(9) What is the relationship between CHOW motives and motivic Galois groups?

Here is <u>P. O'Sullivan's answer.</u>

We <u>assume</u> the standard cj. $\sim_{hom} = \sim_{num}$ and finite-dim. of Chow motives (in the sense of Kimura - O'Sullivan)

[But of course, the following applies to some $\otimes$-subcategories for which one knows the cj holds $\rightsquigarrow$ unconditional results]

We choose a field of coefficients F s.t. $\mathcal{M}_{num}(k)_F$ is tannakian <u>neutral</u>

$\simeq Rep_F\, G$ $\qquad G = G_{mot}$: absolute pure motivic Galois gp (a pro-reductive gp$/F$)

$$\mathcal{M}_{num}(k)_F \simeq Rep_F(G, -id)$$

(acting as -1 on odd rep's.

$$\begin{array}{c} CHM(k)_F \\ \parallel \\ \mathcal{M}_{rat}(k)_F \end{array} \twoheadrightarrow \mathcal{M}_{num}(k)_F$$

Under the above cj :

<u>Thm</u> (O'Sullivan). There exists an affine G-super-scheme $\operatorname{Spec} A$ and a closed point $0 \in \operatorname{Spec} A$ fixed by G, such that $A^G = F$ and

$$\begin{array}{ccc} CHM(k)_F & \xrightarrow{\sim} & \operatorname{Vec}(\operatorname{Spec} A;\ G, -\mathrm{id}) \\ \downarrow & & \downarrow \text{fiber at } 0 \\ M_{num}(k)_F & \xrightarrow{\sim} & \operatorname{Rep}_F(G, -\mathrm{id}) \end{array}$$

Here $\operatorname{Vec}(\operatorname{Spec} A;\ G, -\mathrm{id})$ is the $\otimes$-category of G-equivariant super-vector bundles over $\operatorname{Spec} A$ (for which the action of $-\mathrm{id} \in G$ defines parity).

So understanding chow motives amounts to understanding G and

$A \in \operatorname{Ind} M_{num}(k)_F$, objects which belong to the "numerical world" !

$p^t\ 0 \leftrightarrow a : A \to F$.

- Voevodsky's nilp cj $\Longleftrightarrow$ ker a nil-ideal
- weight grading $A = \bigoplus_{-\infty}^{+\infty} A_i$

Bloch-Beilinson-Murre's conj.

$\Longleftrightarrow$ $A_i = 0$ for $i < 0$ $\Longleftrightarrow$ $A_0 = 1$ and A generated by A_1.

MIXED MOTIVIC GALOIS THEORY

(10) Construction of "tannakian" categories of mixed motives.

(A) Via conjectural t-structure on $DM_{gm}(k)_{\mathbb{Q}}$

$$DM_{gm}(k)_{\mathbb{Q}}^{\leq 0} \cap DM_{gm}(k)_{\mathbb{Q}}^{\geq 0} = MM_{ab.}(k)_{\mathbb{Q}}$$

$$^{\tau}H^{i} : DM_{gm}(k)_{\mathbb{Q}} \longrightarrow MM(k)_{\mathbb{Q}}$$

tannakian after change of comm. constraint à la Koszul.

$$\begin{tikzcd}
 & P(k) \arrow[dl, "h"'] \arrow[dr, "M^{\vee}"] & \\
CHM(k)_{\mathbb{Q}} \arrow[rr, hook] \arrow[d, "\substack{\text{mod.}\\ \text{maxi}\\ \otimes\text{-ideal}}"'] & & DM(k)_{\mathbb{Q}} \arrow[d, "\oplus\, ^{\tau}H^{i}"] \\
M_{num}(k)_{\mathbb{Q}} \arrow[rr, hook] & & MM(k)_{\mathbb{Q}}
\end{tikzcd}$$

full subcategory of semi-simple objects.

absolute mixte motivic Galois group (attached t any realization).

Extension of the pure o by a prounipotent group.

Unconditional ex: mixed Tate motives over k = number field.

$$D^b(TM(k)_{\mathbb{Q}}) \hookrightarrow DM_{gm}(k)_{\mathbb{Q}}.$$

Tannakian gp is an extension of $\mathbb{G}_m$ by a pro-unipotent group.

B) Nori's category. $/k \subset \mathbb{C}$.

Q quiver ("category without composition of morphisms")

"representation" $T : Q \to Ab$

$\rightsquigarrow$ $Q \to (\mathrm{End}\,T)$-Mod (where $\mathrm{End}\,T$ is the ring of natural transf. of T) whenever Q is finite,

and $Q \to C(T)$-Mod in general,

where $C(T) = \lim$ of $(\mathrm{End}\,T_{|Q'})$-Mod over all finite subquivers.

Apply to Q: objects: (X, Y, i) $i \geq 0$

morphisms: $(X, Y, i) \to (X', Y', i)$ $Y \hookrightarrow X$ affine (obvious ones)

$+$ $(X, Y, i) \to (Y, Z, i-1)$ $Z \subset Y \subset X$ $i > 0$

$T : (X, Y, i) \mapsto H_i(X_{\mathbb{C}}, Y_{\mathbb{C}}, \mathbb{Q})$

invert "Tate object" in $C(T) \rightsquigarrow MM'(k)_{\mathbb{Q}}$ tannakian neutral.

(11) Back to Grothendieck's period cj.

$k \subset \bar{\mathbb{Q}}$.

$M \in MM(k)_{\mathbb{Q}} \longrightarrow$ (mixed) periods Ω_M (integrals of alg. forms)

(matrix of the comp. iso w.r.t. "rat." bases:

$$H_{DR}(M) \otimes_k \mathbb{C} \xrightarrow{\sim} H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}.$$

$\downarrow$

(mixed) motivic Galois gp $G_{mot}(M)$

period torsor $G\mathcal{P}(M)$.

$G_{mot}(M)_k$

One can extend Grothendieck's period cj. to this setting:

"all alg. relations between periods (with coef. in k) are of motivic origin"

$$\text{tr.deg}_{\mathbb{Q}}\, k(\Omega_M) \overset{?}{=} \dim G_{mot}(M).$$

Kontsevich viewpoint: if one uses Nori's category, this cj. takes the following form:

"all alg. relations between periods come from alg. changes of variables and Stokes formula for integrals".

More precisely:

Consider $\mathbb{Q}$-space gen. by

$[(X, D, \omega, \gamma)]$ $\quad$ X affine smooth $/\mathbb{Q}$

$D \subset X$ NCD

$\omega \in \Omega^{\dim X}(X)$

$\gamma \in H_{\dim X}(X_{\mathbb{C}}, D_{\mathbb{C}}, \mathbb{Q})$

modulo relations:

i) linearity in ω and γ,

ii) $\forall f : (X, D) \to (X', D')$, $\forall \omega \in \Omega^{\dim X'}(X')$

$\gamma \in H_{\dim X}(X_{\mathbb{C}}, D_{\mathbb{C}}, \mathbb{Q})$,

$$[(X, D, f^*\omega, \gamma)] = [(X, D, \omega, f_*\gamma)]$$

iii) $D^{[1]}$ (resp. $D^{[2]}$) normalization of D (resp. of 2^{nd} intersection stratum), $\forall \eta \in \Omega^{\dim X-1}$ (

$$[(X, D, d\eta, \gamma)] = [(D^{(1)}, D^{(2)}, \eta|_{D^{[1]}}, \partial\gamma)]$$

---

this is a $\mathbb{Q}$-alg. $\rightsquigarrow$ alg. $\hat{P}$ after inversion of $[(\mathbb{A}^1, 0, \frac{dz}{z}, \gamma)]$ $\quad$ γ "formal mixed periods"

$$\hat{P} \to \mathbb{C} : [(X, D, \omega, \gamma)] \mapsto \int_\gamma \omega$$

period cj: $\quad$ this is injective.

Comment. Toward a Galois theory for transcendental numbers?

alg. nbr α $\begin{array}{l}\nearrow \alpha' \\ \to \alpha'' \\ \searrow \vdots\end{array}$ conjugates, which are permuted by the Galois gp of the normal closure of $\mathbb{Q}[\alpha]$ over $\mathbb{Q}$.

Is there anything similar for transcendental cplex numbers α? If α is a period, motivic Galois theory suggests a positive answ:

normal closure of $\mathbb{Q}[\alpha]$? : $\mathbb{Q}[\beta(M)]$ for M minimal (s.t. $\mathbb{Q}[\beta]$ contains $\mathbb{Q}[\alpha]$).

$G = G_{mot}^{(H_0)}(M)(\mathbb{Q})$ $\circlearrowright$

conjugates of α: orbit of α under this action of G.

Ex: - $\mathbb{Q}[\alpha]$ nbr field M : corresp. Artin motive

$\mathbb{Q}[\beta]$ = normal closure of $\mathbb{Q}[\alpha]$ and G is the usual Galois gp.

- $\alpha = 2\pi i$, $M = 1(-1)$, $G = \mathbb{Q}^*$
  conj. of α : non-zero rational multiples.

- $\alpha = \omega_1$ elliptic period of the 1st kind without cplex multiplication
  $M = h'(X)$ $G \simeq GL_2(\mathbb{Q})$,
  conj. of $\alpha = \omega_1$ are el^{ts} of $\mathbb{Q}\omega_1 \oplus \mathbb{Q}\omega_2$.

(12) Polyzêtas, ~~mixed~~ Tate motives and their motivic Galois groups.

(A) Polyzêtas.

$\underline{s} = (s_1, \dots, s_k)$ $\quad s_i > 0$, $s_1 > 1$

$$\zeta(\underline{s}) := \sum_{n_1 > \dots > n_k > 0} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}$$

$$= \int_{1 > t_1 > \dots > t_s > 0} \omega_{s_1} \wedge \dots \wedge \omega_{s_k}$$

$s = |\underline{s}| = s_1 + \dots + s_k$

$\omega_0 = \frac{dt}{t}$, $\omega_1 = \frac{dt}{1-t}$, $\omega_\epsilon = \omega_0^{\epsilon-1}\wedge\omega_1$, $\epsilon > 1$

Chen iterated integral

period of the ind-Tate motive

$h(\pi_1^{uni}(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}, \vec{01})) \in \mathrm{Ind}\ TM(\mathbb{Q})$

(Goncharov, Deligne...)

Actually $\in \mathrm{Ind}\ TM(\mathbb{Z})_{\mathbb{Q}}$

mixed Tate motives unramified at every prime p.

$\mathfrak{Z}_s \subset \mathbb{R}$ = $\mathbb{Q}$-subspace gen. by $\zeta(\underline{s})$ $|\underline{s}| = s$

$\mathfrak{Z} = \Sigma\, \mathfrak{Z}_s$

(B) $TM(\mathbb{Z})_{\mathbb{Q}}$.

in $TM(\mathbb{Q})_{\mathbb{Q}}$, $\mathrm{Ext}^i(1, 1(r)) = 0$ $\quad i > 1$, $r > 0$

$$\mathrm{Ext}^1(1, 1(r)) = H^1(\mathrm{Spec}\,\mathbb{Q}, \mathbb{Q}(r))$$
$$= K_{2r-1}(\mathbb{Q}) \otimes \mathbb{Q} = \begin{cases} 0 & r \text{ even} > 0 \\ \mathbb{Q}^* \otimes \mathbb{Q} & r = 1 \\ \mathbb{Q} & r \text{ odd} > 1 \end{cases}$$

in $TM(\mathbb{Z})_{\mathbb{Q}}$, same but $K_{2r-1}(\mathbb{Z}) \otimes \mathbb{Q} \dots \to 0$ if $r = 1$

<u>Theorem</u> (Deligne - Goncharov) The motivic Galois gp attached to $TM(\mathbb{Z})_{\mathbb{Q}}$ is of the form $G_{TM(\mathbb{Z})} = \mathbb{G}_m \ltimes \underset{\substack{\uparrow \\ \text{pro-unipotent}}}{G'_{TM(\mathbb{Z})}}$

1) Lie $G^1_{TM(\mathbb{Z})}$, graded by the $\mathbb{G}_m$ action, is the free graded Lie algebra with one generator in each odd degree ≤ -3.

2) $TM(\mathbb{Z})_{\mathbb{Q}} \cong \{ \text{f.d.}^{\ell} \ (\text{Lie}\, G^1_{TM(\mathbb{Z})})\text{-graded modules} \}$

$TM'(\mathbb{Z})_{\mathbb{Q}} \subseteq TM(\mathbb{Z})_{\mathbb{Q}}$ sub-Tannakian category generated by f.d. pieces of $h(\pi_1^{\text{uni}}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \vec{01}))$.

so that $\mathfrak{Z}$ (which is a $\mathbb{Q}$-algebra as we shall see) is the $\mathbb{Q}$-alg. of real periods of objects of $TM'(\mathbb{Z})_{\mathbb{Q}}$.

<u>Cor</u>. (Goncharov, Terasoma)

$\dim_{\mathbb{Q}} \mathfrak{Z}_s \leq d_s$, where $d_s = d_{s-2} + d$
$d_0 = d_2 = 1, d_1 = 0$

Rmk: there is no non-motivic proof of this inequality!

Hint : as a graded comm. alg., $\big(U(\mathrm{Lie}\, G_{TM(\mathbb{Z})})\big)^\vee$

$$\simeq \mathbb{Q}[T_1] \otimes \underbrace{U(L(T_{-3}, T_{-5}, \dots))^\vee}_{= \text{ graded Hopf alg. of functions on } G'_{TM(\mathbb{Z})}}$$

via $G_{TM(\mathbb{Z})} \twoheadrightarrow G_{TM'(\mathbb{Z})}$,

$$U(\mathrm{Lie}\, G_{TM'(\mathbb{Z})})^\vee \subseteq \mathbb{Q}[T_1] \otimes U(L(T_{-3}, T_{-5} \dots))^\vee$$

period torsor

$\rightsquigarrow \quad \mathcal{U}(\mathrm{Lie}\, G_{TM'(\mathbb{Z})})^\vee \overset{\varphi}{\twoheadrightarrow} \mathfrak{Z}[2\pi i]$

$$= \mathfrak{Z} \oplus 2\pi i\, \mathfrak{Z}$$

$\mathfrak{Z} =$ image of $\varphi \cap \underbrace{\mathbb{Q}[T_1^2] \otimes U(L(T_{-3}, \dots))^\vee}_{\text{graded piece } s \text{ of dim } d_s}$.

<u>Rmk</u> : $\quad TM(\mathbb{Z})_{\mathbb{Q}} \overset{?}{=} TM'(\mathbb{Z})_{\mathbb{Q}}$

+ Grothendieck's period conj. for $TM'(\mathbb{Z})_{\mathbb{Q}}$

$$\Longleftrightarrow \quad \mathfrak{Z} = \bigoplus_s \mathfrak{Z}_s$$

and $\dim \mathfrak{Z}_s = d_s$.

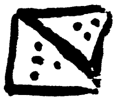
© Explicit relations between polyzetas.

two sets of known relations:

- regularized double shuffle relation (RDS)
- Drinfel'd's associator relations (Ass).

RDS : $\zeta(\underline{s}) \cdot \zeta(\underline{s}')$

$$= \underbrace{\sum_{n_1 > \cdots} \sum_{n_1' > \cdots}}_{\text{decompose the index set}} \frac{1}{\cdots} \, \frac{1}{\cdots}$$



= lin. comb. of $\zeta(\sigma)$'s

$$\| \qquad \underbrace{\int_{1 > t_1 > \cdots} \int_{1 > t_1' > \cdots}}_{\substack{\text{decompose the} \\ \text{integr. domain} \\ \text{into simplexes}}} \omega \cdots\cdots$$

= another lin comb. of $\zeta(\sigma)$'s

($\rightarrow$ $\mathcal{Z}$ is a $\mathbb{Q}$-algebra).

Can be extended to $s_1 = 1$ (regularization)

Thm (Goncharov) RDS relations are of motivic origin.

Thm (Racinet) they define a torsor under some affine gp scheme G_{RDS} which contains $G_{MTM'}(\mathbb{Z})$.

Ass : $$\frac{dG(z)}{dz} = \left(\frac{X_0}{z} + \frac{X_1}{1-z}\right) G(z)$$

sol. $G_0(z) \sim z^{X_0}$, $G_1(z) \sim (1-z)^{-X_1}$

$G_1(z)^{-1} G_0(z) = \phi(X_0, X_1)$ indep. of z

$\phi_{KZ} = \phi\left(\frac{X_0}{2\pi i}, \frac{-X_1}{2\pi i}\right)$ Drinfel'd's associator.

- exp. of a Lie series in X_0, X_1
- $\phi_{KZ}(X_1, X_0) = \phi_{KZ}(X_0, X_1)^{-1}$
- $e^{X_0/2} \phi_{KZ}(X_{-1}, X_0)\, e^{X_{-1}/2} \phi_{KZ}(X_1, X_{-1}).$

$$e^{X_1/2} \phi_{KZ}(X_0, X_1) = 1$$

with $X_{-1} = -X_0 - X_1$,

- $\phi_{KZ}(X_{01}, X_{12} + X_{13}) \cdot \phi_{KZ}(X_{02} + X_{12}, X_{23})$

$$= \phi_{KZ}(X_{12}, X_{23})\, \phi_{KZ}(X_{01} + X_{02}, X_{13} + X_{23})\, \phi_{KZ}(X_{01}, X$$

X_{ij}, $0 \le i < j \le 3$ non-comm. var.

$X_{ij} X_{kl} = X_{kl} X_{ij}$ $([X_{ij} + X_{ik}, X_{jk}] = $
i, j, k, l distinct.

Point : coeff. of $\phi = 1 + \zeta(2) X_0 X_1 + \dots$
are polyzetas

Drinfel'd's rel $\Rightarrow$ assoc. relations between polyzetas.

Fact : (Ass) relations are of motivic origin, and define a torsor under the so-called Grothendieck-Teichmüller gp GT

$$G_{TM(\mathbb{Z})} \twoheadrightarrow G_{TM'(\mathbb{Z})} \begin{matrix} \hookrightarrow G_{RDS} \\ \hookrightarrow GT \end{matrix}$$

Conj : these gps coincide. (Ass) and (RDS) are, independently, defining equations for polyzetas.

Ⓓ Hodge and Tate conjectures for $TM(\mathbb{Z})_{\mathbb{Q}}$

$$TM(\mathbb{Z})_{\mathbb{Q}} \xrightarrow{H_B} MHS_{\mathbb{Q}}$$
$$TM(\mathbb{Z})_{\mathbb{Q}_\ell} \xrightarrow{H_\ell} \mathrm{Rep}_\ell\, \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

<u>fully faithful</u>.

$$\mathrm{Ext}^1(1, 1(r)) \begin{matrix} \hookrightarrow \mathrm{Ext}^1_{MHS}(1, 1(r)) \simeq \mathbb{C}/(2\pi i) \\ \hookrightarrow \mathrm{Ext}^1_{\substack{\mathrm{Rep}_\ell \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \\ \text{unr. outside } \ell}}(1, 1(r)) \simeq K_{2r-1}(\mathbb{Z}) \otimes \mathbb{Q}_\ell \end{matrix}$$

(Soulé).

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the END