A Variant of a Theorem by Springer*

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The main aim of the preprint is to prove the following result which is a variant of the Springer theorem [La] for quadratic spaces over local rings.

Theorem. *Let R be a local Noetherian domain which has an infinite residue field and let $S = R[T]/(F(T))$ be an integral extension which is étale over R. Let (V, q) be a quadratic space over R such that the S-quadratic space $(V \otimes_R S, q \otimes_R S)$ contains a hyperbolic plane $\mathbb{H}_S$. If the degree of the polynomial $F(T)$ is odd, then the space (V, q) contains a hyperbolic plane already over R.*

This theorem is one of the main ingredient in the proof of the following result proved in [P].

0.1 Theorem. *Let R be a regular local ring, K its field of fractions and (V, φ) a quadratic space over R. Suppose R contains a field of characteristic zero. If $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R, that is there exists a unimordular vector $v \in V$ with $\varphi(v) = 0$.*

It is well-known that any finite étale extension S of R has the form $S = R[T]/(F(T))$, where $F(T)$ is a monic separable polynomial. If A is a semi-local ring and (W, ϕ) is a quadratic space over A, then W contains a hyperbolic plane if and only if W contains a unimodular isotropic vector w. A vector w is called unimodular if w can be taken as the first vector w_1 of a free A-base $w_1, \ldots, w_n$ of the A-module W.

1 Preliminaries

In this section we formulate two results which will be used in the proof of the main theorem. These two results will be proved in Section 3. We need to fix certain notation

k is an infinite field ($\mathrm{char}(k) \neq 2$)

$f(t)$ is a monic separable polynomial of degree n over k

$l = k[t]/(f(t))$ is a separable k-algebra

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$\theta = t \mod f(t)$ is an element of the algebra l

(W, ϕ) is a quadratic space over k

(W_l, ϕ_l) is the quadratic space $(W \otimes_k l, \phi \otimes_k l)$ over l

$W[t]^{(m)} = W \cdot 1 \oplus W \cdot t \oplus \ldots \oplus W \cdot t^{m-1} \subset W[t] = W \otimes_k k[t]$

$k[t]^{(m)} = k \cdot 1 \oplus k \cdot t \oplus \ldots \oplus k \cdot t^{m-1} \subset k[t]$

$ev : W[t]^{(n)} \to W_l$ is a map given by

$$ev(v_0 + v_1 t + \ldots + v_{n-1} t^{n-1}) = v_0 + v_1 \theta + \ldots + v_{n-1} \theta^{n-1}$$

$v(\theta) \stackrel{def}{=} ev(v(t))$ for any $v(t) \in W[t]^{(n)}$

$\phi^{(n)} : W[t]^{(n)} \to k[t]^{(2n-1)}$ is the map given by

$$\phi^n(v_0 + v_1 t + \ldots + v_{n-1} t^{n-1}) = \phi(v_0) + 2(v_0, v_1) \cdot t + \phi(v_1) \cdot t^2 + \ldots + \phi(v_{n-1}) \cdot t^{2n-2}$$

1.1 Proposition. *Suppose the quadratic space (W_l, ϕ_l) contains a hyperbolic plane $\mathbb{H}_l$ as a direct summand. Then there exists a unimodular isotropic vector $w \in W_l$ and for any such a vector w there exists an element $v(t) \in W[t]^{(n)}$ satisfying the following conditions*

(1) $\phi^{(n)}(v(t)) \in k[t]^{(2n-1)}$ is a polynomial of degree $2n-2$

(2) $\phi^{(n)}(v(t))$ is a separable polynomial

(3) $\langle w, v(\theta) \rangle \in l^*$, where $l^* \subset l$ is the group of units of l

(4) $\phi_l(v(\theta)) = 0$ in l

1.2. Let A be a semi-local ring and let $\bar{A} = A/Rad(A)$ and let (U, ψ) be a quadratic space over A and suppose (U, ψ) contains a hyperbolic plane $\mathbb{H}_A$ as a direct summand. We will use "bar" for the reduction modulo the radical $Rad(A)$. For example, $(\bar{U}, \bar{\psi})$ is the quadratic space $(U/Rad(A) \cdot U, \bar{\psi})$ over $\bar{A}$.

1.3 Lemma. *Let $u \in U$ be a unimodular isotropic vector. Then for any isotropic vector $\mathrm{v} \in \bar{U}$ with the property $\langle \mathrm{v}, \bar{u} \rangle \in A^*$ there exists a vector $v \in U$ such that*

(1) $\bar{v} = \mathrm{v}$ in $\bar{U}$ and

(2) $\psi(v) = 0$.

Proof. Let v_0 be any vector in U with $\bar{v}_0 = \mathrm{v}$. Then take

$$v = \frac{-\langle v_0, v_0 \rangle}{2\langle v_0, u \rangle} \cdot u + v_0.$$

Then v is the desired isotropic vector. In fact, $\psi(v) = -\langle v_0, v_0 \rangle + \langle v_0, v_0 \rangle = 0$ and $\bar{v} = \bar{v}_0$, because $\langle \bar{v}_0, \bar{v}_0 \rangle = \langle \mathrm{v}, \mathrm{v} \rangle = 0$. Since $\bar{v}_0 = \mathrm{v}$ one has $\bar{v} = \mathrm{v}$. Lemma is proved. $\square$

1.4 Lemma. *Let $k[t]$ be the polynomial ring over the field k and let (W, ϕ) be the quadratic space over k. Let $w(t) = w_0 + w_1 \cdot t + \ldots + w_{n-1} \cdot t^{n-1}$ be an element of $W[t]$ with $w_i \in W$. Suppose the polynomial $\phi^{(n)}(w(t)) \in k[t]$ is separable and let $g(t) \in k[t]$ be an irreducible polynomial dividing $\phi^{(n)}(w(t))$. Then $w(t)$ does not vanish modulo $g(t)$.*

Proof. If $w(t)$ vanishes modulo $g(t)$, then $w(t) = g(t) \cdot u(t)$ for an element $u(t) \in W[t]$. In this case one has

$$\phi^{(n)}(w(t)) = \phi^{(n)}(g(t) \cdot u(t)) = g(t)^2 \cdot \phi^{(n)}(u(t)) \in k[t].$$

This relation contradicts with the separability of $\phi^{(n)}(w(t))$. Thus $w(t)$ does not vanish modulo $g(t)$. Lemma is proved. □

1.5 Lemma. *Let A be a semi-local ring and let (U, ψ) be a quadratic space over A. Let $v \in U$ be a unimodular isotropic vector. Then one can split a hyperbolic plane of (U, ψ), i.e.,*

$$(U, \psi) \cong (U', \psi') \perp \mathbb{H}.$$

Proof. It is easy. □

2 The proof of the main theorem

2.1 Theorem. *Let R be a local Noetherian domain which has an infinite residue field of characteristic different of 2. Let $S = R[T]/(F(T))$ be an integral extension which is étale over R. Let (V, q) be a quadratic space over R such that the S-quadratic space $(V \otimes_R S, q \otimes_R S)$ contains a hyperbolic plane $\mathbb{H}_S$. If $\deg F(T)$ is odd then the space (V, q) contains a hyperbolic plane $\mathbb{H}_R$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and let k be the residue field $R/\mathfrak{m}$ of R. Let $l = S/\mathfrak{m}S = k[t]/(f(t))$, where $f(t) = F[T] \mod \mathfrak{m}$. Since S is étale over R so is the k-algebra l. In particular the k-algebra l is separable.

We will write (V_R, q_R) for (V, q) and write $(\bar{V}, \bar{q})$ for the reduction modulo $\mathfrak{m}$ of the R-quadratic space (V_R, q_R). Let (V_S, q_S) be the scalar extension of (V_R, q_R) up to S and let $(\bar{V}_l, \bar{q}_l)$ be the reduction modulo $\mathfrak{m}S$ of the S-quadratic space (V_S, q_S). Clearly, $(\bar{V}_l, \bar{q}_l) = (\bar{V}, \bar{q}) \otimes_k l$. Set now $(W, \phi) = (\bar{V}, \bar{q})$. Then $(W \otimes_k l, \phi \otimes_k l) = (\bar{V}_l, \bar{q}_l)$ and we will write (W_l, ϕ_l) for $(\bar{V}_l, \bar{q}_l)$.

By the hypotheses of the theorem the space (W_l, ϕ_l) contains a hyperbolic plane $\mathbb{H}_l$ as a direct summand. Thus we are under the hypotheses of Proposition 1.1. So using notation of Section 1 one can find a vector $w \in W_l$ (unimodular and isotropic) and an element $v(t) = v_0 \cdot 1 + v_1 \cdot t + \ldots + v_{n-1} \cdot t^{n-1} \in W[t]^{(n)}$ satisfying the following conditions

(1) $\phi^{(n)}(v(t)) \in k[t]$ has degree $2n - 2$

(2) $\phi^{(n)}(v(t))$ is a separable polynomial over k

(3) $\langle w, v(\theta) \rangle \in l^*$ is an invertible element of l

(4) $\phi_l(v(\theta)) = 0 \in l$, where $\theta = t \mod f(t)$ is the element of l

Recall now that the quadratic space (W_l, ϕ_l) is the reduction modulo $\mathfrak{m}S$ of the quadratic space (V_S, q_S). Thus by Lemma 1.3 one can lift the element $v = v(\theta) \in W_l$ up to a unimodular isotropic vector $\mathrm{v} \in V_S$.

Since $S = R[T]/(F(T))$ we have $V_S = V_R \otimes_R R[T]/(F(T))$. If we set $\theta = T \mod F(T)$, then one can find elements $\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1} \in V_R$ such that $\mathrm{v} = \mathrm{v}_0 \cdot 1 + \mathrm{v}_1 \cdot \theta + \ldots + \mathrm{v}_{n-1} \cdot \theta^{n-1}$. Consider now the element $\mathrm{v}(T) = \mathrm{v}_0 \cdot 1 + \mathrm{v}_1 \cdot T + \ldots + \mathrm{v}_{n-1} \cdot T^{n-1}$ in $V_R \otimes_R R[T]$ and consider the diagram

$$\begin{tikzcd}
W_l & W \times \ldots \times W \arrow[l, "ev"'] \arrow[r, "\phi^{(n)}"] & k \times \ldots \times k \arrow[r, "disc"] & k \\
\bar{V}_l \arrow[u, equal] & \bar{V} \times \ldots \times \bar{V} \arrow[l, "ev"'] \arrow[u, equal] \arrow[r, "\bar{q}^{(n)}"] & k \times \ldots \times k \arrow[r, "disc"] \arrow[u, equal] & k \arrow[u, equal] \\
V_S \arrow[u] & V_R \times \ldots \times V_R \arrow[l, "Ev"'] \arrow[u] \arrow[r, "q_R^{(n)}"] & R \times \ldots \times R \arrow[r, "Disc"] \arrow[u] & R \arrow[u]
\end{tikzcd}$$

where $Ev(u_0, \ldots, u_{n-1}) = u_0 \cdot 1 + u_1 \cdot \theta + \ldots + u_{n-1} \cdot \theta^{n-1} \in V_S$ and $q_R^{(n)}(u_0, \ldots, u_{n-1}) = q_R(u_0) + 2 \cdot (u_0, u_1) \cdot T + q_R(u_1) \cdot T^2 + \ldots + q_R(u_n) \cdot T^{2n-2}$ and $Disc(a_0, a_1, \ldots, a_{2n-2})$ is the discriminant of the polynomial $a_0 + a_1 T + \ldots + a_{2n-2} T^{2n-2}$. The vertical arrows are the canonical maps.

Clearly, this diagram commutes and maps Ev, ev are isomorphisms. In particular, $v_i = \mathrm{v}_i \mod \mathfrak{m}$ for the components $v_0, \ldots, v_{n-1}$ (resp. $\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}$) of the element $v(t) \in W[t]$ (resp. $\mathrm{v}(T) \in V_R[T]$). Thus the reduction modulo $\mathfrak{m}$ of the polynomial $q_R^{(n)}(\mathrm{v}(T))$ coincides with the polynomial $\phi^{(n)}(v(t))$. The last polynomial is separable and has degree $2n-2$ by the choice of $v(t)$.

Since $\mathrm{v}(\theta) = \mathrm{v}$ by the very choice of the element $\mathrm{v}(T)$ and since v is q_R-isotropic one has the relation $q_R(\mathrm{v}(T)) = 0$. Thus $q_R(\mathrm{v}(T))$ vanishes modulo $F(T)$ in the ring $R[T]$, i.e., there exists a polynomial $H(T)$ in $R[T]$ such that

$$q_R^{(n)}(\mathrm{v}(T)) = F(T) \cdot H(T). \qquad (*)$$

Taking the reduction modulo $\mathfrak{m}$ we get the relation

$$\phi^{(n)}(v(t)) = f(t) \cdot h(t), \qquad (**)$$

where $h(t)$ is the reduction of $H(t)$ modulo $\mathfrak{m}$. Since $\deg \phi^{(n)}(v(t)) = 2n-2$ and $\deg f(t) = n$ one gets that $\deg h(t) = n-2$. Since $\deg H(T) \leq n-2$ we see that $\deg H(T) = n-2$ and $H(T)$ is monic (the highest coefficient is invertible). Further $\phi(v(t))$ is separable. Thus $h(t)$ is separable as well. This shows that the R-algebra $S' = R[T]/(H(T))$ is an étale R-algebra.

Denote by A the class of T modulo $H(T)$ in the ring S'. Then the vector $\mathrm{v}(A) = \mathrm{v}_0 \cdot 1 + \mathrm{v}_1 \cdot A + \ldots + \mathrm{v}_{n-1} \cdot A^{n-1}$ of the quadratic S'-space $(V_{S'}, q_{S'})$ is isotropic. In fact, $q_{S'}(\mathrm{v}(A)) = F(A) \cdot H(A) = 0$ on S'. If the vector $\mathrm{v}(A)$ is unimodular then one can split

a hyperbolic plane $\mathbb{H}_{S'}$ of the quadratic space $(V_{S'}, q_{S'})$ over S' (see Lemma 1.5). So in this case we constructed a finite étale extension $S' = R[T]/(H(T))$ such that

$$\deg H(T) = \deg F(T) - 2$$

and such that the space $(V_{S'}, q_{S'})$ contains a hyperbolic plane $\mathbb{H}_{S'}$ as a direct summand. Repeating this procedure several times we finally get a direct hyperbolic summand of the quadratic space (V_R, q_R) itself. Thus to complete the proof of the theorem it remains to check that the vector $\mathrm{v}(A) \in V_{S'}$ is unimodular.

For this denote by k' the ring $S'/\mathfrak{m}S'$ and observe that $k' = k[t]/(h(t))$, where as above $h(t)$ is the reduction modulo $\mathfrak{m}$ of the polynomial $H(t)$. Further denote by $\bar{V}_{k'}$ the k'-module $\bar{V} \otimes_k k'$ and consider the commutative diagram (with the same elements $\mathrm{v}(T)$ and $v(t)$ as above in this proof)

$$\begin{array}{ccccccc} v(\alpha) & \in & \bar{V}_{k'} & \xleftarrow{\overline{Ev}} & \bar{V}[t]^{(n)} & \ni & v(t) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathrm{v}(A) & \in & V_{S'} & \xleftarrow{Ev} & V_R[T]^{(n)} & \ni & \mathrm{v}(T) \end{array} .$$

Here $Ev(u_0 + u_1T + \ldots + u_{n-1}T^{n-1}) = u_0 + u_1 \cdot A + \ldots + u_{n-1} \cdot A^{n-1}$ and $\overline{Ev}(w_0 + w_1 \cdot t + \ldots + w_{n-1} \cdot t^{n-1}) = w_0 + w_1 \cdot \alpha + \ldots + w_{n-1} \cdot \alpha^{n-1}$ and $\alpha = t \mod h(t) \in k'$. To check that $\mathrm{v}(A)$ is unimodular it suffices to verify that $v(\alpha)$ is unimodular. Observe that $\bar{V}_{k'} = \bar{V} \otimes_k k[t]/(h(t))$. Let $h(t) = h_1(t) \cdot \ldots \cdot h_r(t)$ be the decomposition of $h(t)$ in a product of irreducible polynomials. Since $h(t)$ is separable one has $h_i(t) \neq h_j(t)$ for $i \neq j$. Then $\bar{V}_{k'} = \prod_{i=1}^r \bar{V} \otimes_k (k[t]/(h_i(t)))$ and thus $v(\alpha)$ is unimodular in $\bar{V}_{k'}$ if and only if the elements $v(t) \in \bar{V}[t]$ does not vanish modulo any of $h_i(t)$ $(i = 1, \ldots, r)$.

The polynomial $h_i(t)$ divides $h(t)$ and the polynomial $h(t)$ divides $\phi^{(n)}(v(t))$ (see $(*)$). Thus $h_i(t)$ divides $\phi^{(n)}(v(t))$. Since $h_i(t)$ is irreducible and $\phi^{(n)}(v(t))$ is separable Lemma 1.4 proves that $v(t)$ does not vanish modulo $h_i(t)$. Thus indeed the element $v(\alpha)$ is a unimodular vector in $\bar{V}_{k'}$ and the element $\mathrm{v}(A)$ is a unimodular vector in $V_{S'}$. This completes the proof of the theorem.

□

3 The proof of Proposition 1.1

3.1. Since the quadratic space (W_l, ϕ_l) contains a hyperbolic plane $\mathbb{H}_l$ as a direct summand one can find a unimodular isotropic vector $w \in W_l$. Choose and fix such a vector $w \in W_l$.

Now set $X(l) = \{v \in W_l \mid \phi(v) = 0\}$ and consider a map $\rho_w : X(l) \to l$ taking v into the scalar product $\langle w, v \rangle \in l$. Consider then a diagram of sets and their polynomial maps

$$\begin{array}{ccccccccc} l & \xleftarrow{\rho_w} & X(l) & & \xleftarrow{\qquad\qquad} & & Y & & \\ & & \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle j} & & \\ & & W_l & \xleftarrow{ev} & W \times \ldots \times W = W[t]^{(n)} & \xrightarrow{\phi^{(n)}} & k[t]^{(2n-1)} & \xrightarrow{disc} & k \end{array} ,$$

where ev is the map defined in the beginning of Section 1 (this map is clearly an isomorphism) and where $Y = ev^{-1}(X(l))$ and maps i, j are the inclusions and $\phi^{(n)}$ is defined in the beginning of Section 1. The map $disc$ takes a polynomial $g(t) \in k[t]^{(2n-1)}$ into its discriminant. It is well-known that $disc(g(t))$ has a polynomial expression in terms of coefficients of the polynomial $g(t)$.

This diagram is the diagram of k-rational points and their maps induced by the following one

$$\begin{tikzcd}
R_{l/k}(\mathbb{A}^1_l) & R_{l/k}(X_l) \arrow[l, "\mathcal{E}_w"'] \arrow[d, hook, "i"] & \mathbb{Y} \arrow[l, "\cong"'] \arrow[d, hook, "j"] & & & \\
& R_{l/k}(\mathbb{W}_l) & \mathbb{W} \times \ldots \times \mathbb{W} \arrow[l, "ev"', "\cong"] \arrow[r, "\phi^{(n)}"] & \mathbb{A}^1_k \times \ldots \times \mathbb{A}^1_k \arrow[r, "disc"] & \mathbb{A}^1_k &
\end{tikzcd}$$

of algebraic varieties. Here $R_{l/k}$ is the Weil restriction functor. Here $\mathbb{W}$ (resp. $\mathbb{W}_l$) is the k-vector space (resp. the l-vector space) considered as the k-variety (resp. as the l-variety) and X_l is the affine quadric in $\mathbb{W}_l$ given by the equation $\phi_l = 0$ and $\mathcal{E}_w$ is the Weil restriction of the morphism $\rho_w : X_l \to \mathbb{A}^1_l$ taking v into $\langle w, v\rangle$ and the variety $\mathbb{Y}$ is the preimage of $R_{l/k}(X_l)$ under the evaluation isomorphism $ev : \mathbb{W} \times \ldots \times \mathbb{W} \to R_{l/k}(\mathbb{W}_l)$ and morphisms i and j are the closed embeddings. The product $\mathbb{W} \times \ldots \times \mathbb{W}$ consists of n factors and the product $\mathbb{A}^1_k \times \ldots \times \mathbb{A}^1_k$ consists of $2n-1$ factors.

Define open subsets of the variety $\mathbb{Y}$ as follows

$$U_1 = j^{-1}((\phi^{(n)})^{-1}(\mathbb{A}^1_k \times \ldots \times \mathbb{A}^1_k \times (\mathbb{A}^1_k - \{0\})))$$

$$U_2 = j^{-1}((\phi^{(n)})^{-1}(disc^{-1}(\mathbb{A}^1_k - \{0\})))$$

$$U_3 = ev^{-1}(\mathcal{E}_w^{-1}(R_{l/k}(\mathbb{A}^1_l - \{0\}_l)))$$

We need in the following Lemmas

3.2 Lemma. *$\mathbb{Y}$ is k-rational variety.*

3.3 Lemma. $U_1 \cap U_2 \neq \emptyset$

3.4 Lemma. $U_3 \neq \emptyset$

Having these three Lemmas one can prove Proposition 1.1 as follows. Since $\mathbb{Y}$ is k-rational variety any its non-empty open subset has a k-rational point (even infinitely many). Thus by Lemma 3.3 and Lemma 3.4 one can find a point

$$v \in (U_1 \cap U_2 \cap U_3)(k) \subset \mathbb{Y}(k) \subset (\mathbb{W} \times \ldots \times \mathbb{W})(k).$$

Set $v(t) = j(v)$. Then the element $v(t)$ satisfies the condition (1), (2) and (3) of Proposition 1.1 by the very definition of the open sets U_1, U_2 and U_3. Since $v(\theta) \stackrel{def}{=} ev(v(t))$ thus $v(\theta)$ is a k-rational point of the variety $R_{l/k}(X_l)$, i.e., $v(\theta) \in X(l)$. Thus $\phi_l(v(\theta)) = 0$ and the condition (4) is satisfied as well. To complete the proof of Proposition 1.1 it remains to prove Lemmas 3.2 – 3.4.

3.5 (Proof of Lemma 3.2)**.** Since $\mathbb{Y}$ is isomorphic to the variety $R_{l/k}(X_l)$ it suffices to check the k-rationality of the last one. The quadric X_l has an l-rational point and thus X_l is an l-rational variety. This implies that there exist non-empty open subvarieties V_1 in X_l and V_2 in $\mathbb{A}_l^{r-1}$ ($r = dim_k W$) and an isomorphism $\alpha : V_1 \xrightarrow{\cong} V_2$ of l-varieties. Consider now the diagram of l-varieties

$$X_l \supset V_1 \xrightarrow{\cong} V_2 \subset \mathbb{A}_l^{r-1}$$

and apply the Weil restriction functor to this diagram. One obtains a diagram

$$R_{l/k}(X_l) \supset R_{l/k}(V_1) \xrightarrow{\cong} R_{l/k}(V_2) \subset R_{l/k}(\mathbb{A}_l^{r-1})$$

of k-varieties. Since the left and the right hand side inclusions are open imbeddings and since the variety $R_{l/k}(\mathbb{A}_l^{r-1})$ is an affine space over k one concludes that $R_{l/k}(X_l)$ is k-rational. Lemma is proven.

3.6 (Proof of Lemma 3.4)**.** The morphism $ev : \mathbb{Y} \to R_{l/k}(X_l)$ is an isomorphism by the very definition of the variety $\mathbb{Y}$. Therefore it suffices to verify that the variety $\mathcal{E}_w^{-1}(R_{l/k}(\mathbb{A}_l^1 - \{0\}_l))$ is non-empty. We show now that the last variety has k-rational points. For this recall that the morphism $\mathcal{E}_w : R_{l/k}(X_l) \to R_{l/k}(\mathbb{A}_l^1)$ induces a map of k-rational points which coincides with the one

$$\rho_w : X(l) \to l \qquad (v \mapsto \langle v, w\rangle).$$

Take any $v \in X(l)$ with $\langle w, v\rangle \in l^*$ and observe that for this element v one has $\rho_w(v) \in l^*$ and moreover the group l^* coincides with the set of k-rational points of the variety $R_{l/k}(\mathbb{A}_l^1 - \{0\}_l)$. Thus the element v is a k-rational point of the variety $\mathcal{E}_w^{-1}(R_{l/k}(\mathbb{A}_l^1 - \{0\}_l))$. Lemma is proved.

3.7 (Proof of Lemma 3.3)**.** We show that already $(U_1 \cap U_2)(l) \neq \emptyset$. For this consider an l-basis $e_1, e_2, \ldots, e_r$ of the free l-module W_l such that $e_1^2 = 0$, $e_2^2 = 0$, $(e_1, e_2) = 1$ and $(e_1, e_i) = (e_2, e_i) = 0$ for $i \geq 3$. For any polynomial $h(t) \in l[t]^{(n)}$ and for any element $e \in W_l$ we set $h(t) \cdot e = (a_0 \cdot e) \cdot 1 + (a_1 \cdot e) \cdot t + \ldots + (a_{n-1} \cdot e) \cdot t^{n-1}) \in l \otimes_k W[t]^{(n)} = W_l[t]^{(n)}$. Since $l = k[t]/(f(t))$ and $\theta = t \mod f(t)$ (see page 3 for the definition of θ) one has a unique decomposition $f(t) = f^{(1)}(t)(t - \theta)$ in $l[t]$. Let $g(t) \in k[t]$ be a separable of degree $n - 2$ polynomial which is coprime with $f(t)$. Consider now the element

$$v(t) = f^{(1)}(t) \cdot e_1 + (t - \theta)g(t) \cdot e_2 \in l \otimes_k W[t]^{(n)} = W_l[t]^{(n)}.$$

3.8 Claim. $v(t) \in (U_1 \cap U_2)(l)$

To check this claim observe first that

$$\phi^{(n)}(v(t)) = 2 \cdot f^{(1)}(t) \cdot (t - \theta) \cdot g(t) \cdot \langle e_1, e_2\rangle = 2f(t) \cdot g(t) \qquad (*)$$

Thus $\phi(v(\theta)) = 2 \cdot f(\theta) \cdot g(\theta) = 0$ in l and therefore $v(t) \in \mathbb{Y}(l)$. Further $(*)$ shows that $\phi^{(n)}(v(t)) \in k[t] \subset l[t]$ and it has degree $2n - 2$. Therefore $v(t) \in U_1(l)$.

Finally, $\phi^{(n)}(v(t))$ is separable because $f(t)$, $g(t)$ are separable and coprime polynomials in $k[t]$. Thus $disc(\phi^{(n)}(v(t))) \in k^* \hookrightarrow l^*$ and, therefore, $v(t) \in U_2(l)$. So $(U_1 \cap U_2)(l) \ni v(t)$ and, hence, $(U_1 \cap U_2)(l) \neq \emptyset$. Lemma is proved.

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