EXTERIOR POWERS OF SYMMETRIC BILINEAR FORMS

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Abstract. We study exterior powers of classes of symmetric bilinear forms in the Witt-Grothendieck ring of a field of characteristic not equal to 2, and derive their basic properties. The exterior powers are used to obtain annihilating polynomials for quadratic forms in the Witt ring.

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1. Introduction

Throughout this paper, K will be a field of characteristic different from 2.

It is well-known that given a finite-dimensional K-vector space V and a non-negative integer k we may define $\Lambda^k V$, the k-fold exterior power of V. Then the ring of isomorphism classes of finite-dimensional K-vector spaces under direct sum and tensor product is a λ-ring, with the exterior powers acting as the λ-operations. The exterior powers and the related symmetric powers are in fact functors on the category of K-vector spaces and K-linear maps, special cases of the Schur functors (see, for example, [3] or [4]).

The concept of exterior power of a symmetric bilinear form is defined in Bourbaki (see [2, Ch. 9 eqn. (37)]). In [10], Serre remarks that the Grothendieck group of the category of finite rank $\mathbb{Z}$-symmetric bilinear modules is a λ-ring using exterior powers, but the subject does not appear to have been treated in detail in the literature. In this paper we establish the basic facts about the exterior powers of a symmetric bilinear form, including formulas for their classical invariants, and apply these to deriving annihilating polynomials in the Witt ring. We work interchangeably with K-quadratic spaces and K-symmetric bilinear spaces to achieve as much simplicity as possible; all results carry across, by the correspondence in characteristic not equal to 2.

2. Notation

For the definitions of such terms as bilinear and quadratic forms, isometry, the Witt-Grothendieck ring, Witt ring, etc., see, for example, [9].

We will denote by $\widehat{W}(K)^+$ the commutative cancellation semi-ring of isometry classes of symmetric bilinear forms under orthogonal sum and tensor product, and let the Witt-Grothendieck ring $\widehat{W}(K)$ be the Grothendieck completion of $\widehat{W}(K)^+$. Then the Witt ring $W(K)$ is the quotient of $\widehat{W}(K)$ by the ideal generated by hyperbolic spaces.

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Throughout, juxtaposition will denote—according to context—a (tensor) product of forms $\varphi\psi = \varphi \otimes \psi = \varphi \cdot \psi$, or a scalar multiple of a form: if $\varphi = \langle a_1, \dots, a_n \rangle$, then $\lambda\varphi = \langle \lambda a_1, \dots, \lambda a_n \rangle$. We will use a cross to denote an integer times a form, i. e. $n \times \varphi$ means the orthogonal sum of φ with itself n times.

For the definitions of k-fold exterior power and symmetric power of a finite-dimensional vector space V see, for example, [4] or [3].

3. Exterior and symmetric powers of symmetric bilinear forms

Let V be a vector space of dimension n over K. Recall that if k is a non-negative integer, then the k-fold exterior power of V, $\Lambda^k V$, has dimension $\binom{n}{k}$, where we take $\binom{n}{k}$ to be 0 for all $k > n$. In particular, if $\{v_1, \dots, v_n\}$ is a basis for V, then a basis for $\Lambda^k V$ is given by the set of k-fold wedge products $\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ and there are $\binom{n}{k}$ such expressions.

Definition 3.1. Let $\varphi : V \times V \longrightarrow K$ be a bilinear form and let k be a positive integer not greater than n. We define the **k-fold exterior power of** φ,

$$\Lambda^k\varphi : \Lambda^k V \times \Lambda^k V \longrightarrow K,$$

by

$$\Lambda^k\varphi(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det(\varphi(x_i, y_j))_{1 \leq i,j \leq k}.$$

We define $\Lambda^0\varphi := \langle 1 \rangle$, the identity form of dimension 1. For $k > n$ we define $\Lambda^k\varphi$ to be the zero form, since $\Lambda^k V = 0$ for all $k > n$.

It is easily seen that $\Lambda^k\varphi$ is a bilinear form, and is symmetric if φ is symmetric. Also $\Lambda^1\varphi = \varphi$.

If q is the quadratic form associated to φ, we write $\Lambda^k q$ for the quadratic form associated to $\Lambda^k\varphi$.

Remark 3.2. Similarly, given any positive integer k, we may define another bilinear form, the **k-fold symmetric power of** φ, on the k-fold symmetric power of V,

$$S^k\varphi : S^k V \times S^k V \longrightarrow K,$$

by

$$S^k\varphi(x_1 \cdots x_k, y_1 \cdots y_k) = \operatorname{per}(\varphi(x_i, y_j))_{1 \leq i,j \leq k},$$

where $\cdot$ is the multiplication in the symmetric algebra of V and per is the permanent of the matrix $(\varphi(x_i, y_j))$.

In fact, given any partition π of k, we may define an associated "Schur power" of φ in a similar fashion. Results on these other powers will appear in a future paper.

Remark 3.3. Let $\varphi : V \times V \longrightarrow K$ be a symmetric bilinear form on V and let G be a group acting on V. Then G acts on $\Lambda^k V$ also, via

$$g(v_1 \wedge \cdots \wedge v_k) = (gv_1) \wedge \cdots \wedge (gv_k).$$

Thus if V is a representation module for G, so is $\Lambda^k V$.

Suppose that φ is a G-**form**, that is, for all $v, w \in V$ and all $g \in G$, $\varphi(gv) = \varphi(v)$ (e. g. for G a subgroup of the orthogonal group $O(\varphi)$). Then an easy computation shows $\Lambda^k\varphi$ is a G-form also.

4. Diagonalisation of an exterior power

Proposition 4.1. *Let V be a vector space of dimension n over F and let $\varphi = \langle a_1, \dots, a_n\rangle$ be a diagonalisation of a symmetric bilinear form on V. Let $k \leq n$. Then $\Lambda^k\varphi$ is a symmetric bilinear form of dimension $\binom{n}{k}$ and has a diagonalisation of the form*

$$\Lambda^k\varphi = \perp_{1\leq i_1<\cdots<i_k\leq n} \langle a_{i_1}\cdots a_{i_k}\rangle.$$

In particular,

$$\Lambda^k(n \times \langle 1\rangle) = \tbinom{n}{k} \times \langle 1\rangle.$$

Proof. Let $\{v_1, \dots, v_n\}$ be an orthogonal basis for V, with $\varphi(v_i, v_i) = a_i$ for $i = 1, \dots, n$ and $\varphi(v_i, v_j) = 0$ for $i, j \in \{1, \dots, n\}$, $i \neq j$.

Let $k \leq n$. Since $\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ is a basis for $\Lambda^k V$, we have immediately that the form $\Lambda^k\varphi$ has dimension $\binom{n}{k}$.

Let $v_{i_1} \wedge \cdots \wedge v_{i_k}$ and $v_{j_1} \wedge \cdots \wedge v_{j_k}$ be two basis elements of $\Lambda^k V$, and consider

$$\Lambda^k\varphi(v_{i_1} \wedge \cdots \wedge v_{i_k}, v_{j_1} \wedge \cdots \wedge v_{j_k}) = \det(\varphi(v_{i_l}, v_{j_m}))_{1\leq l,m\leq k}.$$

Firstly suppose the $\{v_{i_l}\}_{l=1,\dots,k}$ and $\{v_{j_m}\}_{m=1,\dots,k}$ are the same set of k vectors. Then linear independence of the basis gives $i_l = j_l$, $l = 1, \dots, k$ and using orthogonality of $\{v_1, \dots, v_n\}$ we get

$$\begin{aligned}\Lambda^k\varphi(v_{i_1} \wedge \cdots \wedge v_{i_k}, v_{i_1} \wedge \cdots \wedge v_{i_k}) &= \det(\varphi(v_{i_l}, v_{i_m}))_{1\leq l,m\leq k} \\ &= \varphi(v_{i_1}, v_{i_1}) \cdots \varphi(v_{i_k}, v_{i_k}) \\ &= a_{i_1} \cdots a_{i_k}.\end{aligned}$$

Next suppose the $\{v_{i_l}\}$ and $\{v_{j_m}\}$ are not the same set of vectors. Choose a v_{j_m} which is not in $\{v_{i_l}\}_{l=1,\dots,k}$. Then by orthogonality of the basis $\{v_1, \dots, v_n\}$ of V, the ${j_m}^{\text{th}}$ column of the matrix $(\varphi(v_{i_l}, v_{j_m}))_{1\leq l,m\leq k}$ will be a zero column. Thus $\Lambda^k\varphi(v_{i_1}\wedge\cdots\wedge v_{i_k}, v_{j_1}\wedge\cdots\wedge v_{j_k})$, the determinant of this matrix, will be 0. This completes the proof. $\square$

Remark 4.2. Note that if φ is a hyperbolic form then $\Lambda^k\varphi$ need not be hyperbolic. This is easily seen from the above expression for dimension, since a hyperbolic form must have even dimension.

For example, consider the hyperbolic forms $\varphi = 2\times\langle 1, -1\rangle$ of dimension 4, $\psi = 3\times\langle 1, -1\rangle$ of dimension 6. Then $\Lambda^3\psi = 10\times\langle -1, 1\rangle$ is hyperbolic, while $\Lambda^2\psi$ has dimension $\binom{6}{2} = 15$ and so cannot be hyperbolic. Also $\Lambda^2\varphi = \langle -1, 1, -1, -1, 1, -1\rangle$ has anisotropic part $(\Lambda^2\varphi)_{\text{an}} = \langle -1, -1\rangle$, of even dimension.

Thus, though φ and ψ are Witt equivalent (in fact $\varphi \sim \psi \sim 0$), $\Lambda^k\varphi$ and $\Lambda^k\psi$ are not in general Witt equivalent, so Λ^k is not well-defined on elements of the Witt ring. However, Corollary 6.8 later shows that Λ^k *is* well-defined on elements of the Witt-Grothendieck ring. For this reason we restrict ourselves to elements of the Witt-Grothendieck ring

$\widehat{W}(K)$ when considering exterior powers, though some results in $\widehat{W}(K)$ may be carried over to $W(K)$.

Remark 4.3. Note that if φ is an isotropic form of dimension $n \geq 2$, then $\Lambda^k\varphi$ will also be isotropic for $0 < k < n$. For, φ will contain a hyperbolic plane and so $\varphi \simeq \varphi' \perp \langle 1, -1 \rangle$ for some symmetric bilinear form $\varphi' = \langle a_1, \ldots, a_{n-2} \rangle$. Then for any $a_1, \ldots, a_{k-1} \in \{a_1, \ldots, a_{n-2}\}$, the subform $\langle a_1 \cdots a_{k-1}, -a_1 \cdots a_{k-1} \rangle$ of $\Lambda^k\varphi$ will be hyperbolic (see [9, Corollary I.4.6(iii)]) and so $\Lambda^k\varphi$ will be isotropic. In fact, there will be a hyperbolic plane for all such choices of $a_1, \ldots, a_{k-1}$ and so the Witt index of $\Lambda^k\varphi$ will be at least $\binom{n-2}{k-1}$.

Remark 4.4. If φ is an anisotropic form then $\Lambda^k\varphi$ need not be anisotropic. To see this, let K be a field of u-invariant $u = 4$ (recall that the **u-invariant** of a field K is the maximal dimension of an anisotropic form over K). For example, we may take K to be a local field (of characteristic not equal to 2), or a non-real algebraic number field, or a p-adic field. Let φ be of dimension 4 over K and anisotropic. Then $\Lambda^2\varphi$ has dimension $\binom{4}{2} = 6$ and so must be isometric since $u < 6$. The same argument works for any field with finite u-invariant $u > 2$ and any anisotropic form φ of dimension u, since $\binom{u}{2} > u$ for all positive integers $u > 3$.

Remark 4.5. Note that non-isometric forms *may* have isometric k-fold exterior powers.

For example, let $\varphi = \langle 1, 2, 3 \rangle$, $\psi = \langle 1, 6, 3 \rangle$ over a field K in which 3 is not a square (such as $\mathbb{Q}$). Then φ and ψ are not isometric, since their determinants in $\dot{K}/\dot{K}^2$ are 6 and 2, respectively, and these differ by a factor of 3.

However, $\Lambda^2\varphi = \langle 2, 3, 6 \rangle \simeq \langle 6, 3, 2 \rangle = \Lambda^2\psi$.

Corollary 4.6. *Let φ be a symmetric bilinear form of dimension n over K. Then*

$$(a)\quad \Lambda^n\varphi = \langle \det\varphi \rangle \qquad \textit{and}$$
$$(b)\quad \Lambda^k\varphi = (\det\varphi)\,\Lambda^{n-k}\varphi, \qquad 0 \leq k \leq n.$$

Remark 4.7. Corollary 4.6 gives us a "duality principle" for exterior powers of a symmetric bilinear form.

5. Determinant of an exterior power

Proposition 4.1 also allows us to easily compute the determinant of $\Lambda^k\varphi$ in terms of the determinant of φ.

Proposition 5.1. *Let φ be a symmetric bilinear form of dimension n. Then*

$$\det(\Lambda^k\varphi) = (\det\varphi)^{\binom{n-1}{k-1}}.$$

In particular, for $k = 2$,

$$\det(\Lambda^2\varphi) = (\det\varphi)^{n-1}.$$

Proof. Write $\varphi = \langle a_1, \ldots, a_n \rangle$ in diagonal form. Then by Proposition 4.1,

$$\det(\Lambda^k\varphi) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k},$$

There are $\binom{n}{k}$ terms in the product, each term being itself a product of k of the a_i and an easy computation shows

$$\begin{aligned}\det(\Lambda^k\varphi) &= (a_1a_2\cdots a_n)^{\binom{n-1}{k-1}}\\ &= (\det\varphi)^{\binom{n-1}{k-1}}\end{aligned}$$

which completes the proof. □

Corollary 5.2. *Let $k \geq 1$ and let φ be a symmetric bilinear form. Then φ is non-singular if and only if $\Lambda^k\varphi$ is non-singular.*

Proof. φ is non-singular if and only if $\det\varphi \neq 0$. By Proposition 5.1 this is equivalent to $\det(\Lambda^k\varphi) = (\det\varphi)^{\binom{n-1}{k-1}} \neq 0 \iff \Lambda^k\varphi$ is non-singular. □

6. Functorial properties of Λ^k

Let $\mathcal{V}$ denote the category whose objects are finite-dimensional K-vector spaces and whose morphisms are K-linear maps. It is well-known that the Schur functors (see [3, Appendix A2], [4, Lecture 6]), of which Λ^k and S^k are examples, are functors on $\mathcal{V}$. In this section we show that Λ^k is in fact a functor on the category of finite-dimensional K-symmetric bilinear spaces (equivalently finite-dimensional K-quadratic spaces). We first define some terms.

Definition 6.1. Let $\mathcal{B}$, the **category of K-symmetric bilinear spaces**, be the category whose objects are finite-dimensional K-symmetric bilinear spaces and whose morphisms are isometries.

Remark 6.2. It is easy to see that this category is well-defined: if $f \in \mathrm{Hom}_{\mathcal{B}}((V,\varphi),(W,\psi))$ and $g \in \mathrm{Hom}_{\mathcal{B}}((W,\psi),(U,\chi))$ then $\varphi = \psi \circ f$ and $\psi = \chi \circ g$, so $\varphi = \chi \circ g \circ f$ and $g \circ f \in \mathrm{Hom}_{\mathcal{B}}((V,\varphi),(U,\chi))$; further, the composition of bijective maps is bijective.

Remark 6.3. Except in cases where confusion is possible, we drop the $\circ$ notation for composition of morphisms henceforth.

We state some facts which we require (see e. g. [3, A2.2(a), A2.3(e)]).

Proposition 6.4. *If $f : V \longrightarrow W$ is a linear map, then $\Lambda^k f : \Lambda^k V \longrightarrow \Lambda^k W$ is a linear map and $\Lambda^k f(v_1 \wedge \cdots \wedge v_k) = fv_1 \wedge \cdots \wedge fv_k$.*

Moreover, if f has matrix A with respect to bases $\{v_1,\ldots,v_n\}$ and $\{w_1,\ldots,w_m\}$ of V and W respectively, then $\Lambda^k f$ has matrix whose entry corresponding to the basis elements $v_{i_1}\wedge\cdots\wedge v_{i_k}$ and $w_{j_1}\wedge\cdots\wedge w_{j_k}$ is the determinant of the submatrix of A involving columns $i_1,\ldots,i_k$ and rows $j_1,\ldots,j_k$.

Proposition 6.5. *If f is an injective K-linear map, then $\Lambda^k f$ is also an injective K-linear map.*

Proof. Suppose that V and W are K-vector spaces of dimensions n and m respectively, and that $f : V \longrightarrow W$ is an injective K-linear map, so that $n \leq m$ and $\operatorname{rank} f = n$. Then any matrix for f will have $\dim V = n$ linearly independent columns.

Choose a basis $\{v_1, \ldots, v_n\}$ for V, and complete (if necessary) $\{fv_1, \ldots, fv_n\}$ to a basis $\{fv_1, \ldots, fv_n, w_{n+1}, \ldots, w_m\}$ for W.

Then the matrix for f with respect to these bases is

$$\begin{pmatrix} I_n \\ O_{m-n,n} \end{pmatrix}$$

where the upper block is the $n \times n$ identity matrix I_n and the lower block is the $(m-n)\times n$ zero matrix $O_{m-n,n}$ (possibly not present i. e. if $m = n$).

We first look at the case where $k \leq n$.

Consider $\Lambda^k V$ with basis $\{v_1 \wedge \cdots \wedge v_k, \ldots, v_{n-k+1} \wedge \cdots \wedge v_n\}$ of order $\binom{n}{k}$ (all choices of k out of the n vectors v_i) and extend the set $\{fv_1 \wedge \cdots \wedge fv_k, \ldots, fv_{n-k+1} \wedge \cdots \wedge fv_n\}$ to a basis for $\Lambda^k W$.

We observe that the determinant of the submatrix of I_n involving columns $i_1, \ldots, i_k$ and rows $j_1, \ldots, j_k$ is 1 if $\{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\}$ and 0 otherwise. Then by 6.4 we have that the matrix of $\Lambda^k f$ with respect to the above bases for $\Lambda^k V$ and $\Lambda^k W$ is the $\binom{m}{k} \times \binom{n}{k}$ matrix

$$\begin{pmatrix} I \\ O \end{pmatrix}$$

where the upper block is the $\binom{n}{k} \times \binom{n}{k}$ identity matrix and the lower block is the $\left(\binom{m}{k} - \binom{n}{k}\right) \times \binom{n}{k}$ zero matrix (not present if $m = n$).

This matrix clearly has rank $\binom{n}{k}$, so $\Lambda^k f$ is injective as required, when $k \leq n$.

If $k > n$, then $\Lambda^k V = \{0\}$, so $\Lambda^k f : \Lambda^k V \longrightarrow \Lambda^k W$ is trivially injective, and we are done. □

Corollary 6.6. *If f is a bijective K-linear map, then $\Lambda^k f$ is also a bijective K-linear map.*

Proof. The statement follows by taking $m = n$ in the proof of Proposition 6.5. □

Theorem 6.7. *$\Lambda^k : \mathcal{B} \longrightarrow \mathcal{B}$ is a (covariant) functor.*

Proof. From the definition of Λ^k, if $(V, \varphi) \in \operatorname{Ob}\mathcal{B}$ then $(\Lambda^k V, \Lambda^k \varphi) \in \operatorname{Ob}\mathcal{B}$.

Next, if $f : (V, \varphi) \longrightarrow (W, \psi)$ is a morphism in $\mathcal{B}$ then $\Lambda^k f : (\Lambda^k V, \Lambda^k \varphi) \longrightarrow (\Lambda^k W, \Lambda^k \psi)$ is a morphism in $\mathcal{B}$.

For, if $\varphi = \psi \circ f$ with f bijective, then for all $v_1, \ldots, v_k, w_1, \ldots, w_k \in V$,

$$\begin{aligned}
\Lambda^k \varphi(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) &= \det(\varphi(v_i, w_j))_{1 \leq i,j \leq k} \\
&= \det((\psi \circ f)(v_i, w_j))_{1 \leq i,j \leq k} \\
&= \det(\psi(fv_i, fw_j))_{1 \leq i,j \leq k} \\
&= \Lambda^k \psi(fv_1 \wedge \cdots \wedge fv_k, fw_1 \wedge \cdots \wedge fw_k) \\
&\overset{6.4}{=} \Lambda^k \psi(\underbrace{\Lambda^k f(v_1 \wedge \cdots \wedge v_k)}_{\in \Lambda^k W}, \underbrace{\Lambda^k f(w_1 \wedge \cdots \wedge w_k)}_{\in \Lambda^k W}) \\
&= \Lambda^k \psi \circ \Lambda^k f(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k),
\end{aligned}$$

giving $\Lambda^k\varphi = \Lambda^k\psi \circ \Lambda^k f$, and $\Lambda^k f$ is bijective by Corollary 6.6.

Further, if $f \in \mathrm{Hom}_{\mathcal{B}}((V,\varphi),(W,\psi))$ and $g \in \mathrm{Hom}_{\mathcal{B}}((W,\psi),(U,\chi))$ are morphisms in $\mathcal{B}$, so $gf \in \mathrm{Hom}_{\mathcal{B}}((V,\varphi),(U,\chi))$, then

$$\begin{aligned} \Lambda^k f &: (\Lambda^k V, \Lambda^k\varphi) \longrightarrow (\Lambda^k W, \Lambda^k\psi), \\ \Lambda^k g &: (\Lambda^k W, \Lambda^k\psi) \longrightarrow (\Lambda^k U, \Lambda^k\chi) \\ \text{and}\quad \Lambda^k(gf) &: (\Lambda^k V, \Lambda^k\varphi) \longrightarrow (\Lambda^k U, \Lambda^k\chi) \end{aligned}$$

are all morphisms in $\mathcal{B}$. By [3, page 591], the Schur functors, in particular Λ^k, act in afunctorial way on linear maps, so the composition property $\Lambda^k(gf) = \Lambda^k g \Lambda^k f$ holds.

Finally, if $(V,\varphi) \in \mathrm{Ob}\,\mathcal{B}$, let $\mathrm{id}_{(V,\varphi)}$ be the identity map, whose matrix with respect to any basis of V is I_n where $n = \dim V$. Then $\mathrm{id}_{(\Lambda^k V, \Lambda^k\varphi)} :=$ identity map on $\Lambda^k V$ (with matrix $I_{\binom{n}{k}}$ with respect to any basis of $\Lambda^k V$) is clearly $\Lambda^k(\mathrm{id}_{(V,\varphi)})$, by Proposition 6.4. This completes the proof. $\square$

Corollary 6.8. *If (V,φ) and (W,ψ) are isometric symmetric bilinear spaces, then $(\Lambda^k V, \Lambda^k\varphi)$ and $(\Lambda^k W, \Lambda^k\psi)$ are also isometric symmetric bilinear spaces.*

Remark 6.9. One can also define two other categories, with more morphisms than $\mathcal{B}$.

Let $\mathcal{B}_{\mathrm{n}}$ be the category whose objects are those of $\mathcal{B}$ and whose morphisms are K-linear maps $f : (V,\varphi) \longrightarrow (W,\psi)$ such that $\varphi = \psi \circ f$.

Let $\mathcal{B}_{\mathrm{i}}$ be the category whose objects are those of $\mathcal{B}$ and whose morphisms are *injective* morphisms of $\mathcal{B}_{\mathrm{n}}$.

Clearly $\mathcal{B}_{\mathrm{i}}$ and $\mathcal{B}$ are subcategories of $\mathcal{B}_{\mathrm{n}}$, but not full subcategories. The above results also hold for $\mathcal{B}_{\mathrm{n}}$ and $\mathcal{B}_{\mathrm{i}}$, and each Λ^k is a functor on both $\mathcal{B}_{\mathrm{n}}$ and $\mathcal{B}_{\mathrm{i}}$.

7. Algebraic properties of exterior powers

We derive some basic properties of Λ^k: how it behaves with respect to scalar multiplication, orthogonal sum and tensor product of forms.

Proposition 7.1. *Let φ be a symmetric bilinear form over K, let $\alpha \in K$ and let k be a positive integer. Then*

$$\Lambda^k(\alpha\varphi) = \alpha^k \Lambda^k\varphi = \begin{cases} \alpha\Lambda^k\varphi & \text{if } k \text{ is odd;} \\ \Lambda^k\varphi & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let φ, α and k be as in the statement. Suppose that φ acts on a vector space V and let $x_1, \dots, x_k, y_1, \dots, y_k \in V$. Then

$$\begin{aligned} \Lambda^k(\alpha\varphi)(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) &= \det(\alpha\varphi(x_i, y_j))_{1\le i,j\le k} \\ &= \alpha^k \det(\varphi(x_i, y_j))_{1\le i,j\le k} \\ &= \alpha^k \Lambda^k\varphi(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k), \end{aligned}$$

completing the proof. $\square$

Remark 7.2. Proposition 7.1 says that Λ^k is k-homogeneous with respect to scalar multiplication.

Proposition 7.3. *Let φ and ψ be symmetric bilinear forms over K and let k be a positive integer. Then*

$$\Lambda^k(\varphi \perp \psi) = \perp_{i+j=k} \Lambda^i\varphi \cdot \Lambda^j\psi.$$

Proof. This follows from Proposition 9.3 below. □

Corollary 7.4. *Let $\varphi_1, \dots, \varphi_m$ be symmetric bilinear forms over K and let k be a positive integer. Then*

$$\Lambda^k(\varphi_1 \perp \cdots \perp \varphi_m) = \perp_{i_1+\cdots+i_m=k} \Lambda^{i_1}\varphi_1 \cdots \Lambda^{i_m}\varphi_m.$$

Proof. Use Proposition 7.3 and induction on m. □

Lemma 7.5. *Let (V, φ) be a symmetric bilinear space over K of dimension n. Then $\varphi^2 = n \times \langle 1\rangle \perp 2 \times \Lambda^2\varphi$.*

Proof. Let $\varphi = \langle a_1, \dots, a_n\rangle$. Then

$$\begin{aligned}
\varphi^2 &= \langle a_1, \dots, a_n\rangle \otimes \langle a_1, \dots, a_n\rangle = \perp_{1\le i,j\le n} \langle a_i a_j\rangle \\
&= \left(\perp_{1\le i\le n} \langle a_i a_i\rangle\right) \perp \left(\perp_{1\le i<j\le n} \langle a_i a_j\rangle\right) \perp \left(\perp_{1\le i<j\le n} \langle a_j a_i\rangle\right) \\
&= n \times \langle 1\rangle \perp 2 \times \Lambda^2\varphi
\end{aligned}$$

which completes the proof. □

Proposition 7.6. *Let (V, φ) and (W, ψ) be symmetric bilinear spaces over K of dimensions n and m respectively. Then*

$$\Lambda^2(\varphi\psi) = m \times \Lambda^2\varphi + n \times \Lambda^2\psi + 2 \times \Lambda^2\varphi\Lambda^2\psi$$

Proof. Let $\varphi = \langle a_1, \dots, a_n\rangle$ be a diagonalisation of φ. Then

$$\begin{aligned}
\Lambda^2(\varphi\psi) &= \Lambda^2(\langle a_1\rangle\psi \perp \cdots \perp \langle a_n\rangle\psi) \\
&= \Lambda^2(a_1\psi \perp \cdots \perp a_n\psi) \\
&= \perp_{i_1+\cdots+i_n=2} \Lambda^{i_1}(a_1\psi) \cdots \Lambda^{i_n}(a_n\psi) && \text{by Corollary 7.4} \\
&= \Lambda^2(a_1\psi) \perp \cdots \perp \Lambda^2(a_n\psi) \perp \perp_{1\le i_1<i_2\le n} a_1\psi a_2\psi \\
&= a_1^2\Lambda^2\psi \perp \cdots \perp a_n^2\Lambda^2\psi \perp \psi^2 \perp_{1\le i_1<i_2\le n} a_1 a_2 && \text{by Proposition 7.1} \\
&= n \times \Lambda^2\psi \perp \psi^2\Lambda^2\varphi
\end{aligned}$$

and substituting $\psi^2 = m \times \langle 1\rangle \perp 2 \times \Lambda^2\psi$ using the last Lemma gives the result. □

Remark 7.7. Similar formulas may be proven for higher exterior powers $\Lambda^k(\varphi\psi)$ and for exterior powers of exterior powers using the polynomials in Definition 9.6 and the results of Corollary 9.11 and Example 9.12 in the next section.

8. Pfister forms and exterior powers

Recall that for $a_1, \ldots, a_n \in K^*$, we can define the n**-fold Pfister form**

$$\psi = \langle\langle a_1, \ldots, a_n \rangle\rangle := \bigotimes_{i=1}^{n} \langle 1, a_i \rangle = \perp_{\substack{\text{all subsets} \\ \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}}} \langle a_{i_1} \cdots a_{i_k} \rangle$$

Thus

$$\psi = \langle\langle \varphi \rangle\rangle = \perp_{k=0}^{n} \Lambda^k \varphi$$

where $\varphi = \langle a_1, \ldots, a_n \rangle$. If φ is a form on V then $\langle\langle \varphi \rangle\rangle$ is naturally a form on $\Lambda(V)$, the exterior algebra of V, viewed as a vector space.

Remark 8.1. Corollary 6.8 and the last fact give a quick way of seeing that if φ_1 is isometric to φ_2, then $\langle\langle \varphi_1 \rangle\rangle$ is isometric to $\langle\langle \varphi_2 \rangle\rangle$.

Noting that the Clifford algebra of φ, $C(\varphi) = \perp_{k=0}^{n} \Lambda^k V$ as vector spaces, it is reasonable to look for a connection between $C(\varphi)$ and $\langle\langle \varphi \rangle\rangle$.

Recall that the **standard involution** on $C(\varphi)$ is the involution σ which is the identity on V, with $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in V$. An easy induction shows that for all positive integers m, $\sigma(x_1 \cdots x_m) = \sigma(x_m) \cdots \sigma(x_1)$.

Then the **involution trace form** on $C(\varphi)$ associated to σ is the map

$$T_{C(\varphi)} : C(\varphi) \times C(\varphi) \longrightarrow K : (x, y) \longmapsto T_{C(\varphi)}(x, y) := \mathrm{Trace}_{C(\varphi)/K}\, \sigma(x)y.$$

Proposition 8.2. *With the notation above,*

$$T_{C(\varphi)} = 2^n \langle\langle \varphi \rangle\rangle = \begin{cases} \langle\langle \varphi \rangle\rangle, & \text{for } n \text{ even;} \\ 2\langle\langle \varphi \rangle\rangle, & \text{for } n \text{ odd.} \end{cases}$$

Proof. Pick an orthogonal basis $\{v_1, \ldots, v_n\}$ for V such that $\varphi(v_i, v_i) = a_i$ for $i \in \{1, \ldots, n\}$. Then a basis for $C(\varphi)$ is given by the set of 2^n products

$$B = \{v_{i_1} \cdots v_{i_k} : 1 \le i_1 < \cdots < i_k \le n, 0 \le k \le n\} = \{v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n} : \varepsilon_i = 0, 1,\ i = 1, \ldots, n\}.$$

We have that for all distinct $x, y \in B$, with $x = v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n}$,

$$\begin{aligned}
\sigma(x)x &= \sigma(v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n}) v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n} \\
&= v_n^{\varepsilon_n} \cdots v_1^{\varepsilon_1} v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n} \\
&= v_n^{\varepsilon_n} \cdots v_2^{\varepsilon_2} \varphi(v_1, v_1)^{\varepsilon_1} v_2^{\varepsilon_2} \cdots v_n^{\varepsilon_n} \\
&= \varphi(v_1, v_1)^{\varepsilon_1} \varphi(v_2, v_2)^{\varepsilon_2} \cdots \varphi(v_n, v_n)^{\varepsilon_n} \\
&= a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n}
\end{aligned}$$

which is a scalar, and conversely, by linear independence, $\sigma(x)y$ is not a scalar. Thus, left multiplication by $\sigma(x)y$ will move any element z of B to a scalar multiple of a *different* element of B. It follows that the matrix for left multiplication by $\sigma(x)y$ will have all zeros on the main diagonal and so be of trace zero; while the matrix for left multiplication by $\sigma(x)x$ will be the scalar matrix $a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n} I_{2^n}$ which has trace $2^n a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n}$.

Hence, as a form, $T_{C(\varphi)}$ has diagonalisation $\perp_{k=0}^{n} 2^n \Lambda^k \varphi$ with respect to the basis B, completing the proof. $\square$

9. Properties of $\widehat{W}(K)$ as a λ-ring

Definition 9.1. A **pre-λ-ring** R is a commutative ring with identity, and with unary operations $\lambda^n : R \longrightarrow R$, for $n = 0, 1, 2, \dots$ such that for all $x, y \in R$:

$$\begin{array}{ll} (i) & \lambda^0(x) = 1; \\ (ii) & \lambda^1(x) = x; \\ (iii) & \lambda^n(x+y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y). \end{array}$$

An equivalent definition is: for $x \in R$, consider the formal power series in the variable t defined by

$$\lambda_t(x) = \lambda^0(x) + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots$$

Then the conditions are that:

$$\begin{array}{ll} (i)' & \lambda^0(x) = 1; \\ (ii)' & \lambda^1(x) = x; \\ (iii)' & \lambda_t(x+y) = \lambda_t(x)\lambda_t(y). \end{array}$$

Definition 9.2. Let R be a pre-λ-ring. An element $x \in R$ is of **finite degree** n if $\lambda_t(x)$ is a polynomial of degree n, i. e. $\lambda^i(x) = 0$ for all $i > n$ but $\lambda^n(x) \neq 0$. R is **finitary** if each element of R is a difference of elements of finite degree.

Proposition 9.3. *The Witt-Grothendieck ring $\widehat{W}(K)$ under the operations of orthogonal sum and tensor product forms a pre-λ-ring with the exterior powers Λ^k acting as the λ-operations.*

Proof. We show that (i), (ii) and $(iii)'$ of Definition 9.1 above hold. First, Corollary 4.6(a) and the definition of $\Lambda^0\varphi$ to be the identity form imply (i) and (ii).

From the definition of a Pfister form we have, for $b_1, \dots, b_{n+m}$ elements of a field L, that

$$\langle\langle b_1, \dots, b_n, b_{n+1}, \dots, b_{n+m}\rangle\rangle = \langle\langle b_1, \dots, b_n\rangle\rangle \otimes \langle\langle b_{n+1}, \dots, b_{n+m}\rangle\rangle \qquad (*).$$

We work over the field $K(t)$ of rational functions in t. Given forms $\varphi = \langle a_1, \dots, a_n\rangle$, $\psi = \langle a_{n+1}, \dots, a_{n+m}\rangle$ over K, consider the forms $\langle ta_1, \dots, ta_n\rangle$, $\langle ta_{n+1}, \dots, ta_{n+m}\rangle$ over $K(t)$. Then

$$\langle\langle ta_1, \dots, ta_n\rangle\rangle = 1 + t\varphi + t^2\Lambda^2\varphi + \cdots + t^n\Lambda^n\varphi = \lambda_t(\varphi).$$

Similarly, we have that $\langle\langle ta_{n+1}, \dots, ta_{n+m}\rangle\rangle = 1 + t\psi + t^2\Lambda^2\psi + \cdots + t^m\Lambda^m\psi = \lambda_t(\psi)$ and $\langle\langle ta_1, \dots, ta_{n+m}\rangle\rangle = 1 + t(\varphi+\psi) + \cdots + t^m\Lambda^m(\varphi+\psi) = \lambda_t(\varphi+\psi)$. Then applying $(*)$ above shows that $\lambda_t(\varphi+\psi) = \lambda_t(\varphi)\lambda_t(\psi)$, which is just $(iii)'$ of Definition 9.1.

The equivalence of this to (iii) proves Proposition 7.3. □

Remark 9.4. Each element of $\widehat{W}(K)^+$ is of finite degree, the degree being its dimension, since $\Lambda^i\varphi = 0$ for all $i > \dim\varphi$ but $\Lambda^{\dim\varphi}\varphi = \langle\det\varphi\rangle$. Thus $\widehat{W}(K)$, whose elements are formal differences of elements of $\widehat{W}(K)^+$, is a finitary pre-λ-ring.

Remark 9.5. Recall that the k^{th} **elementary symmetric function** of symbols $x_1, \dots, x_n$ is $a_k = \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}$.

Definition 9.6. Let $\xi_1, \ldots, \xi_q$ and $\eta_1, \ldots, \eta_r$ be indeterminates. Define b_n and c_m, where $n, m \in \{1, 2, 3, \ldots\}$, by

$$1 + b_1 t + b_2 t^2 + \cdots = \prod_{i=1}^{q} (1 + \xi_i t),$$

and

$$1 + c_1 t + c_2 t^2 + \cdots = \prod_{j=1}^{r} (1 + \eta_j t).$$

Then the b_n are the elementary symmetric functions of the ξ_i and the c_m are the elementary symmetric functions of the η_j. We define polynomials in the b_n and c_m as follows. Let

$$P_k(b_1, \ldots, b_k; c_1, \ldots, c_k) := \text{coefficient of } t^k \text{ in } \prod_{i,j} (1 + \xi_i \eta_j t)$$

and

$$Q_{k,l}(b_1, \ldots, b_{kl}) := \text{coefficient of } t^k \text{ in } \prod_{1 \leq i_1 < \cdots < i_l \leq q} (1 + \xi_{i_1} \cdots \xi_{i_l} t).$$

Then P_k and $Q_{k,l}$ are integer polynomials independent of q and r provided both $q, r \geq k$ in the case of P_k, and $q \geq kl$ in the case of $Q_{k,l}$. Being integer polynomials, they are well-defined over any ring with 1, in particular the Witt and Witt-Grothendieck rings.

Definition 9.7. A **λ-ring** is a pre-λ-ring R in which (see [6, page 13])

$$\begin{array}{ll} (i) & \lambda_t(1) = 1 + t; \\ (ii) & \lambda^k(xy) = P_k(\lambda^1 x, \ldots, \lambda^k x; \lambda^1 y, \ldots, \lambda^k y), \quad \text{for all } x, y \in R,\ k \geq 0; \\ (iii) & \lambda^l(\lambda^k(x)) = Q_{k,l}(\lambda^1 x, \ldots, \lambda^{kl} x), \quad \text{for all } x \in R,\ k, l \geq 0. \end{array}$$

Proposition 9.8 ([6], page 17)**.** *Let R be a pre-λ-ring in which*

(i) $\lambda_t(1) = 1 + t$;

(ii) *each $x \in R$ is a finite sum $x = \sum \pm x_i$, where each x_i has degree one*;

(iii) *the product of two elements of degree one is again of degree one.*

Then R is a λ-ring.

Comment 9.9. For a general result on when a pre-λ-ring is a λ-ring, see [1, page 392]. For further details on λ-rings and related topics, see Lectures V and VI of [1].

Corollary 9.10. *The Witt-Grothendieck ring $\widehat{W}(K)$ is a λ-ring, where the k^{th} λ-operation is defined to be the k^{th} exterior power Λ^k.*

Proof. We verify that the hypotheses of Proposition 9.8 hold. For (i), we have that

$$\begin{array}{rcll} \lambda_t(\langle 1 \rangle) & = & \Lambda^0(\langle 1 \rangle) + \Lambda^1(\langle 1 \rangle) t + \Lambda^2(\langle 1 \rangle) t^2 + \cdots & \\ & = & \Lambda^0(\langle 1 \rangle) + \Lambda^1(\langle 1 \rangle) t & \text{since } \Lambda^k(\langle 1 \rangle) = 0 \text{ for all } k \geq 2 \\ & = & \langle 1 \rangle + \langle 1 \rangle t & \text{by Definition 3.1} \end{array}$$

as required.

(ii) follows from the Diagonalisation Theorem (see [9, Theorem I.3.5] — every symmetric bilinear space is an orthogonal sum of one-dimensional symmetric bilinear spaces), and the construction of $\widehat{W}(K)$ as formal differences of elements of $\widehat{W}(K)^+$.

(iii) is clear because $\langle a\rangle\langle b\rangle = \langle ab\rangle$ for all $a, b \in K$. □

Corollary 9.11. *Let P_k and $Q_{k,l}$ be the integer polynomials in Definition 9.6. Then, in the Witt-Grothendieck ring $\widehat{W}(K)$,*

$$\begin{array}{ll} (i) & \Lambda^k(\varphi\otimes\psi) = P_k(\Lambda^1\varphi,\dots,\Lambda^k\varphi;\Lambda^1\psi,\dots,\Lambda^k\psi), \quad \textit{for all } \varphi,\psi\in\widehat{W}(K),\ k\geq 0; \\ (ii) & \Lambda^l(\Lambda^k(\varphi)) = Q_{k,l}(\Lambda^1\varphi,\dots,\Lambda^{kl}\varphi), \quad \textit{for all } \varphi\in\widehat{W}(K),\ k,l\geq 0. \end{array}$$

Example 9.12. The following formulas hold in any λ-ring and in particular, in the Witt-Grothendieck ring.

$$\begin{array}{lrcl} (1) & \lambda^2(xy) & = & x^2\,\lambda^2 y + y^2\,\lambda^2 x - 2\lambda^2 x\,\lambda^2 y \\ (2) & \lambda^3(xy) & = & x^3\,\lambda^3 y + y^3\,\lambda^3 x + 3\lambda^3 x\,\lambda^3 y + xy\lambda^2 x\,\lambda^2 y - 3x\lambda^2 x\,\lambda^3 y - 3y\lambda^2 y\,\lambda^3 x \\ (3) & \lambda^2(\lambda^2(x)) & = & \lambda^3 x\,\lambda^1 x - \lambda^4 x \\ (4) & \lambda^2(\lambda^3(x)) & = & \lambda^6 x - \lambda^5 x\,\lambda^1 x + \lambda^4 x\,\lambda^2 x \end{array}$$

Definition 9.13. Given λ-rings R and S, a **λ-ring homomorphism** is a ring homomorphism $f : R \longrightarrow S$ such that

$$\lambda^n(f(x)) = f(\lambda^n(x)) \qquad \text{for all } x\in R \text{ and all } n\geq 0.$$

A **λ-ring endomorphism** is a λ-ring homomorphism $f : R \longrightarrow R$.

Remark 9.14. As in [6, pages 9–10], given a λ-ring R and $x \in R$ written as a sum of n elements of degree 1, $x = x_1 + \cdots + x_n$ we may regard the k^{th} λ-operation λ^k as the k^{th} elementary symmetric function $a_k = \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1}\cdots x_{i_k}$ of the x_i. Recall that the (symmetric) power sums s_j are defined in terms of the a_k via Newton's formulas

$$s_j \;=\; s_{j-1}a_1 - s_{j-2}a_2 + \cdots + (-1)^{j-2}s_1a_{j-1} + j(-1)^{j-1}a_j$$

and solving for a_k by Cramer's rule as in [6, page 36] gives

$$k!\,\times a_k = \det\begin{pmatrix} s_1 & 1 & & & & \\ s_2 & s_1 & 2 & & 0 & \\ s_3 & s_2 & s_1 & 3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ s_{k-1} & s_{k-2} & \ddots & s_2 & s_1 & k-1 \\ s_k & s_{k-1} & \dots & s_3 & s_2 & s_1 \end{pmatrix} \qquad (*).$$

Definition 9.15. Let R be a λ-ring, let $x \in R$ and let k be a positive integer. We define the k^{th} **Adams operation**, Ψ^k, by

$$\Psi^k(x) := s_k(x).$$

Remark 9.16. Alternatively, (see [5, Definition 12.2.1]) we may define

$$\Psi_t(x) = \sum_{k\geq 1}\Psi^k(x)t^k$$

by the relation

$$\Psi_{-t}(x) = -\frac{t}{\lambda_t(x)}\frac{d}{dt}\lambda_t(x).$$

This definition leads to Newton's formulas (see [5, Theorem 12.2.5]):

$$\Psi^k(x) = \lambda^1(x)\Psi^{k-1}(x) + \cdots + (-1)^{k-2}\lambda^{k-1}(x)\Psi^1(x) + k(-1)^{k-1}\lambda^k(x).$$

Remark 9.17. Thus equation $(*)$ in Remark 9.14 becomes, for $x \in R$,

$$k! \times \lambda^k(x) = \det \begin{pmatrix} \Psi^1(x) & 1 & & & & \\ \Psi^2(x) & \Psi^1(x) & 2 & & 0 & \\ \Psi^3(x) & \Psi^2(x) & \Psi^1(x) & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ \Psi^{k-1}(x) & \Psi^{k-2}(x) & \ddots & \ddots & \Psi^1(x) & k-1 \\ \Psi^k(x) & \Psi^{k-1}(x) & \Psi^{k-2}(x) & \dots & \Psi^2(x) & \Psi^1(x) \end{pmatrix}$$

Proposition 9.18 ([6], page 48)**.** *Let R be a λ-ring. Then each Adams operation is a λ-ring endomorphism of R. In particular, $\Psi^1 = \lambda^1 =$ the identity function on R.*

Lemma 9.19. *Let k be a positive integer, let $\langle a\rangle$ be a 1-dimensional symmetric bilinear form and consider the λ-ring $\widehat{W}(K)$, equipped with Adams operations derived from the λ-operations Λ^k. Then*

$$\Psi^k(\langle a\rangle) = \begin{cases} \langle 1\rangle, & \text{for } k \text{ even;} \\ \langle a\rangle, & \text{for } k \text{ odd.} \end{cases}$$

Proof. First note that $\Psi^1(\langle a\rangle) = \langle a\rangle$ since Ψ^1 is the identity function by Proposition 9.18. Next, by Remark 9.16, for any $k > 1$ we have

$$\Psi^k(\langle a\rangle) = \langle a\rangle\Psi^{k-1}(\langle a\rangle) + \Lambda^2(\langle a\rangle)\Psi^{k-2}(\langle a\rangle) + \cdots + (-1)^{k-1}\Lambda^{k-1}(\langle a\rangle)\langle a\rangle + k(-1)^k\Lambda^k(\langle a\rangle).$$

Now every term except the first involves a $\Lambda^j(\langle a\rangle)$ for some $j \geq 2$, and so is zero, since $\Lambda^j(\langle a\rangle) = 0$ for all $j \geq 2$. Thus we get

$$\Psi^k(\langle a\rangle) = \langle a\rangle\Psi^{k-1}(\langle a\rangle) \text{ for all } k > 1$$

and an easy induction starting from the first note above gives

$$\Psi^k(\langle a\rangle) = \langle a\rangle^k.$$

Since $\langle a\rangle\langle a\rangle = \langle 1\rangle \in \widehat{W}(K)$, the result follows. $\square$

Proposition 9.20. *Let n and k be positive integers and let φ be an n-dimensional symmetric bilinear form. Then, in the the λ-ring $\widehat{W}(K)$,*

$$\Psi^k(\varphi) = \begin{cases} n \times \langle 1\rangle, & \text{for } k \text{ even;} \\ \varphi, & \text{for } k \text{ odd.} \end{cases}$$

Proof. Let $\varphi = \langle a_1, \ldots, a_n\rangle$. Since, by Proposition 9.18, Ψ^k is a λ-ring endomorphism of $\widehat{W}(K)$ for all positive integers k, we may write

$$\Psi^k(\varphi) = \Psi^k\Bigg(\mathop{\perp}_{i=1}^{n}\langle a_i\rangle\Bigg) = \sum_{i=1}^{n}\Psi^k(\langle a_i\rangle)$$

and the result follows from Lemma 9.19. □

Remark 9.21. For convenience, we will usually write n for the element $n \times \langle 1 \rangle$ of the Witt-Grothendieck ring $\widehat{W}(K)$. Then Proposition 9.20 says that

$$\Psi^k(\varphi) = \begin{cases} n, & \text{for } k \text{ even;} \\ \varphi, & \text{for } k \text{ odd.} \end{cases}$$

Remark 9.22. Clearly the dimension of a form (see Remark 9.4) extends to a homomorphism $\deg : \widehat{W}(K) \longrightarrow \mathbb{Z}$. Using this, and the fact that Ψ^k is a λ-ring endomorphism, we can extend Proposition 9.20 as follows.

For any element $x = \varphi - \psi \in \widehat{W}(K)$ where $\varphi, \psi \in \widehat{W}(K)^+$, and any positive integer k:

$$\Psi^k(x) = \Psi^k(\varphi) - \Psi^k(\psi) = \begin{cases} \deg\varphi - \deg\psi & = & \deg x, & \text{for } k \text{ even;} \\ \varphi - \psi & = & x, & \text{for } k \text{ odd.} \end{cases}$$

10. Exterior powers and annihilating polynomials

Definition 10.1. We define two $(n+1) \times (n+1)$ matrices $M_n(t)$ and $N_n(t)$ as follows:

$$M_n(t) = \begin{pmatrix} t & 1 & & & & \\ n & t & 2 & & O & \\ t & n & t & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & n-1 & \\ & & & n & t & n \\ & * & & t & n & t \end{pmatrix}, \quad N_n(t) = \begin{pmatrix} t & 1 & & & & \\ n & t & 2 & & O & \\ & n-1 & t & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & O & & 2 & t & n \\ & & & & 1 & t \end{pmatrix}.$$

We also define

$$D_n(t) = \det M_n(t)$$

and

$$T_n(t) = \det N_n(t).$$

Remark 10.2. Applying Proposition 9.20 to the determinant in Remark 9.17 we have that, for a λ-ring R and $x \in R$,

$$(n+1)! \times \lambda^{n+1}(x) = \det \begin{pmatrix} x & 1 & & & & \\ n & x & 2 & & O & \\ x & n & x & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & n-1 & \\ & & & n & x & n \\ & * & & x & n & x \end{pmatrix} = D_n(x).$$

Lemma 10.3. *The determinants $D_n(t)$ and $T_n(t)$ are equal.*

Proof. We may use elementary row operations $\mathrm{row}_i \longrightarrow \mathrm{row}_i - \mathrm{row}_{i-2}$, for $i = 3, \dots, n+1$ to convert $M_n(t)$ to $N_n(t)$. Since adding a multiple of one row to another does not change the determinant, we have that $\det M_n(t) = \det N_n(t)$ as required. □

Lemma 10.4. *The determinant $D_n(t)$ is equal to $(t^2 - n^2)D_{n-2}(t)$.*

Proof. We use elementary column operations $\mathrm{col}_i \longrightarrow \mathrm{col}_i - \mathrm{col}_{i+2}$, for $i = 1, \ldots, n-1$ to convert $M_n(t)$ to

$$\left(\begin{array}{cccccc|cc} t & 1 & & & & & 0 & 0 \\ n-2 & t & 2 & & O & & 0 & 0 \\ & n-3 & t & \ddots & & & \vdots & \vdots \\ & & \ddots & \ddots & n-3 & & \vdots & \vdots \\ & O & & 2 & t & n-2 & 0 & 0 \\ & & & & 1 & t & n-1 & 0 \\ \hline 0 & \cdots & \cdots & \cdots & \cdots & 0 & t & n \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & n & t \end{array}\right)$$

which we see is just

$$\left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & N_{n-2}(t) & & \vdots & \vdots \\ & & & 0 & 0 \\ & & & n-1 & 0 \\ \hline 0 & \cdots & 0 & t & n \\ 0 & \cdots & 0 & n & t \end{array}\right).$$

Thus $\det M_n(t) = \det N_{n-2}(t)(t^2 - n^2)$ which is $\det M_{n-2}(t)(t^2 - n^2)$ by Lemma 10.3 and the proof is complete. $\square$

Definition 10.5. Let n be a positive integer. Define the polynomial $p_n \in \mathbb{Z}[t]$ by

$$p_n(t) = (t-n)(t-n+2)\cdots(t+n).$$

So, for n even,

$$p_n(t) = t(t^2 - 2^2)\cdots(t^2 - n^2),$$

and, for n odd,

$$p_n(t) = (t^2 - 1^2)(t^2 - 3^2)\cdots(t^2 - n^2).$$

We note that for n even, p_n is an odd polynomial, and for n odd, p_n is an even polynomial.

Lemma 10.6. *The polynomials p_n satisfy the following recurrence relation:*

$$p_n(t) = (t+n)p_{n-1}(t-1) \textit{ for all positive integers } n.$$

Proof. An easy exercise. $\square$

Proposition 10.7. *The polynomial $p_n(t)$ is equal to the determinant $D_n(t)$ of the matrix $M_n(t)$.*

Proof. We verify the result by induction on n in steps of 2. The result is clear for $n = 1$ and $n = 2$. For $n > 2$, Lemma 10.4 gives us that

$$\begin{aligned} D_n(t) &= (t^2 - n^2)D_{n-2}(t) \\ &= (t^2 - n^2)p_{n-2}(t) \qquad \text{(by the induction hypothesis)} \\ &= p_n(t), \end{aligned}$$

which completes the proof. $\square$

Theorem 10.8. *Let (V,φ) be a symmetric bilinear space of dimension n. Then*

$$(n+1)!\,\times\Lambda^{n+1}\varphi = p_n(\varphi)\ \text{in}\ \widehat{W}(K).$$

Proof. We know by Corollary 9.10 that $\widehat{W}(K)$ is a λ-ring. By Remark 10.2,

$$(n+1)!\,\times\Lambda^{n+1}\varphi = D_n(\varphi)$$

and by Proposition 10.7, since $D_n(t) = p_n(t)$ this becomes

$$(n+1)!\,\times\Lambda^{n+1}\varphi = D_n(\varphi) = p_n(\varphi),$$

completing the proof. □

Remark 10.9. In fact for any element $x \in \widehat{W}(K)$ of positive degree n we have

$$(n+1)!\,\times\Lambda^{n+1}x = p_n(x)\ \text{in}\ \widehat{W}(K).$$

Corollary 10.10. *Let (V,φ) be a symmetric bilinear space of dimension n. Then*

$$p_n(\varphi) = 0$$

i. e. the polynomial p_n annihilates φ in $\widehat{W}(K)$.

Proof. By Definition 3.1, we have that the $(n+1)^{\text{th}}$ exterior power of an n-dimensional symmetric bilinear form φ is zero. Then Theorem 10.8 gives

$$p_n(\varphi) = (n+1)!\,\times\Lambda^{n+1}\varphi = (n+1)!\,\times 0 = 0$$

which completes the proof. □

Corollary 10.11. *Let (V,φ) be a symmetric bilinear space of dimension n. Then the polynomial p_n annihilates the Witt class of φ in $W(K)$.*

Proof. This follows since the projection map $\pi : \widehat{W}(K) \longrightarrow W(K)$ is a ring homomorphism and $0_{\widehat{W}(K)} \overset{\pi}{\longmapsto} 0_{W(K)}$. □

Remark 10.12. Corollary 10.11 was first proven in [7]. Other proofs have since appeared, see, e.g. [8], also a quick proof due to Leung is given in [7].

Definition 10.13. We define a more general determinant, $D_n^k(t)$, as follows:

$$D_n^k(t) := \det\begin{pmatrix} t & 1 & & & & & \\ n & t & 2 & & & O & \\ t & n & t & 3 & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & & & & t & k-1 & \\ & * & & & n & t & k \\ & & & & t & n & t \end{pmatrix}.$$

Lemma 10.14. *We have the following recurrence relation:*

$$D_n^k(t) = tD_n^{k-1}(t) - k(n-k+1)D_n^{k-2}(t)$$

Proof. An easy computation of the determinant $D_n^k(t)$ along the rightmost column. □

Remark 10.15. When $k = n$ we get $p_n(t) = D_n(t) = D_n^n(t)$. Note that $D_n^k(t) \neq p_k(t)$ when $k \neq n$.

Example 10.16. We easily work out that

$$\begin{aligned} D_n^0(t) &= t \\ D_n^1(t) &= t^2 - n \end{aligned}$$

and we can then work out all of the $D_n^k(t)$ and thus all of the $D_n^k(\varphi)$ using the recurrence relation Lemma 10.14.

Proposition 10.17. *For a symmetric bilinear form φ of dimension n, we have*

$$(k+1)! \times \Lambda^{k+1}\varphi = D_n^k(\varphi) = \varphi D_n^{k-1}(\varphi) - k(n-k+1)D_n^{k-2}(\varphi)$$

and

$$p_n(\varphi) = D_n(\varphi) = D_n^n(\varphi) = \varphi D_n^{n-1}(\varphi) - nD_n^{n-2}(\varphi).$$

Proof. This follows from Remark 9.17 and Lemma 10.14. □

Corollary 10.18. *Let φ be a symmetric bilinear form of dimension n and let $k \geq n$. Then*

$$D_n^k(\varphi) = 0.$$

11. Signature of an exterior power

Throughout this section, K will be an ordered field, and P will be an ordering of K. The notation $p_n(t)$ and $D_n^k(t)$ will continue to denote the polynomials defined in 10.5 and 10.13 respectively. The notation $\operatorname{sign}_P(\varphi)$ or $\operatorname{sign}(\varphi)$ will denote the signature of φ with respect to P.

Proposition 11.1. *Let (V, φ) be a symmetric bilinear space of dimension n and signature* $\operatorname{sign}(\varphi) = r$, *and let k be a positive integer. Then*

$$\operatorname{sign}(\Lambda^k\varphi) = \frac{D_n^{k-1}(r)}{k!}$$

and, in particular, $k!$ divides $D_n^{k-1}(r)$ for any positive integer k.

Proof. This follows from Remark 10.2 and the facts that $\widehat{W}(K)$ is a λ-ring and that $\operatorname{sign} : \widehat{W}(K) \longrightarrow \mathbb{Z}$ is a ring homomorphism. We are justified in dividing $D_n^{k-1}(r)$ by $k!$ since the signature of any form must be an integer. □

Example 11.2. Let φ be a symmetric bilinear form of dimension $n = 8$ and let $k = 5$. Then $\Lambda^k\varphi$ has dimension $\binom{8}{5} = 56$. In this case $D_n^{k-1}(t) = t(t^4 - 60t^2 + 584)$, an odd polynomial.

Now the possible signatures of any form are all congruent to its dimension modulo 2. Let $r = \operatorname{sign}_P \varphi$, so possible values of r are $-8, -6, -4, -2, 0, 2, 4, 6, 8$. These give the possible values of $\operatorname{sign}_P(\Lambda^k\varphi)$, using Proposition 11.1.

$r = 0$ gives $\frac{1}{5!}D_8^4(0) = \frac{0}{120} = 0$.

$r = 2$ gives $\frac{1}{5!}D_8^4(2) = \frac{720}{120} = 6$.

$r = 4$ gives $\frac{1}{5!}D_8^4(4) = \frac{-480}{120} = -4$.

$r = 6$ gives $\frac{1}{5!}D_8^4(6) = \frac{-1680}{120} = -14$.

$r = 8$ gives $\frac{1}{5!}D_8^4(8) = \frac{6720}{120} = 56$.

Since $D_n^{k-1}(t)$ is odd, the other possible signatures are $-6, 4, 14$ and -56. Only forms with these nine signatures can possibly be exterior powers over ordered fields.

Corollary 11.3. *If (V, φ) is a symmetric bilinear space of dimension n and signature r, then r is a root of the polynomial $D_n^k(t)$ for all positive integers $k \geq n$.*

Proof. This follows from Proposition 11.1 and Corollary 10.18. □

Proposition 11.4. *Let n, k be positive integers. Then the polynomial $D_n^k(t)$ is monic. Further,*

$$D_n^k(t) \text{ is } \begin{cases} \text{even} & \text{if } k \text{ is odd;} \\ \text{odd} & \text{if } k \text{ is even.} \end{cases}$$

Proof. First note $\deg D_n^k(t) = k + 1$ which is of opposite parity to k.

We proceed by complete induction on k. Fix n. By Example 10.16 $D_n^0(t) = t$ and $D_n^1(t) = t^2 - n$ so the result is true for $k = 0$ and $k = 1$.

Suppose the result is true for $1, 2, \ldots, k - 1$. The recurrence relation in Lemma 10.14 is

$$D_n^k(t) = tD_n^{k-1}(t) - k(n - k + 1)D_n^{k-2}(t). \tag{5}$$

Comparing degrees in (5) shows that

$$\text{leading term of } D_n^k(t) = t(\text{leading term of } D_n^{k-1}(t))$$

so $D_n^k(t)$ is monic by the induction hypothesis.

Moreover, $D_n^{k-1}(t)$ has opposite parity to $D_n^{k-2}(t)$ by the induction hypothesis, so the parity of $tD_n^{k-1}(t)$ is equal to the parity of $D_n^{k-2}(t)$. Thus the parity of the right hand side of (5) is that of $D_n^{k-2}(t)$, and this is the parity of $D_n^k(t)$, completing the proof. □

Corollary 11.5. *Suppose k is an odd positive integer, and φ is a symmetric bilinear form with $\operatorname{sign}_P \varphi = 0$. Then $\operatorname{sign}_P(\Lambda^k \varphi) = 0$.*

Proof. Since k is odd, $D_n^{k-1}(t)$ is an odd polynomial by Proposition 11.4. Thus

$$D_n^{k-1}(0) = D_n^{k-1}(-0) = -D_n^{k-1}(0) \quad \text{which implies} \quad D_n^{k-1}(0) = 0.$$

Then by Proposition 11.1, $\operatorname{sign}_P(\Lambda^k \varphi) = D_n^{k-1}(0)/k! = 0$ as required. □

Proposition 11.6. *Suppose that k is an even positive integer, and φ is a symmetric bilinear form with $\operatorname{sign}_P \varphi = 0$ (so $n := \dim \varphi$ is even). Then*

$$\operatorname{sign}_P(\Lambda^k \varphi) = \begin{cases} 0 & \text{if } n < k; \\ (-1)^{k/2} \binom{n/2}{k/2} & \text{if } n \geq k. \end{cases}$$

Proof. The result for $n < k$ follows since $\dim \Lambda^k\varphi = 0$. Also, by Proposition 11.1,

$$\operatorname{sign}_P(\Lambda^k\varphi) = \frac{1}{k!}D_n^{k-1}(\operatorname{sign}_P\varphi)$$

which in this case is

$$\frac{1}{k!}D_n^{k-1}(0) = \frac{1}{k!}(\text{constant term of } D_n^{k-1}(t)).$$

For simplicity of notation, let $m = n/2$ and $l = k/2$. We use the recurrence relation in Lemma 10.14 for $k-1$,

$$D_n^{k-1}(t) = tD_n^{k-2}(t) - (k-1)(n-k+2)D_n^{k-3}(t),$$

with $t = 0$ to get

$$\begin{aligned} D_n^{k-1}(0) &= -(k-1)(n-k+2)D_n^{k-3}(0) \;=\; \cdots \\ &= -(k-1)(n-k+2)\big(-(k-3)(n-k+4)\big)\big(\cdots\big)\big(-3(n-2)\big)\big(-1(n)\big) \\ &= (-1)^l\prod_{j=0}^{l-1}(2j+1)(n-2j) \end{aligned}$$

Now by Proposition 11.1, when $n \geq k$ we have

$$\begin{aligned} \operatorname{sign}(\Lambda^k\varphi) &= \frac{D_n^{k-1}(\operatorname{sign}(\varphi))}{k!} \;=\; \frac{1}{k!}D_n^{k-1}(0) \\ &= \frac{1}{k!}(-1)^l\prod_{j=0}^{l-1}(2j+1)(2m-2j) \end{aligned}$$

and writing $k! = (2l)! = \prod_{j=0}^{l-1}(2j+1)(2j+2)$ gives

$$\begin{aligned} \operatorname{sign}(\Lambda^k\varphi) &= (-1)^l\prod_{j=0}^{l-1}\frac{(2j+1)(2m-2j)}{(2j+1)(2j+2)} \\ &= (-1)^l\prod_{j=0}^{l-1}\frac{m-j}{j+1} \\ &= (-1)^l\binom{m}{l} \end{aligned}$$

which completes the proof. □

Corollary 11.7. *Let φ be a hyperbolic form. Then $\Lambda^k\varphi$ is hyperbolic if and only if k is an odd number.*

Proof. Each entry in the diagonalisation of $\Lambda^k\varphi$ is a product of $+1$'s and -1's and so is either $+1$ or -1. Hence $\Lambda^k\varphi$ is hyperbolic precisely when the number of -1's is the same as the number of $+1$'s, that is, precisely when $\operatorname{sign}_P(\Lambda^k\varphi) = 0$ (with respect to any ordering P).

Now $\operatorname{sign}_P\varphi = 0$ since φ is hyperbolic, so the result follows from Corollary 11.5 and Proposition 11.6. □

Proposition 11.8. *Let φ be a hyperbolic form of dimension $n = 2m$, so $\varphi = m \times \langle 1, -1\rangle$ and let $k = 2l$ be an even positive integer, with $k \leq n$. Then*

$$\operatorname{sign}_P(\Lambda^k\varphi) = (-1)^l \binom{m}{l}$$

That is, $\Lambda^k\varphi$ has a Witt decomposition

$$\Lambda^k\varphi = \binom{m}{l} \times \langle (-1)^l\rangle \perp \frac{1}{2}\left(\binom{n}{k} - \binom{m}{l}\right) \times \langle 1, -1\rangle$$

Proof. Let φ be a hyperbolic form of dimension $n = 2m$ and let $k = 2l$ be an even positive integer, with $k \leq n$; thus $l \leq m$. As in Corollary 11.7, each entry in the diagonalisation of $\Lambda^k\varphi$ is either $+1$ or -1. Each pair $\langle 1, -1\rangle$ gives a hyperbolic plane, and the unpaired entries left over give the other part: there are $|\operatorname{sign}_P(\Lambda^k\varphi)|$ unpaired entries.

From the proof of Proposition 11.6, since n and k are both even,

$$\operatorname{sign}_P(\Lambda^k\varphi) = (-1)^l \binom{m}{l}$$

so the other part of $\Lambda^k\varphi$ is $\binom{m}{l} \times \langle (-1)^l\rangle$ and the result follows. □

Remark 11.9. In fact, over any field K of characteristic different from 2, the k-fold exterior power of $\varphi = m \times \langle 1, -1\rangle$ will have diagonalisation

$$\Lambda^k\varphi = \binom{m}{l} \times \langle (-1)^l\rangle \perp \frac{1}{2}\left(\binom{n}{k} - \binom{m}{l}\right) \times \langle 1, -1\rangle$$

but the $\binom{m}{l} \times \langle (-1)^l\rangle$ summand need not always be anisotropic.

12. Hasse invariant of an exterior power

We denote by (α, β) or $(\alpha, \beta)_K$ the quaternion algebra defined by α and $\beta \in K$. Let $B(K)$ be the subgroup of exponent 2 of the Brauer group $\operatorname{Br}(K)$ of Brauer-equivalence classes of central simple $K-$algebras (see [9, Corollary 2.13.2]). Recall that $(\cdot\,,\cdot) : \dot{K} \times \dot{K} \longrightarrow B(K)$ is a (Steinberg) symbol: it is bimultiplicative and $(\alpha, 1-\alpha) = 1$ for all $\alpha \in K$, $\alpha \neq 0, 1$.

Definition 12.1. Let (V, φ) be a K-quadratic space with $\varphi = \langle a_1, \ldots, a_n\rangle$. For $n > 1$ the **Hasse invariant** $s(\varphi)$ of φ is defined by the formula

$$s(\varphi) = \prod_{1 \leq i < j \leq n} (a_i, a_j) \in B(K)$$

and for $n = 1$ we define $s(\varphi) = 1$. It can be shown (see [9, Remark 2.12.5]) that $s(\varphi)$ is independent of the diagonalisation of φ.

Theorem 12.2. *Let (V, φ) be a quadratic space of dimension n and determinant d, and let k be a positive integer. Then the Hasse invariant of $\Lambda^k\varphi$ is*

$$s(\Lambda^k\varphi) = s(\varphi)^g (d, -1)^e$$

where

$$g = \binom{n-2}{k-1}, \qquad e = \binom{\binom{n-1}{k-1}}{2}.$$

Proof. Let $\varphi = \langle a_1, \ldots, a_n \rangle$ be of dimension n, and let $d := \det \varphi = a_1 \cdots a_n$. Then the k^{th} exterior power of φ is the form

$$\Lambda^k \varphi = \perp_{1 \leq i_1 < \cdots < i_k \leq n} \langle a_{i_1} \cdots a_{i_k} \rangle, \tag{6}$$

which has dimension $\binom{n}{k}$. The Hasse invariant of a quadratic form $\psi = \langle b_1, \ldots, b_m \rangle$ is

$$s(\psi) = \prod_{1 \leq i < j \leq m} (b_i, b_j) \tag{7}$$

which is a product of $\binom{m}{2}$ terms.

To find the Hasse invariant for $\psi = \Lambda^k \varphi$ we need a well-ordering on terms of the form $a_{i_1} \cdots a_{i_k}$. We define a lexicographic well-ordering on such terms as follows. Given multi-indices $\mathbf{i} = (i_1, \ldots, i_k)$ and $\mathbf{j} = (j_1, \ldots, j_k)$, we say

$\mathbf{i} = \mathbf{j}$ if $i_l = j_l$, for $l = 1, \ldots, k$;
$\mathbf{i} < \mathbf{j}$ if $i_l < j_l$, and the l^{th} position is the first position where $\mathbf{i}$ and $\mathbf{j}$ differ;
$\mathbf{i} > \mathbf{j}$ otherwise.

We write $a_{\mathbf{i}}$ for $a_{i_1} \cdots a_{i_k}$ using this multi-index shorthand. Then the Hasse invariant of $\Lambda^k \varphi$ is

$$s(\Lambda^k \varphi) = \prod_{\mathbf{i} < \mathbf{j}} (a_{\mathbf{i}}, a_{\mathbf{j}}) = \prod_{(i_1, \ldots, i_k) < (j_1, \ldots, j_k)} (a_{i_1} \cdots a_{i_k}, a_{j_1} \cdots a_{j_k}). \tag{8}$$

From Equation 6 and Equation 7, this has $\binom{\binom{n}{k}}{2}$ terms in the product. Since $(\cdot, \cdot)$ is bimultiplicative, each term $(a_{i_1} \cdots a_{i_k}, a_{j_1} \cdots a_{j_k})$ can be written as

$$(a_{i_1}, a_{j_1})(a_{i_1}, a_{j_2}) \cdots (a_{i_k}, a_{j_k})$$

which is a product of k^2 terms of the form (a_p, a_q). Thus $s(\Lambda^k \varphi)$ is a product of

$$N := k^2 \binom{\binom{n}{k}}{2}$$

terms of the form (a_p, a_q). Let the number of occurences of (a_1, a_1) be e.

Since $\varphi \simeq \langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} \rangle$, for any permutation σ on n letters, Corollary 6.8 and [9, Corollary 2.11.11] show that for each $i \in \{1, \ldots, n\}$, there will be e terms of the form (a_i, a_i) in $s(\Lambda^k \varphi)$. Thus there will be $N - ne$ terms of the form (a_i, a_j) for fixed distinct elements i and j of $\{1, \ldots, n\}$. Since there are $\binom{n}{2}$ ways of choosing distinct i and j from $\{1, \ldots, n\}$, we see that for given i and j with $i \neq j$, there are

$$f := \frac{N - ne}{\binom{n}{2}}$$

terms of the form (a_i, a_j) in $s(\Lambda^k \varphi)$.

It remains to determine e. To do this, we establish how many entries in the diagonalisation of $\Lambda^k \varphi$ contain a_1 in the product $a_{i_1} \cdots a_{i_k}$. Let $a_{i_1} \cdots a_{i_k}$ be an entry containing a_1. Then a_1 appears as the first term, that is, $i_1 = 1$ and $a_{i_1} \cdots a_{i_k} = a_1 a_{i_2} \cdots a_{i_k}$. Now a_1 can appear at most once in an entry in $\Lambda^k\varphi$, so the other $k-1$ of the a_i in the product are chosen from the remaining $n-1$ elements of $\{2,\dots,n\}$. This choice can be made in

$$M := \binom{n-1}{k-1}$$

ways. Given a term $a_1 a_{i_2}\cdots a_{i_k}$, Equation 8 and the lexicographic well-ordering show that it will contribute *one* term (a_1, a_1) to the Hasse invariant for each entry $a_1 a_{j_2}\cdots a_{j_k}$ coming *after* it in the diagonalisation. The first entry $a_1 a_2 \cdots a_k$ in $\Lambda^k\varphi$ will have $M-1$ entries after it containing a_1, the second will have $M-2$ entries after it containing a_1, and so on. Then the number of terms (a_1, a_1) in $s(\Lambda^k\varphi)$ is

$$(M-1)+(M-2)+\cdots+2+1 = \sum_{i=1}^{M-1} i = \frac{M(M-1)}{2} = \binom{M}{2}.$$

Thus

$$e = \binom{M}{2} = \binom{\binom{n-1}{k-1}}{2}$$

and $ne = n\binom{M}{2}$.

A straightforward computation with binomial coefficients shows

$$N \;=\; k^2\binom{\binom{n}{k}}{2} \;=\; n^2\binom{M}{2} + \frac{n(n-k)}{2}M.$$

It follows from another easy computation that

$$f \;=\; \frac{N-ne}{\binom{n}{2}} \;=\; 2\binom{M}{2} + \binom{n-2}{k-1}.$$

Since there are f copies of (a_i, a_j) for each $i, j \in \{1,\dots,n\}$ with $i \neq j$, Equation 7 gives

$$\begin{aligned} s(\Lambda^k\varphi) &= s(\varphi)^f \left(\prod_{i=1}^{n}(a_i, a_i)\right)^e \\ &= s(\varphi)^f (a_1\cdots a_n, -1)^e \;=\; s(\varphi)^f (d,-1)^e. \end{aligned}$$

Because symbols (a, b) have order 2 in the Brauer group, we are only interested in the parity of f, which is the same as that of

$$g := \binom{n-2}{k-1}$$

since $g \equiv f \pmod 2$. Then

$$s(\Lambda^k\varphi) = s(\varphi)^g (d,-1)^e$$

and the proof is complete. □

Corollary 12.3. *Let (V,φ) be a quadratic space of dimension n and determinant d. Then the Hasse invariant of $\Lambda^2\varphi$ is*

$$s(\Lambda^2\varphi) = s(\varphi)^n (d,-1)^{(n-1)(n-2)/2}$$

which gives the following table:

$n \pmod 4$	0	1	2	3
$s(\Lambda^2\varphi)$	$(d,-1)$	$s(\varphi)$	$1_{\mathrm{Br}(K)}$	$s(\varphi)(d,-1)$

Proof. This follows immediately from Theorem 12.2 on noting that when $k = 2$, $M = \binom{n-1}{1} = n-1$, so $e = \binom{M}{2} = (n-1)(n-2)/2$; and $g = \binom{n-2}{1} = n-2 \equiv n \pmod 2$. To construct the table, we note that e is even if 4 divides $n-1$ or $n-2$, i. e. if $n \equiv 1, 2 \pmod 4$; otherwise e is odd. $\square$

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The idea of using a λ-ring structure on the Witt-Grothendieck ring to obtain annihilating polynomials for quadratic forms (such as those in Corollary 10.10) was also described by Serre in talks at Luminy (1994) and Besancon (1997), but he has not published this work.

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