$ be an Azumaya algebra over X_0. Let $\mathcal{B}_0$ and $\mathcal{C}_0$ be the restrictions of $\mathcal{A}_0$ to U_0 and V_0. By [Ci], Theorem 3, we can find, for any n, Azumaya algebras $\mathcal{B}_n$ over U_n and $\mathcal{C}_n$ over V_n which restrict to $\mathcal{B}_0$ and $\mathcal{C}_0$. The algebra $\mathcal{A}_0$ defines an isomorphism $\varphi_0 : \mathcal{B}_0|_{W_0} \to \mathcal{C}_0|_{W_0}$. Following the proof of Prosposition 5 of [Ci], we construct successively isomorphisms $\varphi_n : \mathcal{B}_n|_{W_n} \to \mathcal{C}_n|_{W_n}$ such that, for $n \geq 1$, $\varphi_n|_{W_{n-1}} = \varphi_{n-1}$. Using [EGA III_1], 5.1.4, we obtain a vector bundle $\mathcal{A}$ on X and a homomorphism $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which restricts to the multiplication on $\mathcal{A}_n$ for each n. Hence $\mathcal{A}$ is an Azumaya algebra over X.

Lemma 1.6. *Let C be a one-dimensional noetherian scheme and let $C_{red} \subset C$ be the associated reduced scheme. The natural map*

$$Br_{Az}(C) = Br(C) \to Br_{Az}(C_{red}) = Br(C_{red})$$

is an isomorphism.

Proof. There exists a sequence of closed immersions

$$C_{red} = C_0 \subset C_1 \subset \cdots \subset C_n = C$$

together with ideals $\mathcal{I}_j \subset \mathcal{O}_{C_j}$ such that $\mathcal{O}_{C_{j-1}} = \mathcal{O}_{C_j}/\mathcal{I}_j$ and $\mathcal{I}_j^2 = 0$. On each C_j, we have the following exact sequence of sheaves for the étale topology:

$$0 \to \mathcal{I}_j \to \mathbb{G}_{m,C_j} \to r_*\mathbb{G}_{m,C_{j-1}} \to 1,$$

where the coherent ideal $\mathcal{I}_j$ is viewed as a sheaf for the étale topology, r is the closed immersion $C_{j-1} \to C_j$ and the map $\mathcal{I}_j \to \mathbb{G}_{m,C_j}$ is given by $x \mapsto 1 + x$. For any i, we have $H^i_{\text{ét}}(C_j, \mathcal{I}_j) = H^i_{Zar}(C_j, \mathcal{I}_j)$ (these properties would hold for any noetherian scheme C).

Because the C_j's are curves, for $i \geq 2$, these last groups vanish. Thus

$$H^2_{\text{ét}}(C_j, \mathbb{G}_m) \to H^2_{\text{ét}}(C_j, r_*\mathbb{G}_{m,C_{j-1}})$$

is an isomorphism. We have $R^1 r_*(\mathbb{G}_m) = 0$ because r is a closed immersion and $H^1_{\text{ét}}(A, \mathbb{G}_m) = Pic(A) = 0$ for any local ring A. We also have $R^2 r_*(\mathbb{G}_m) = 0$, because $H^2_{\text{ét}}(A, \mathbb{G}_m) = 0$ for any one-dimensional strictly henselian local ring A (combine [GB I], Cor. 6.2 and [GB II], Cor. 2.2). The Leray spectral sequence for the immersion $C_{j-1} \to C_j$ and the sheaf $\mathbb{G}_m$ now yields

$$H^2_{\text{ét}}(C_j, r_*\mathbb{G}_{m,C_{j-1}}) \simeq H^2_{\text{ét}}(C_{j-1}, \mathbb{G}_m)$$

Thus

$$H^2_{\text{ét}}(C_j, \mathbb{G}_m) \simeq H^2_{\text{ét}}(C_{j-1}, \mathbb{G}_m).$$

We may now state:

Theorem 1.7. *Let A be a complete local ring and k its residue field. Let $\pi : X \to Spec(A)$ be a proper morphism whose special fibre X_0 is of dimension at most one.*

(a) The natural map of Azumaya Brauer groups $Br_{Az}(X) \to Br_{Az}(X_0)$ is an isomorphism.

(b) If X is of dimension two, reduced, excellent and such that any finite set of closed points is contained in an affine open set, then the natural map $Br(X) \to Br(X_0)$ induces an isomorphism of the torsion group of $Br(X)$ with $Br(X_0)$.

(c) If X is of dimension two and regular, then the natural map $Br(X) \to Br(X_0)$ is an isomorphism.

Proof. Combining Lemmas 1.5 and 1.6 yields (a). Statements (b) and (c) follow from (a) and Lemma 1.2.

We now discuss the second case where we may get hold of the p-part. The key ingredient here is Artin's approximation theorem.

Theorem 1.8. *Let R be a field or an excellent discrete valuation ring, and let A be a henselization of an R-algebra of finite type at a prime ideal. Let k be the residue field. Let $\pi : X \to Spec(A)$ be a proper map whose special fibre $X_0 \to Spec(k)$ is of dimension at most one.*

(a) The restriction map $Br_{Az}(X) \to Br_{Az}(X_0)$ is an isomorphism.

(b) If X is of dimension two, reduced and excellent, and any finite set of closed points is contained in an affine open set, then the natural map $Br(X) \to Br(X_0)$ induces an isomorphism of the torsion group of $Br(X)$ with the torsion group $Br(X_0)$.

(c) If X is of dimension two and regular, then the natural map $Br(X) \to Br(X_0)$ is an isomorphism.

Proof. Given a commutative ring A, a covariant functor F from commutative A-algebras to sets is said to be of finite presentation if it commutes with filtering direct limits, *i.e.* given a filtering system A_i of commutative A-algebras, the natural map $\varinjlim F(A_i) \to F(\varinjlim A_i)$ is an isomorphism. For A and k as above, and $\hat{A}$ the completion of A, a special case of Artin's approximation theorem ([Art$_1$]) says that for any element $\hat{\xi} \in F(\hat{A})$ there exists an element $\xi \in F(A)$ which has the same image as $\hat{\xi}$ in $F(k)$ under the obvious reduction maps. In particular, if $F(\hat{A})$ is not empty, the same holds for $F(A)$.

Given $X \to Spec(A)$ as above, and any smooth A-group scheme G over A, the functor from commutative A-algebras to sets which sends an A-algebra B to $H^1_{\text{ét}}(X \times_A B, G_B)$ is of finite presentation (see [SGA 4], Tome 2, VII 5.9 and Remark 5.14; this also follows from [EGA IV$_3$], Thm. 8.8.2.).

For any $n > 0$, we have an exact sequence of group schemes over $\mathbb{Z}$

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

which induces an exact sequence of group schemes and of étale sheaves

$$1 \to \mathbb{G}_{m,Y} \to GL_{n,Y} \to PGL_{n,Y} \to 1$$

over any scheme Y. This sequence in turn induces an exact sequence of pointed Čech cohomology sets (see [Mi] p. 143)

$$H^1_{\text{ét}}(Y, GL_n) \to H^1_{\text{ét}}(Y, PGL_n) \to Br_{Az}(Y).$$

By Theorem 1.7, the reduction map $Br_{Az}(X \times_A \hat{A}) \to Br_{Az}(X_0)$ is injective. To prove injectivity in (a), it is thus enough to prove that the restriction map $Br_{Az}(X) \to Br_{Az}(X \times_A \hat{A})$ is injective. Let c be an element in the kernel of that map. There exists an integer $n > 0$ and a class $\xi \in H^1_{\text{ét}}(X, PGL_n)$ such that the boundary map $H^1_{\text{ét}}(X, PGL_n) \to Br_{Az}(X)$ sends ξ to $c \in Br_{Az}(X)$. Let us introduce the functor F_ξ from commutative A-algebras to sets which to an A-algebra B associates

$$F_\xi(B) = \{\eta \in H^1_{\text{ét}}(X \times_A B, GL_n) \mid \eta \mapsto \xi_B \in H^1_{\text{ét}}(X \times_A B, PGL_n)\}.$$

One again checks that this functor is of finite presentation. From the exact sequence of sets

$$H^1_{\text{ét}}(X \times_A \hat{A}, GL_n) \to H^1_{\text{ét}}(X \times_A \hat{A}, PGL_n) \to Br_{Az}(X \times_A \hat{A})$$

we conclude that $\xi_{\hat{A}}$ is in the image of the first map. Thus $F_\xi(\hat{A}) \neq \emptyset$. By Artin's theorem, this implies $F_\xi(A) \neq \emptyset$. From the exact sequence of sets

$$H^1_{\text{ét}}(X, GL_n) \to H^1_{\text{ét}}(X, PGL_n) \to Br_{Az}(X)$$

we conclude $c = 0 \in Br_{Az}(X)$.

Let us now show that the map $Br_{Az}(X) \to Br_{Az}(X_0)$ is surjective. By Theorem 1.7, the reduction map $Br_{Az}(X \times_A \hat{A}) \to Br_{Az}(X_0)$ is an isomorphism. Let $\hat{c} \in Br_{Az}(X \times_A \hat{A})$ and let $\hat{\xi} \in H^1_{\text{ét}}(X \times_A \hat{A}, PGL_n)$ be a lift for some $n > 0$. Since the functor $B \to H^1_{\text{ét}}(X \times_A B, PGL_n)$ from commutative A-algebras to sets is of finite presentation, by Artin's theorem there exists $\xi \in H^1_{\text{ét}}(X, PGL_n)$ such that the images of ξ and of $\hat{\xi}$ in $H^1(X_0, PGL_n)$ coïncide. Thus the image c of ξ under the boundary map $H^1_{\text{ét}}(X, PGL_n) \to Br_{Az}(X)$ has same image as $\hat{c}$ when pushed into $Br_{Az}(X_0)$. This completes the proof of (a).

From Lemma 1.2 we get (b) and (c).

We apply the previous theorems in the case where the residue field k is either separably closed or finite.

Corollary 1.9. *Let A be a henselian local ring and k its residue field. Assume that k is separably closed of characteristic $p \geq 0$. Let $\pi : X \to Spec(A)$ be a proper map whose special fibre $X_0 \to Spec(k)$ is of dimension at most one. Then:*

(a) The torsion subgroup of $Br(X)$ is a p-primary torsion group.

(b) If X is regular, then $Br(X)$ is a p-primary torsion group.

(c) If A is excellent, two-dimensional and integral, with quotient field K, then the unramified Brauer group $Br_{nr}(K)$ is a p-primary torsion group.

Proof. For any proper curve X_0 over a separably closed field, $Br(X_0) = 0$ ([GB III], Cor. 5.8, p. 132). Statements (a) and (b) immediately follow from Theorem 1.3. Statement (c) follows upon taking a regular model X of $Spec(A)$ and applying Lemma 1.2.

Corollary 1.10. *Let A be a henselian local ring and k its residue field. Assume that A is complete, or that it is the henselization of an R-algebra of finite type at a prime ideal, where R is a field or an excellent discrete valuation ring. Let $\pi : X \to Spec(A)$ be a proper map whose special fibre $X_0 \to Spec(k)$ is of dimension at most one. Assume that k is separably closed. Then:*

(a) The group $Br_{Az}(X)$ is trivial.

(b) If X is two-dimensional and regular, then $Br(X) = 0$.

(c) If A is excellent, two-dimensional and integral, with quotient field K, the unramified Brauer group $Br_{nr}(K)$ is trivial.

Proof. As above, using Theorem 1.7 and Theorem 1.8.

Corollary 1.11. *Let A be a henselian local ring. Assume that its residue field k is finite of characteristic p. Let $\pi : X \to Spec(A)$ be a proper map whose special fibre $X_0 \to Spec(k)$ is of dimension at most one. Then:*

(a) The torsion subgroup of $Br(X)$ is a p-primary torsion group.

(b) If X is regular, then $Br(X)$ is a p-primary torsion group.

(c) If A is excellent, two-dimensional and integral, with quotient field K, then the unramified Brauer group $Br_{nr}(K)$ is a p-primary torsion group.

Proof. For any proper curve X_0 over a finite field, $Br(X_0) = 0$ ([GB III], p. 97). The rest of the proof is as in Corollary 1.9.

Remark. It would be worth comparing this result with those of [Sai_1].

Using Theorems 1.7 and 1.8, we similarly obtain:

Corollary 1.12. *Let A be a henselian local ring and k its residue field. Assume that A is complete, or that it is the henselization of an R-algebra of finite type at a prime ideal, where R is a field or an excellent discrete valuation ring. Let $\pi : X \to Spec(A)$ be a proper map whose special fibre $X_0 \to Spec(k)$ is of dimension at most one. Assume that k is finite. Then:*

(a) The group $Br_{Az}(X)$ is trivial.

(b) If X is two-dimensional and regular, then $Br(X) = 0$.

(c) If A is excellent, two-dimensional and integral, with quotient field K, the unramified Brauer group $Br_{nr}(K)$ is trivial.

We now consider the case in which the residue field of A is real closed.

Lemma 1.13. *Let A be a regular local ring, K its field of fractions and k its residue field. Let $\alpha \in Br(A)$. If α vanishes in $Br(R)$ for every real closed field R containing K, then its restriction to $Br(k)$ vanishes when pushed over to any real closed field containing k.*

Proof. If k is not formally real, the statement is empty. We therefore assume k formally real, hence in particular 2 invertible in A.

The first, well-known, step is the reduction to the case of a discrete valuation ring. Let $d = \dim(A) \geq 2$. Asssume the theorem has been proved for rings of dimension at most $d-1$. Let t be a regular parameter in the maximal ideal of A. Let L be the residue field of the discrete valuation ring $A_{(t)}$. Applying the theorem to $A_{(t)}$, we see that the image of α in $Br(L)$ vanishes in each real closed field containing L. The ring A/t is a $(d-1)$-dimensional regular local ring and its fraction field is L. Applying the theorem to the image of α in $Br(A/t)$ yields the result.

To prove the statement when A is a discrete valuation ring, it is enough to prove it when A is complete. Since k is assumed formally real, its characteristic is zero, hence A is isomorphic to $k[[t]]$. But any embedding of k in a real closed field R may be extended to an embedding of $k((t))$ into a real closed field R_1 with $R \subset R_1$. The natural map $\mathbb{Z}/2 = Br(R) \to Br(R_1) = \mathbb{Z}/2$ is an isomorphism, which completes the proof.

Proposition 1.14. *Let C be a reduced quasi-projective curve over a field k. Let $f : C' \to C$ be its normalization and D the closed subscheme of C defined by the conductor of f. The canonical homomorphism*

$$Br(C) \to Br(C') \times Br(D)$$

is injective.

Proof. Since C is of dimension 1 and D is of dimension 0, the statement is equivalent to : $Br_{Az}(C) \to Br_{Az}(C') \times Br_{Az}(D)$ is injective. Let $S \subset C$ be a finite set of closed points containing at least one point of each component of C and containing all the points whose local ring is not regular. Let A be the semilocal ring of C at S and let A' be its inverse image under f. We have a Milnor patching diagram (see [Ba], Chapter IX, §5 and in particular Example 5.6)

$$\begin{CD} A @>>> A' \\ @VVV @VVV \\ A/\mathfrak{c} @>>> A'/\mathfrak{c} \end{CD}$$

where $\mathfrak{c} = \{a \in A \mid aA' \subseteq A\}$ is the conductor of A' in A. Let $\mathcal{A}$ be an Azumaya algebra over A which becomes trivial over A' and over $A/\mathfrak{c}$. We may assume that $\mathcal{A}$ is of constant rank n^2. In this case $\mathcal{A}$ is obtained by patching $M_n(A')$ and $M_n(A/\mathfrak{c})$ with an automorphism α of $M_n(A'/\mathfrak{c})$. Since A' is semilocal, the canonical map $GL_n(A') \to GL_n(A'/\mathfrak{c})$ is surjective and thus α is induced by an automorphism of $M_n(A')$. This implies that $\mathcal{A} \simeq M_n(A)$ and proves the injectivity of $Br_{Az}(A) \to Br_{Az}(A') \times Br_{Az}(A/\mathfrak{c})$. The proof now follows from the commutative diagram

$$\begin{CD} Br_{Az}(C) @>>> Br_{Az}(C') \times Br_{Az}(D) \\ @VVV @VVV \\ Br_{Az}(A) @>>> Br_{Az}(A') \times Br_{Az}(A/\mathfrak{c}) \end{CD}$$

in which the left vertical map is injective by passing to the limit in Lemma 1.1 and where the bottom map is injective by the above discussion.

Proposition 1.15. *Let C be a quasi-projective curve over a real closed field k. If $\alpha \in Br(C)$ vanishes at all k-rational points of C, then it vanishes.*

Proof. By Lemma 1.6 we may assume that C is reduced. By Proposition 1.14 it suffices to show that the images of α in $Br(D)$ and in $Br(C')$ are trivial. For $Br(D)$ this is clear by a zero-dimensional variant of Lemma 1.6 because the reduced scheme underlying D is just a set of closed points of C. Since every real point of C' maps to a real point of C, the image of α in $Br(C')$ is trivial at every real point of C', thus we are reduced to the case of a smooth curve. In this case the proposition was proved by Witt for $k = \mathbb{R}$ and can be deduced from Remark 10.6 in [Kn] for an arbitrary real closed k.

Remark 1.15.1. For affine singular curves over $\mathbb{R}$ this result was proved by Demeyer and Knus ([DK]). For affine varieties of arbitrary dimension there is a generalization for suitable higher cohomology groups ([Sch] Thm. 20.2.11 p. 235).

Theorem 1.16. *Let A be a henselian local domain, K its quotient field and k its residue field. Assume that k is a real closed field. Let $\pi : X \to Spec(A)$ be a proper birational map, where X is regular integral and the special fibre $X_0 \to Spec(k)$ has dimension at most one. Let $\alpha \in Br(X)$. Assume that for any real closed field R with $K \subset R$, the image of α in $Br(R)$ vanishes. Then $\alpha = 0$.*

Proof. Since X is regular, Lemma 1.13 implies that, for any real closed field R and any morphism $Spec(R) \to X$, the inverse image of α on $Spec(R)$ vanishes. Let α_0 be the image of α in $Br(X_0)$. It vanishes at all rational points of X_0. Therefore, by Proposition 1.15, α_0 vanishes. By Theorem 1.3, the restriction map $Br(X) \to Br(X_0)$ is an isomorphism. Hence $\alpha = 0$ in $Br(X)$.

2. Every algebra is cyclic

Let X be an integral scheme with function field K and let $n > 0$ be invertible on X. Given a regular codimension one point $x \in X$ with residue field $\kappa(x)$, there is a natural (and classical) residue map

$$\partial_x : H^2_{\text{ét}}(K, \mu_n) = {}_nBr(K) \to H^1_{\text{ét}}(\kappa(x), \mathbb{Z}/n) .$$

A class $\alpha \in {}_nBr(K)$ is unramified at x if and only if $\partial_x(\alpha) = 0$ ([GB II], Prop. 2.1).

Given a class $\xi \in {}_nBr(K)$, the ramification divisor of ξ on X is by definition the sum

$$\mathrm{ram}_X(\xi) = \sum_x \overline{\{x\}} ,$$

where x runs through the codimension one points where $\partial_x(\xi) \neq 0$ and $\overline{\{x\}}$ is the closure of x in X.

Let us recall the following special case of a very general fact (Kato, [Ka], §1). On any excellent integral scheme X with field of functions K, given an integer $n > 0$ invertible on X, there is a natural *complex*

$$H^2_{\text{ét}}(K, \mu_n^{\otimes 2}) \to \bigoplus_{x \in X^{(1)}} H^1_{\text{ét}}(\kappa(x), \mu_n) \to \bigoplus_{x \in X^{(2)}} \mathbb{Z}/n \qquad (\mathcal{C}) .$$

The set $X^{(i)}$ is the set of points of codimension i on X. Assume x is a regular point of codimension one on X. For any $a, b \in K^*$ with cup-product $(a, b) \in H^2_{\text{ét}}(K, \mu_n^{\otimes 2})$, the first map in $(\mathcal{C})$ is given by the tame symbol formula

$$\delta_x(a, b) = (-1)^{v_x(a).v_x(b)} \overline{(a^{v_x(b)}/b^{v_x(a)})} \in \kappa(x)^*/\kappa(x)^{*n}.$$

If y is a regular point of codimension one on X and x is a point of codimension two on X which is a regular point on the closure $Y \subset X$ of y, then the map $\kappa(y)^*/\kappa(y)^{*n} = H^1(\kappa(y), \mu_n) \to \mathbb{Z}/n$ associated to y and x in $(\mathcal{C})$ is simply the valuation modulo n associated to the discrete valuation ring $\mathcal{O}_{Y,x}$.

Suppose we are given an isomorphism $\mathbb{Z}/n \simeq \mu_n$ over X. Then for any regular codimension one point $x \in X$, the map

$$H^2_{\text{ét}}(K, \mu_n^{\otimes 2}) \to \bigoplus_{x \in X^{(1)}} H^1_{\text{ét}}(\kappa(x), \mu_n)$$

is the residue map ∂_x mentioned above.

Theorem 2.1. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is separably closed. Let Δ be a central division algebra over K whose class in the Brauer group of K has order n, prime to the characteristic of k. Then Δ is a cyclic algebra of index n.*

Proof. The assumptions on A allow us to choose an identification of $\mathbb{Z}/n$ with μ_n over $Spec(A)$.

Let $\xi \in {}_nBr(K)$ be the class of Δ. As recalled at the beginning of §1, there exists a regular model $X \to Spec(A)$ of A such that the ramification divisor of ξ is of the form $C + E$ where C and E are (not necessarily connected) regular closed curves on X, and $C + E$ has normal crossings. If $C + E$ is empty, *i.e.* if ξ is unramified on X, since ${}_nBr(X) = 0$ (Cor. 1.9), then $\xi = 0$ and the theorem is clear. We thus assume $C + E$ not empty.

Let S be a finite set of closed points of X including all points of intersection of C and E and at least one point of each component of $C+E$. Since X is projective over $Spec(A)$, there exists an affine open $U \subset X$ containing S. The semi-localization of U at S is a semi-local regular domain, hence a unique factorization domain. Thus there exists an $f \in K^*$ such that the divisor of f on X is of the form $\mathrm{div}_X(f) = C + E + G$, where the support of G does not contain any point of S, hence in particular has no common component with $C + E$. Let L be the cyclic field $L = K(f^{1/n})$. At each generic point of a component of $C + E$, the extension L/K is totally ramified of degree n. In particular, L/K is of degree n. To prove the theorem, it suffices to show that the image ξ_L of ξ in ${}_nBr(L)$ is zero.

Let X' be the normalization of X in L and let $\pi : Y \to X'$ be a projective birational morphism such that Y is regular and integral. Let B be the integral closure of A in L. The ring B is an excellent henselian two-dimensional local domain with the same residue field k. By the universal property of the normalization the composite morphism $X' \to X \to Spec(A)$ factorizes though a projective birational morphism $X' \to Spec(B)$, hence induces a birational projective morphism $Y \to Spec(B)$. By Corollary 1.9, ${}_nBr(Y) = 0$.

It is thus enough to show that ξ_L is unramified on Y. Let $y \in Y$ be a codimension one point. We show that $\partial_y(\xi_L) = 0$. Let $x \in X$ be the image of y under the composite map $Y \to X' \to X$.

Suppose first that $\mathrm{codim}(x) = 1$. If $\overline{\{x\}}$ is not a component of $C+E$, then $\partial_x(\xi) = 0$, hence $\partial_y(\xi_L) = 0$. Suppose that $D = \overline{\{x\}}$ is a component of $C + E$. Then f is a uniformizing parameter of the discrete valuation ring $\mathcal{O}_{X,x}$. The extension L/K is totally ramified at x. The restriction map $Br(K) \to Br(L)$ induces multiplication by the ramification index on the character groups of the residue fields. Hence ξ_L is unramified at y.

Suppose now that $\mathrm{codim}(x) = 2$. If x does not belong to C or E, then ξ belongs to $Br(\mathcal{O}_{X,x})$, hence ξ_L is unramified at y. Suppose x belongs to C but not to E. Let $\pi \in \mathcal{O}_{X,x}$ be a local equation of C at x. Since C is regular we can choose a δ such that (π, δ) is a regular system of parameters of $\mathcal{O}_{X,x}$. Since the ramification of ξ in $\mathcal{O}_{X,x}$ is only along π, using the complex $(\mathcal{C})$, or rather its restriction over the local ring $\mathcal{O}_{X,x}$, one finds that $\partial_\pi(\xi) \in \kappa(\pi)^*/\kappa(\pi)^{*n}$ has image zero under the map $\kappa(\pi)^*/\kappa(\pi)^{*n} \to \mathbb{Z}/n$ induced by the valuation defined by x on the field $\kappa(\pi)$, which is the fraction field of the discrete valuation ring $\mathcal{O}_{X,x}/\pi$. Thus $\partial_\pi(\xi)$ is the class of a unit of $\mathcal{O}_{X,x}/\pi$, and such a unit lifts to a unit μ of $\mathcal{O}_{X,x}$. Now the residues of $\xi - (\mu, \pi)$ at all points of codimension one of $\mathcal{O}_{X,x}$ are trivial. Since $\mathcal{O}_{X,x}$ is a regular two-dimensional ring, this implies that $\xi - (\mu, \pi)$ is the class of an element $\eta \in Br(\mathcal{O}_{X,x})$. Now

$$\partial_y(\xi_L) = \partial_L((\mu, \pi)) = \overline{\mu}^{v_y(\pi)} \text{ modulo } \kappa(y)^{*n} ,$$

where $\kappa(y)$ is the residue field of y and $\overline{\mu}$ is the class of μ in $\kappa(y)$. This class comes from $\kappa(x) = k$, which is separably closed of characteristic prime to n, therefore $\overline{\mu}$ is an n-th power and $\partial_y(\xi_L) = 0$.

Suppose now that x belongs to $C \cap E$. There exists a regular system of parameters (π, δ) defining (C, E) such that $f = u\pi\delta$, with $u \in \mathcal{O}_{X,x}^*$. Since the ramification of ξ on $Spec(\mathcal{O}_{X,x})$ is only along π and δ, a variant of the above argument using the complex $(\mathcal{C})$ ([Sal$_1$], Prop. 1.2) shows that one we may write

$$\xi = \eta + (\pi, \mu_1) + (\delta, \mu_2) + r(\pi, \delta) ,$$

with $\mu_1, \mu_2 \in \mathcal{O}_{X,x}^*$, with $\eta \in Br(\mathcal{O}_{X,x})$ and some $r \in \mathbb{Z}$. Since $f = u\pi\delta$, we get

$$(\pi, \delta) = (\pi, fu^{-1}\pi^{-1}) = (\pi, f) + (\pi, -u) .$$

The symbol (π, f) vanishes over L and the other symbols, as in the previous case, become unramified at y.

Remark. The technique used in the proof is essentially the one used in the papers [FS], [Sal$_1$] and [Sal$_2$].

Recall a conjecture of Serre: for any semisimple simply connected linear algebraic group G over a perfect field K of cohomological dimension two, $H^1(K, G) = 0$.

Corollary 2.2. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is algebraically closed of characteristic zero. Let G be a semisimple simply connected linear algebraic group over K without E_8-factors. Then $H^1(K, G) = 0$.*

Proof. Since any finite field extension of K is the field of fraction of an excellent henselian two-dimensional local domain whose residue field is algebraically closed of characteristic zero, a standard argument allows us to assume that G/K is absolutely almost simple. By a theorem of Gabber and Kato (see Theorem 3.2 below), the cohomological dimension of K is two. The statement now follows for groups of type A_n^1 (Merkurjev-Suslin, see [BP]) and for all groups of classical type and of type G_2 and F_4 ([BP]).

By theorem 2.1, the field K has the additional property that a division algebra over K of exponent n has index n. By Gille's results ([Gi], §IV.2) this implies $H^1(K, G) = 0$ for the other exceptional groups except possibly those of type E_8.

3. Quadratic forms

In this section we shall use the standard notation in the algebraic theory of quadratic forms ([La]).

Theorem 3.1. *Let A be an excellent henselian two-dimensional local domain in which 2 is invertible, K its field of fractions and k its residue field. Assume that k is either separably closed or finite. Every quadratic form φ of rank 3 or 4 over K which is isotropic in all completions of K with respect to discrete valuations is isotropic.*

Proof. The isotropy of the rank 3 form $<a,b,c>$ is equivalent to the isotropy of the rank 4 form $<a,b,c,abc>$. Thus we may assume that φ is 4-dimensional and after scaling that $\varphi =< 1,a,b,abd >$ with $a,b,d \in K^*$. If d is a square, then φ is the norm form of a quaternion algebra $\mathcal{A}$. The condition that φ is isotropic at all completions implies that $\mathcal{A}$ is split at all completions of K. In particular $\mathcal{A}$ is unramified in $Br(K)$ and hence, by Corollaries 1.9 and 1.11, is trivial. In particular, φ is hyperbolic.

Suppose now that d is not a square. Let $L = K(\sqrt{d})$. The field L and the integral closure B of A in L satisfy the same assumptions as K and A. The form φ_L over L has trivial discriminant and is isotropic at all completions of L at discrete valuations. By the previous case, φ_L is hyperbolic. By [La], Ch. 7, Lemma 3.1, the form φ contains a multiple of $<1,-d>$ and, being of discriminant d, also contains a rank 2 subform of discriminant 1. Hence it is isotropic.

Remark 3.1.1. For A as in Theorem 3.1, any discrete valuation ring R on the fraction field K is centered on A, *i.e.*, A is contained in R. Indeed, since k is separably closed or finite, there exists a prime l different from the characteristic of k such that $k^* = k^{*l}$. Hence, since A is henselian, $A^* = A^{*l}$, hence $A^* = A^{*l^n}$ for any $n > 0$. For any $x \in A^*$, the valuation $v(x) \in \mathbb{Z}$ is thus divisible by arbitrarily high powers of l, hence $v(x) = 0$ and $A^* \subset R^* \subset R$. Now $A = A^* + A^*$, hence $A \subset R$.

Remark 3.1.2. Theorem 3.1 does not in general hold for quadratic forms of rank 2 (this was also observed by Jaworski [Ja]). Let A be as in the theorem, with k algebraically closed of characteristic not 2. Let $X \to Spec(A)$ be a regular model, and let X_0 be the special fibre. By the proper base change theorem ([Mi], VI.2.7), there is an isomorphism $H^1(X,\mathbb{Z}/2) \simeq H^1(X_0,\mathbb{Z}/2)$. One may produce examples where X_0 is the union of smooth projective curves of genus zero C_i, with $i \in \mathbb{Z}/n$ ($n \geq 2$), C_i intersecting C_{i+1} transversally in one point, and $C_i \cap C_j = \emptyset$ for $j \notin \{i-1, i, i+1\}$. We then have $H^1(X,\mathbb{Z}/2) = H^1(X_0,\mathbb{Z}/2) = \mathbb{Z}/2$.

Let $\xi \in H^1(X,\mathbb{Z}/2)$ be the nontrivial class. Since X is regular hence normal, the map $H^1(X,\mathbb{Z}/2) \to H^1(K,\mathbb{Z}/2) = K^*/K^{*2}$ given by restriction to the function field is injective, hence the image $\xi_K \in K^*/K^{*2}$ is nontrivial. On the other hand let $v : K^* \to \mathbb{Z}$ be a discrete valuation on K and let R be the associated valuation ring. Let K_v be the completion of K at v. By Remark 3.1.1, we have $A \subset R$. Since $X \to Spec(A)$ is proper, there exists a point x of the scheme X on which R is centered, *i.e.* the local ring $B = \mathcal{O}_{X,x}$ is contained in R and the inclusion is a morphism of local rings. We claim that ξ_K has trivial restriction to each K_v^*/K_v^{*2}. This will produce an anisotropic quadratic form of rank 2 over K which is isotropic over each completion K_v. To prove the claim, it is enough to show that the image

ξ_κ of ξ under the composite map

$$H^1(X,\mathbb{Z}/2) \to H^1(B,\mathbb{Z}/2) \to H^1(\kappa_x,\mathbb{Z}/2) \to H^1(\kappa,\mathbb{Z}/2)$$

is trivial. If x is of codimension 2 on X, then the residue field κ_x coincides with k, hence $H^1(\kappa_x,\mathbb{Z}/2)=0$ and the result is clear. Suppose x is a codimension one point of X which is not on X_0. Let $Y \subset X$ be the Zariski closure of x in X. This is a connected one-dimensional scheme which is proper and quasi-finite, hence finite over $Spec(A)$, hence $Y = Spec(T)$ where T is a one-dimensional henselian local ring with residue field k. Hence $H^1(T,\mathbb{Z}/2)=0$. The map $H^1(X,\mathbb{Z}/2) \to H^1(\kappa_x,\mathbb{Z}/2)$ factors through $H^1(T,\mathbb{Z}/2)$, hence is trivial. Let us now assume that x is the generic point of one of the components of X_0. By assumption, any such component is isomorphic to the projective line $\mathbb{P}^1_k$. The map $H^1(X,\mathbb{Z}/2) \to H^1(\kappa_x,\mathbb{Z}/2)$ factors through the group $H^1(\mathbb{P}^1_k,\mathbb{Z}/2)=0$, hence is zero.

The following theorem is due independently to Gabber and Kato.

Theorem 3.2. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is algebraically closed. Then, for every prime $l \neq \mathrm{char}(k)$, $\mathrm{cd}_l(K)=2$.*

Proof. See [Sai$_1$], Theorem 5.1.

Corollary 3.3. *For A, K and k as in Theorem 3.2, if $\mathrm{char}(k) \neq 2$ any 3-fold Pfister form over K is split. The group $I^3(K) \subset W(K)$ vanishes.*

Proof. For any field F of characteristic $\neq 2$ and any $a,b,c \in F^*$ the form

$$<<a,b,c>>=<1,-a> \otimes <1,-b> \otimes <1,-c>$$

is split if and only if the element $(a)\cup(b)\cup(c)$ of $H^3_{\text{ét}}(F,\mathbb{Z}/2)$ vanishes (Merkurjev, see [Ara$_2$], Proposition 2). In our case, $H^3_{\text{ét}}(K,\mathbb{Z}/2)=0$, whence the first result. We then have $I^3K=0$, since $I^3(K)$ is spanned by multiple of 3-fold Pfister forms.

Theorem 3.4. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is algebraically closed and of characteristic $\neq 2$. Then every quadratic form of rank at least 5 over K is isotropic.*

Proof. It suffices to prove the theorem for a form φ of rank 5. In this case the form $\psi = \varphi \perp <-\det(\varphi)>$, having discriminant 1, is similar to a so called Albert form $<a,b,-ab,-c,-d,cd>$. We refer to [KMRT], §16 for the theory of Albert forms. We recall that an Albert form $<a,b,-ab,-c,-d,cd>$ is isotropic if and only if the biquaternion algebra $(a,b)\otimes(c,d)$ is not a division algebra. In our case, by the cyclicity result (Theorem 2.1) no such algebra is a division algebra and therefore ψ is isotropic. This means that φ represents $\det(\varphi)$ and hence is of the form $<\det(\varphi)>\perp \varphi_0$, where φ_0, having determinant 1, can be written as $\det(\varphi)\cdot <u,v,w,uvw>$ for some $u,v,w \in K^*$. This shows that

$$\varphi = \det(\varphi)\cdot <1,u,v,w,uvw>$$

is similar to a Pfister neighbour of $<<u,v,w>>$. But a 3-fold Pfister forms over K, by Corollary 3.3, contains a 4-dimensional totally isotropic space, which intersects the underlying space of φ in a nontrivial space. This proves that φ is isotropic.

Remark 3.4.1. The same argument would yield a local-global principle for the isotropy of 5-dimensional forms over the field of fractions of an excellent henselian two-dimensional local domain with finite residue field, if the following question over such a field K had a positive answer:

Let D be the tensor product of two quaternion algebras over K. Assume that $D \otimes_K K_v$ is similar to a quaternion algebra over each completion K_v of K at a rank one discrete valuation. Is D similar to a quaternion algebra over K ?

Proposition 3.5. *Let Y be an irreducible algebraic surface over a finite field $\mathbb{F}$ and let A be a local domain which is the henselization of Y at a closed point. Let K be the fraction field of A. For any integer n prime to the characteristic of $\mathbb{F}$, the map $H^3(K, \mu_n^{\otimes 2}) \to \prod_v H^3(K_v, \mu_n^{\otimes 2})$, where v runs through the discrete valuations of K, is injective.*

Proof. We shall prove an a priori stronger statement. Let $X \to Spec(A)$ be a regular model of A. We claim that the group $H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2}))$ vanishes.

Note that since X is regular and essentially of finite type over a field, the Bloch-Ogus theory applies (see [BO], [CT]). We therefore have an exact sequence

$$H^3(X, \mu_n^{\otimes 2}) \to H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})) \to CH^2(X)/n$$

(see [CT] (3.10)). The only codimension two points on X are the closed points of the special fibre X_0. Given any such point M, one may find an integral curve $Y \subset X$ which is not contained in X_0 and on which M is a regular point (indeed the local ring at M is a two-dimensional regular local ring). This regular integral curve Y is proper and quasifinite, hence finite over $Spec(A)$. Thus Y is affine, $Y = Spec(T)$. By one of the definitions of a henselian local ring, T is local, hence is a discrete valuation ring. Hence on this curve M is rationally equivalent to zero, hence also on X.

The above exact sequence now reduces to a surjective map

$$H^3(X, \mu_n^{\otimes 2}) \to H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})).$$

Going over to multiples of n prime to the characteristic of $\mathbb{F}$, and passing to the direct limit, we obtain a commutative diagram

$$\begin{array}{ccc} H^3(X, \mu_n^{\otimes 2}) & \longrightarrow & H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})) \\ \downarrow & & \downarrow \\ H^3(X, \mathbb{Q}/\mathbb{Z}'(2)) & \longrightarrow & H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}'(2))) \end{array}$$

Here for any $j \geq 0$, we let $\mathbb{Q}/\mathbb{Z}'(j)$ be the direct limit of all $\mu_n^{\otimes j}$ for n running through the integers prime to the characteristic of $\mathbb{F}$. The map $H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})) \to H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}'(2)))$ is injective: indeed, the map $H^3(K, \mu_n^{\otimes 2}) \to H^3(K, \mathbb{Q}/\mathbb{Z}'(2))$ is injective by the Merkurjev-Suslin theorem. To prove our claim, it is enough to show that the group $H^3(X, \mathbb{Q}/\mathbb{Z}'(2))$ vanishes. By the proper base change theorem ([Mi], VI.2.7), we have $H^3(X, \mathbb{Q}/\mathbb{Z}'(2)) \simeq H^3(X_0, \mathbb{Q}/\mathbb{Z}'(2)))$. The Hochschild-Serre spectral sequence for the curve X_0 over the finite field $\mathbb{F}$ yields an isomorphism

$H^3(X_0, \mathbb{Q}/\mathbb{Z}'(2)) \simeq H^1(\mathbb{F}, H^2(\overline{X}_0, \mathbb{Q}/\mathbb{Z}'(2)))$. Because the Brauer group of the (possibly singular) proper curve $\overline{X}_0$ is trivial ([GB III], Cor. 5.8, p. 132), we have

$$Pic(\overline{X}_0) \otimes \mathbb{Q}/\mathbb{Z}'(1) \simeq H^2(\overline{X}_0, \mathbb{Q}/\mathbb{Z}'(1))).$$

Thus $H^1(\mathbb{F}, H^2(\overline{X}_0, \mathbb{Q}/\mathbb{Z}'(2))) = H^1(\mathbb{F}, P \otimes \mathbb{Q}/\mathbb{Z}'(1))$ for $P = Pic(\overline{X}_0)$. Now for any discrete Galois module P over $\mathbb{F}$, we have $H^1(\mathbb{F}, P \otimes \mathbb{Q}/\mathbb{Z}'(1)) = 0$. Let us recall the proof of this well-known lemma: reduce to P finitely generated, use the fact that $\mathbb{F}$ is of cohomological dimension 1 to reduce to the case where P a permutation lattice, use Shapiro's lemma and finally use $H^1(\mathbb{F}_1, \mathbb{Q}/\mathbb{Z}'(1)) \simeq \mathbb{F}_1^* \otimes \mathbb{Q}/\mathbb{Z}' = 0$ for any finite extension $\mathbb{F}_1$ of $\mathbb{F}$.

Remark 3.5.1. Proposition 3.5 should be compared with Theorem 5.2 of Saito [Sai$_2$]. When A is normal, Saito's theorem computes the kernel of the map

$$H^3(K, \mu_n^{\otimes 2}) \to \prod_v H^3(K_v, \mu_n^{\otimes 2})$$

when the product is restricted to the valuations given by primes of height one on A. That kernel need not be zero.

Theorem 3.6. *Let Y be an algebraic surface over a finite field $\mathbb{F}$ of characteristic different from 2. Let A be a local domain which is the henselization of Y at a closed point. Let K be the fraction field of A. The map $I^2(K) \to \prod_v I^2(K_v)$, where v runs through the discrete valuations of K, is injective.*

Proof. By Merkurjev's theorem, the classical invariant $e_K^2 : I^2(K) \to H^2(K, \mathbb{Z}/2)$ has kernel $I^3(K)$. By Corollary 1.11, the map $H^2(K, \mathbb{Z}/2) \to \prod_v H^2(K_v, \mathbb{Z}/2)$ is an injection. Hence the kernel of $I^2(K) \to \prod_v I^2(K_v)$ is contained in the kernel of $I^3(K) \to \prod_v I^3(K_v)$. The field K is a C_3-field, hence $I^4(K)$=0. By Prop. 3.1 of [AEJ], this implies that the Arason invariant $e_K^3 : I^3(K) \to H^3(K, \mathbb{Z}/2)$ is injective. Proposition 3.5 shows that $H^3(K, \mathbb{Z}/2) \to \prod_v H^3(K_v, \mathbb{Z}/2)$ is an injection. Therefore the kernel of $I^3(K) \to \prod_v I^3(K_v)$ is zero and the theorem follows.

4. The real case

A quadratic form φ over a field K is said to be torsion if, for some integer n, the form $n \cdot \varphi = \varphi \perp \cdots \perp \varphi$ is hyperbolic. By a well-known theorem of Pfister, φ is torsion if and only if φ_R is hyperbolic for every real closed extension $K \subset R$.

By a result of Arason (see [AEJ], Lemma 2.2), for an element $\xi \in H^n_{\text{ét}}(K, \mathbb{Z}/2)$, ξ_R is zero in $H^n_{\text{ét}}(R, \mathbb{Z}/2)$ for all real closed extensions $K \subset R$ if and only if there exists a natural integer i such that the cup-product $\xi \cup (-1) \cup \cdots \cup (-1)$ is zero in $H^{n+i}_{\text{ét}}(K, \mathbb{Z}/2)$; here (-1) denotes the class of -1 in $K^*/K^{*2} = H^1(K, \mathbb{Z}/2)$. We say that such a class is (-1)-torsion.

Theorem 4.1. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is real closed. Every (-1)-torsion element $\xi \in H^2_{\text{ét}}(K, \mathbb{Z}/2)$ is the class of a quaternion algebra.*

Proof. Let $\overline{K} = K(\sqrt{-1})$. By Theorem 3.2, the field $\overline{K}$ has cohomological dimension 2. Consider the long exact cohomology sequence

$$\cdots \to H^i_{\text{ét}}(\overline{K}, \mathbb{Z}/2) \xrightarrow{Cor_{\overline{K}/K}} H^i_{\text{ét}}(K, \mathbb{Z}/2) \xrightarrow{\cup(-1)} H^{i+1}_{\text{ét}}(K, \mathbb{Z}/2) \to \cdots$$

(see [Ara$_1$], Corollary 4.6) where $Cor_{\overline{K}/K}$ denotes the corestriction map. Since $H^i_{\text{ét}}(\overline{K}, \mathbb{Z}/2) = 0$ for $i \geq 3$, the group $H^3_{\text{ét}}(K, \mathbb{Z}/2)$ is (-1)-torsion free. This implies $\xi \cup (-1) = 0$, hence from the same sequence we conclude that there exists a $\widetilde{\xi} \in H^2_{\text{ét}}(\overline{K}, \mathbb{Z}/2)$ such that

$$Cor_{\overline{K}/K}(\widetilde{\xi}) = \xi \ .$$

Resolution of singularities and uninhibited blowing up yield an integral regular scheme X and a projective birational morphism $\pi : X \to Spec(A)$ such that the ramification locus $ram_X(\xi)$ of ξ on X is contained in $C + E$ with C and E regular curves with normal crossings ([Sh], Theorem, page 38 and Remark 2, page 43). Similarly, one can ensure that on $ram_{\overline{X}}(\widetilde{\xi}) \subset \overline{C} + \overline{E}$ on $\overline{X} = X \times_{\mathbb{Z}} Spec(\mathbb{Z}(\sqrt{-1}))$, where $\overline{C}$ and $\overline{E}$ are the preimages of C and E. Since the projection $\overline{X} \to X$ is étale, $\overline{C}$ and $\overline{E}$ are also regular, with normal crossings. As in the proof of Theorem 2.1, we can find an $f \in K^*$ such that, $\widetilde{\xi}_{\overline{K}(\sqrt{f})}$ is zero in $Br(\overline{K}(\sqrt{f}))$. From the commutative diagram

$$\begin{array}{ccc} H^2_{\text{ét}}(\overline{K}, \mathbb{Z}/2) & \longrightarrow & H^2_{\text{ét}}(\overline{K}(\sqrt{f}), \mathbb{Z}/2) \\ \downarrow & & \downarrow \\ H^2_{\text{ét}}(K, \mathbb{Z}/2) & \longrightarrow & H^2_{\text{ét}}(K(\sqrt{f}), \mathbb{Z}/2) \end{array}$$

we see that $\xi_{K(\sqrt{f})} = 0$. This proves that ξ is the class of a quaternion algebra.

Theorem 4.2. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is real closed. Then every 4-dimensional torsion form over K which is isotropic in each completion with respect to a discrete valuation of K is isotropic.*

Proof. Let φ be such a form and let d be its discriminant. Let $L = K(\sqrt{d})$. It suffices to show that φ_L is isotropic (see [La], chap. 7, Lemma 3.1). Scaling φ_L we may assume that it is of the form $<1, -a, -b, ab>$, hence it suffices to show that the associated quaternion algebra $(a, b)_L$ is trivial. Let B be the integral closure of A in L and $\pi : X \to Spec(B)$ a projective birational morphism, with X regular and integral. The quaternion algebra (a, b) is unramified at each codimension one point of X because it is trivial in all completions with respect to the discrete valuations of L. By Lemma 1.2(c), it comes from a class $\alpha \in Br(X)$. If X_0 is the closed fiber of π, by Theorem 1.3 we have $Br(X) \xrightarrow{\sim} Br(X_0)$; thus, to show that $\alpha = 0$ it suffices to show that its restriction to X_0 is trivial. By Proposition 1.15 it suffices to show that α vanishes at all real points of X_0. Now, the torsion assumption on φ implies that $(a, b)_L$ vanishes at all real closures of L. By Lemma 1.13 this implies that it also vanishes at all real closed points of X, in particular at all rational points of X_0.

Proposition 4.3. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is real closed. Then every 6-dimensional torsion form over K is isotropic.*

Proof. Let $\varphi = <a, b, c, d, e, f>$ be a 6-dimensional torsion form. For any real closed extension R of K the discriminant of φ_R is -1, hence $\psi = <a, b, c, d, e, -abcde>$ is torsion as well. Now, ψ is a scalar multiple of a torsion Albert form. By Theorem

4.1 and the basic property of Albert forms, such forms are isotropic, hence ψ is isotropic. In this case $< a, b, c, d, e >$ is a neighbour of a 3-fold Pfister form χ. This Pfister form is isotropic hence hyperbolic at all real closures of K, hence it is torsion. But, as we already saw in the proof of 4.1, $H^3_{\text{ét}}(K, \mathbb{Z}/2)$ is torsion free and thus χ is trivial. This implies that $<a, b, c, d, e>$ is isotropic.

Theorem 4.4. *Let A be an excellent henselian two-dimensional local domain, K its field of fractions and k its residue field. Assume that k is real closed. Then every torsion form φ over K of even rank ≥ 6 is isotropic.*

Proof. By Proposition 4.3 we may assume that φ is of rank at least 8. Its Clifford invariant is torsion, hence, by Theorem 4.1, this Clifford invariant is represented by a torsion 2-fold Pfister form ψ.The form $\varphi \perp -\psi$ is a torsion form in $I^3(K)$. Since $K(\sqrt{-1})$ has cohomological dimension 2, $I^3(K(\sqrt{-1})) = 0$. By Prop. 1.24 of [AEJ], this implies that $I^3(K)$ is torsion free hence $\varphi \perp \psi$ is hyperbolic and φ, being of rank at least 8, must be isotropic.

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