Pfister Involutions

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Introduction

Pfister forms play a prominent role in the algebraic theory of quadratic forms. On the other hand, involutions on central simple algebras share many properties with quadratic forms. Hence it is natural to look for an analog of the notion of Pfister form in the framework of algebras with involution.

The aim of the present paper is to propose such a notion. An n–fold Pfister involution (or Pfister involution, for short) will be by definition a central simple algebra with an orthogonal involution which is a tensor product of n quaternion algebras with involution. We show that if $n = 4$, then after passing to any splitting field of the algebra, the involution is induced by a Pfister form. For $n \leq 3$, this was already proved by D. Tao. We also compute cohomological invariants of 2-fold Pfister involutions, and raise some open questions.

1. Definitions and notation

Let F be a field of characteristic $\neq 2$. Let D be a central simple algebra over F. We say that an involution $\sigma : D \to D$ is of the *first kind* if its restriction to F is the identity. After extension to a splitting field of D, any involution of the first kind is induced by a symmetric or by a skew-symmetric form. We say that the involution is of the *orthogonal type* in the first case, and of the *symplectic type* in the second case.

Definition. Let (D, σ) be a central simple algebra endowed with an involution of the first kind. We say that (D, σ) is an *n–fold Pfister involution* if σ is of the orthogonal type, and if there exist quaternion algebras $H_1, \ldots, H_n$, and involutions $\sigma_i : H_i \to H_i$ such that

$$(D, \sigma) \simeq (H_1, \sigma_1) \otimes ... \otimes (H_n, \sigma_n).$$

Let F_s be a separable closure of F, and set $H^n(F) = H^n(\mathrm{Gal}(F_s/F), \mathbf{Z}/2\mathbf{Z})$.

2. Statement of results and open questions

Let (D, σ) be a central simple algebra with an orthogonal involution of the first kind. Let K be a splitting field of D. Then after tensoring with K, the involution is induced by a quadratic form q defined over K. It is natural to ask whether (D, σ) is a Pfister involution if and only if q is similar to a Pfister form. This is proved by Tao [5] for algebras of degree ≤ 8 (cf. [4], [2], p.150).

In the present paper, we show that for a 4–fold Pfister involution the associated quadratic form is a 4–fold Pfister form (see §5, th. 2). It would be interesting to know whether the converse also holds : this is still an open question.

It is also natural to try to define n–dimensional cohomological invariants of n–fold Pfister involutions. More precisely, suppose that (D, σ) is an n–fold Pfister involution. Let X_D be the Brauer–Severi variety of D and $q_{F(X_D)}$ the associated quadratic form. For $n \leq 4$, we know that $q_{F(X_D)}$ is an n–fold Pfister form. Let $E_n(D) = Ker(H^n(F) \to H^n(F(X_D)))$. Does there exist an invariant $e_n(D, \sigma) \in H^n(F)/E_n(D)$ with the property that $e_n(q_{F(X_D)})$ is equal to the image of $e_n(D, \sigma)$ in $H^n(F(X_D))$? In §4, we define such an invariant for 2–fold Pfister involutions.

3. A lemma

The following lemma will be used in §4 and §5 :

Lemma 1. *Let τ be an orthogonal involution on D. Let $\phi_\tau : D \otimes_F D \to \mathrm{End}_F(D)$ be the isomorphism induced by $\phi_\tau(v, w)(x) = vx\tau(w)$. Let $u \in D^*$ be such that $\tau(u) = u$. Then the involution $\tau \otimes \mathrm{Int}(u)o\tau$ transports under ϕ_τ to σ_q, where σ_q is the adjoint involution on $\mathrm{End}_F(D)$ induced by the quadratic form $q : D \to F$ given by $q(x) = \mathrm{Trd}(xu^{-1}\tau(x))$.*

Proof. See [2], II.1, page 133.

4. Invariants of 2-fold Pfister involutions

Let (H_1, τ_1) and (H_2, τ_2) be two quaternion algebras with involution, and set $(D, \tau) = (H_1, \tau_1) \otimes (H_2, \tau_2)$. Suppose that τ is an orthogonal involution. By [1] we know that $\mathrm{disc}(\tau) = 1$ and that $(D, \tau) = (H_3, \tau_3) \otimes (H_4, \tau_4)$, where H_3, H_4 are quaternion algebras, and τ_3, τ_4 are their canonical involutions. We have $C(D, \tau) = H_3 \times H_4$ and the factors are unique up to switch.

Let X_D be the Severi–Brauer variety of D, and set

$$E_2(D) = Ker(H^2(F) \to H^2(F(X_D))).$$

Then $E_2(D)$ is the subgroup of $H^2(F)$ generated by the class of D. Let us denote by $[H_3]$ the class of H_3 in $H^2(F)/E_2(D)$. Then $[H_3]$ is an invariant of (D, τ).

If D is split, then $H_3 \simeq H_4$, and $(D, \tau) = (H_3, \tau_3) \otimes (H_4, \tau_4) \simeq (\mathrm{End}(H_3), \tau_q)$, where $q : H_3 \to F$ is the quadratic form $q(x) = \mathrm{Trd}(x\tau_3(x)) = 2\mathrm{Nrd}(x)$, cf. Lemma 1. Thus $q \simeq 2N_{H_3}$, where N_{H_3} is the norm form of the quaternion algebra H_3. This implies that $e_2(q) \in H^2(F)$ is the class of H_3. Set $e_2(D, \tau) = [H_3] \in H^2(F)/E(D)$. This is an invariant of the algebra with involution (D, τ), and it coincides with $e_2(q)$ if $D \simeq M_4(F)$ and $\tau = \tau_q$.

Theorem 1. *Let $(D,\tau)=(H_1,\mathrm{Int}(u)o\tau_1)\otimes(H_2,\mathrm{Int}(v)o\tau_2)$ where the H_i are quaternion algebras, τ_i the canonical involution of H_i and $u\in H_1$, $v\in H_2$ satisfy $\tau_1(u)=-u$, $\tau_2(v)=-v$. Then $e_2(D,\tau)=[H_1]+(-\mathrm{Nrd}(u))\cup(-\mathrm{Nrd}(v))$ in $H^2(F)/E(D)$.*

Proof. Suppose that $H_1\otimes H_2$ is split. We may assume that $H_1=H_2$. Then we have

$$(D,\tau)=(H_1,\mathrm{Int}(u)o\tau_1)\otimes(H_1,\mathrm{Int}(\theta)o\mathrm{Int}(u)o\tau_1),$$

where $\theta=vu^{-1}$. Then $\tau=\tau_q$, where $q:H_1\to F$ is the quadratic form

$$q(x)=\mathrm{Trd}(x\theta^{-1}\mathrm{Int}(u)o\tau_1(x))=\mathrm{Trd}(xuv^{-1}u\tau_1(x)u^{-1}),$$

cf. lemma 1. Let us define $b_q:H_1\times H_1\to F$ by $b_q(x,y)=\mathrm{Trd}(xuv^{-1}u\tau_1(y)u^{-1})$. Set $\lambda=\mathrm{Trd}(uv^{-1})$.

If $\lambda=0$, then $q(1)=\mathrm{Trd}(uv^{-1})=0$. Thus q is isotropic and has discriminant 1, hence it is hyperbolic. As $\lambda=0$, the elements $1,u,v,uv$ form a quaternionic basis for H_1, so that $H_1=(-\mathrm{N}(u))\cup(-\mathrm{N}(v))$ and $[H_1]+(-\mathrm{N}(u))\cup(-\mathrm{N}(v))=0$. This concludes the proof in case $\lambda=0$.

Let us suppose that $\lambda\neq 0$. Then 1, u and $vu^{-1}-\frac{2}{\lambda}$ are mutually orthogonal. Indeed, $b_q(1,u)=-\mathrm{Trd}(uv^{-1}u^2u^{-1})=-\frac{\mathrm{Trd}(uv^{-1}u^{-1})}{u^2}=0$, as $\mathrm{Trd}(v)=0$. We have $b_q(1,vu^{-1}-\frac{2}{\lambda})=\mathrm{Trd}(uv^{-1}u(u^{-1}v-\frac{2}{\lambda})u^{-1})=\mathrm{Trd}(1-\frac{2}{\lambda}uv^{-1})=0$. Further, $b_q(u,vu^{-1}-\frac{2}{\lambda})=\mathrm{Trd}(u^2v^{-1}u(u^{-1}v-\frac{2}{\lambda})u^{-1})=\mathrm{Trd}(-\mathrm{N}(u)u^{-1}+\frac{2}{\lambda}\mathrm{N}(u)v^{-1})=0$.

We have $q(1)=\lambda$, $q(u)=\mathrm{Trd}(-u^2v^{-1}u)=\mathrm{N}(u)\lambda$, and

$$q(vu^{-1}-\frac{2}{\lambda})=\mathrm{Trd}[(vu^{-1}-\frac{2}{\lambda})uv^{-1}u(u^{-1}v-\frac{2}{\lambda})u^{-1})]$$

$$=\mathrm{Trd}[(u-\frac{2}{\lambda}uv^{-1}u)(u^{-1}vu^{-1}-\frac{2}{\lambda}u^{-1})].$$

Hence $q(vu^{-1}-\frac{2}{\lambda})=\mathrm{Trd}(vu^{-1}-\frac{4}{\lambda}+\frac{4}{\lambda^2}uv^{-1})=\mathrm{Trd}(vu^{-1})-\frac{4}{\lambda}=\frac{\lambda}{\mathrm{N}(uv^{-1})}-\frac{4}{\lambda}$.

Since $\mathrm{disc}(q)=1$, we have

$$q\simeq\lambda<1,\mathrm{N}(u),\frac{1}{\mathrm{N}(uv^{-1})}-\frac{4}{\lambda^2},\mathrm{N}(u)(\frac{1}{\mathrm{N}(uv^{-1})}-\frac{4}{\lambda^2})>.$$

We have to check that

$$(-\mathrm{N}(u))\cup(-\mathrm{N}(v))+[H_1]=(-\mathrm{N}(u))\cup-(\frac{1}{\mathrm{N}(uv^{-1})}-\frac{4}{\lambda^2}).$$

Since u, v are trace zero elements, there exists a trace zero element w such that $1,u,w,uw$ is a quaternionic basis for H_1, and $v=au+bw$, $[H_1]=(-\mathrm{N}(u))\cup(-\mathrm{N}(w))$.

Hence we have :

$$(-\mathrm{N}(u)) \cup (-\mathrm{N}(v)) + [H_1] = (-\mathrm{N}(u) \cup (\mathrm{N}(v)\mathrm{N}(w)).$$

It suffices to show that $\mathrm{N}(v)\mathrm{N}(w)(\frac{4}{\lambda^2} - \frac{1}{\mathrm{N}(uv^{-1})})$ is a value of $< 1, \mathrm{N}(u) >$.

We have $\mathrm{N}(v)\mathrm{N}(w)(\frac{4}{\lambda^2} - \frac{1}{\mathrm{N}(uv^{-1})}) = (a^2\mathrm{N}(u) + b^2\mathrm{N}(w))\mathrm{N}(w)(\frac{4}{\mathrm{Trd}(uv^{-1})^2} - \frac{1}{\mathrm{N}(uv^{-1}})).$

Note that

$$\mathrm{Trd}(uv^{-1}) = \frac{\mathrm{Trd}(vu^{-1})}{\mathrm{N}(vu^{-1})} = \frac{2a}{a^2 + b^2\mathrm{N}(wu^{-1})},$$

and

$$\mathrm{N}(uv^{-1}) = \frac{1}{\mathrm{N}(vu^{-1})} = \frac{1}{a^2 + b^2\mathrm{N}(wu^{-1})}.$$

We have

$$\mathrm{N}(v)\mathrm{N}(w)(\frac{4}{\lambda^2} - \frac{1}{\mathrm{N}(uv^{-1})}) = (a^2\mathrm{N}(u) + b^2\mathrm{N}(w))\mathrm{N}(w)(a^2 + b^2\mathrm{N}(wu^{-1}))[\frac{a^2 + b^2\mathrm{N}(wu^{-1})}{a^2} - 1]$$

$$= \frac{[a^2\mathrm{N}(u) + b^2\mathrm{N}(w)]^2}{\mathrm{N}(u)^2a^2}\mathrm{N}(w)^2b^2,$$

which is a square, hence a norm. Therefore we have $e_2(q) = (-\mathrm{N}(u)) \cup (-\mathrm{N}(v)) + [H_1]$, as claimed.

5. 4–fold Pfister involutions

The aim of this section is to prove the following result :

Theorem 2. Let (D, σ) be a 4–fold Pfister involution. Let K be a splitting field of D, and let $q = q_K$ be the quadratic form induced by σ over K. Then q is a Pfister form.

The proof will make use of the following lemmas :

Lemma 2. *Let H_1 and H_2 be two quaternion algebras, and set $D = H_1 \otimes H_2$. Let τ be an orthogonal involution with trivial discriminant on D, and let $u \in D^*$ with $\tau(u) = u$. Then the quadratic form $q : D \to F$ defined by $q(x) = \mathrm{Trd}(xu\tau(x))$ is in $I^3(F)$.*

Proof. By lemma 1, the involution $\tau \otimes (\mathrm{Int}u o\tau)$ on $D \otimes D$ transports into σ_q on $\mathrm{End}_F(D)$. Thus $\mathrm{disc}(q) = 1$ and $C_0(q) \simeq C(D \otimes D, \tau \otimes (\mathrm{Int}u o\tau))$ is split (see for instance [2],p. 150). This implies that $C(q)$ is split and that q is in $I^3(F)$.

Lemma 3. *Let H_1 and H_2 be two quaternion algebras, and set $D = H_1 \otimes H_2$. Let τ_i be the canonical involution on H_i, and set $\tau = \tau_1 \otimes \tau_2$. For any $u \in D^*$ such that* $\mathrm{Nrd}(u) \in F^{*2}$, *we define the quadratic form $q_u : D \to F$ by $q_u(x) = \mathrm{Trd}(xu\tau(x))$. Then q_u is a Pfister form.*

Proof. It suffices to show that if q_u is isotropic then it is split. Let us suppose q_u isotropic. By a general position argument, there exists $y \in D^*$ such that $q_u(y) = 0$. The quadratic forms q_u and $q_{yu\tau(y)}$ are isometric under the map $D \to D$, $x \mapsto xy^{-1}$. Hence replacing u by $yu\tau(y)$, we may assume that u has the additional properties $\mathrm{Trd}(u) = 0$, $\tau(u) = u$.

For $x \in H_1$, we have $\tau(x) = x^{-1}\mathrm{Nrd}(x)$ so that $q_u(x) = \mathrm{Trd}(xux^{-1}\mathrm{Nrd}(x)) = \mathrm{Nrd}(x)\mathrm{Trd}(xux^{-1}) = 0$. Thus H_1 is totally isotropic for q_u : indeed, H_1 is totally isotropic for q_z for any $z \in D^*$ with $\mathrm{Trd}(z) = 0$, $\tau(z) = z$.

The involution $\mathrm{Int}(u)o\tau$ on D has trivial discriminant, hence $D = H_1' \otimes H_2'$ and $\mathrm{Int}(u)o\tau = \tau_1' \otimes \tau_2'$, where τ_i' denotes the canonical involution on H_i'. Let $j \in H_2'$ with $\mathrm{Trd}(j) = 0$ and let $(H_1')^0$ be the set of trace zero elements in H_1'. Then $W = (H_1')^0 j$ is a 3–dimensional subspace of D elementwise fixed by $\mathrm{Int}(u)o\tau$ such that for every $y \in W$, we have $y^2 \in F$. Set $T = \{y \in D,\ \mathrm{Trd}(u\tau y) = 0\}$. Then $\dim(T) = 15$ and $\dim(T \cap W) \geq 2$. Let $V_j \subset T \cap W$ be a 2–dimensional subspace. Then for $y \in V_j$, we have $\mathrm{Trd}(u\tau(y) = 0$, $y^2 \in F$ and $\mathrm{Int}(u)o\tau(y) = y$. This implies that $u\tau(y) = yu$, hence $\tau(u\tau(y) = yu = u\tau(y)$, i.e. $u\tau(y)$ is symmetric under τ. Since for $y \in V_j$ we have $y^2 \in F$, we have $\mathrm{Trd}(yu\tau(y) = \mathrm{Trd}(y^2u) = y^2\mathrm{Trd}(u) = 0$. Thus V_j is totally isotropic for q_u.

Further for $y \in V_j$, the element $u\tau(y)$ being symmetric for τ with trace zero, H_1 is totally isotropic for $q_{u\tau(y)}$. Hence for $x, z \in H_1$, we have $\mathrm{Trd}(xu\tau(y)\tau(z)) = 0$. In particular, setting $z = 1$, we get, for $x \in H_1$, $\mathrm{Trd}(xu\tau(y)) = 0$. Hence H_1 is orthogonal to V_j with respect to q_u.

Let i, j, k be a basis of trace zero elements in H_2' with $i = jk$, $jk = -kj$. Suppose that $V_j \subset H_1$. Then we have $V_j \subset H_1^0$. If $V_k \subset H_1$, then we have $V_j + V_k \subset H_1^0$, and $V_j \cap V_k = 0$. Hence we get $\dim(V_j + V_k) = 4$, contradicting $\dim(H_1^0) = 3$.

Thus there exists $y \in V_j$ or V_k such that $y \notin H_1$, y totally isotropic for q_u and $\mathrm{Trd}(xu\tau(y)) = 0$ for all $x \in H_1$. Thus $H_1 \oplus Fy$ is a 5–dimensional totally isotropic subspace for q_u. Therefore the anisotropic rank of q_u is at most 6 and by Lemma 2, we have $q_u \in I^3(F)$. Hence q_u is hyperbolic.

Proof of theorem 2. Since (D, σ) is Pfister, we may write (D, σ) as

$$(D_1, \sigma_1) \otimes (D_2, \sigma_2) \otimes (D_3, \sigma_3) \otimes (D_4, \sigma_4),$$

where the D_i's are quaternion algebras and σ_i's are orthogonal involutions. Using [1], we may write $(D_1, \sigma_1) \otimes (D_2, \sigma_2)$ as $(H_1, \tau_1) \otimes (H_2, \tau_2)$, where the H_i's are quaternion algebras and the τ_i's their canonical involutions. To prove the theorem, we may assume that D is split.

Using [1] again, we may write $(D_3, \sigma_3) \otimes (D_4, \sigma_4) = (H_1 \otimes H_2, \mathrm{Int}(u^{-1})o(\tau_1 \otimes \tau_2)$, with $u \in H_1 \otimes H_2$ satisfying $\mathrm{Nrd}(u) \in F^{*2}$. By lemma 1, the form q associated to σ is given by $q(x) = \mathrm{Trd}(xu\tau_1 \otimes \tau_2(x))$. By lemma 3, the form q is indeed a 4–fold Pfister form.

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