AYOUB 1

Conservativity of the Vanishing-Cycles functor and The Schur-finitness of DM(k)

Introduction:

The notion of Schur-finitness goes back to Deligne....

Definition: let M be a monoidal symetric additive $\mathbb{Q}$-linear category. An object $X \in M$ is called Schur fini[te] if there exist $n \geq 1$, $p \in \mathbb{Q}[\Sigma_n]$ a non trivial idempotent such that p act by zero on $X^{\otimes n}$.

When M is also a triangulated category, then the schur-finitness is stable by taking cones [C. Mazza]

Motivations: (Guletskii)

1- Schur finitness of motives of surfaces $\Rightarrow$ Bloch Conjecture.

2- Schur finitness of $CHM_{\mathbb{Q}}(k)$ implies that a correspondence

$$\alpha : X \longmapsto X$$

such that $\alpha_{num} = 0$ is nilpotent in $CH^{\dim(X)}(X \times X)$

(This is a special case of Voevodsky's nilpotence conjecture)

What about the Vanishing cycles?

We begin by the Étale case

The étale Case:

Let S be the spectrum of a s. henselian DVR.

Let $f: X \longrightarrow S$ a proper map with X regular:

$$\begin{tikzcd} X_{\bar\eta} \arrow[r] \arrow[d, "f_{\bar\eta}"] & X_\eta \arrow[r] \arrow[d, "f_\eta"] & X \arrow[d, "f"] & X_s \arrow[l] \arrow[d, "f_s"] \\ \bar\eta \arrow[r] & \eta \arrow[r] & S & s \arrow[l] \end{tikzcd}$$

One have a canonical map:

$$H^i_{et}(X_s, \Lambda) \longrightarrow H^i_{et}(X_{\bar\eta}, \Lambda),$$

This map is not an isomorphism (usually)

The vanishing 'cycles' are here to correct the answer.

More precisely, there exist a complex Ψ on $(X_s)_{et}$, with a canonical isomorphism:

$$H^*_{et}(X_s, \Psi) \longrightarrow H^*_{et}(X_{\bar\eta}, \Lambda)$$

In fact: there exist a "system" of functors (One can say a specialisation structure):

$$\Psi_f : D(X_\eta, \Lambda) \longrightarrow D(X_s, \Lambda)$$

attached to $f : X \to S$ (and a choice of a (uniformization of)) and verifying the usual compatibilities of (SGA 7).

In particular:

1) If f is projective,
$$f_* \Psi \simeq \Psi f_*$$

2) If f smooth, $\Psi f^*_\eta \xrightarrow{\simeq} f^*_s \Psi$

Examples:

$f = Id_S$. So we have:

$$\begin{tikzcd} \Psi_f : D(\eta, \Lambda) \arrow[r] \arrow[d, equal] & D(s, \Lambda) \arrow[d, equal] \\ D(\mathcal{O}_\eta\text{-modules}) \arrow[r, "for"'] & D(A\text{-mod}) \end{tikzcd}$$

In particular, we have this trivial theorem:

Theorem 1: $\Psi_{Id_S} : D(\eta, \Lambda) \longrightarrow D(s, \Lambda)$ is a monoidal and a conservative functor.

II – What is The Link?

The link between the two subjects is provided by the following observation:

Observation :

Suppose we have the following :

1) A specialisation structure on DM ($\mathbb{Q}$-linear) : Ψ, wich is good in the following sense :

a) $\Psi_{Id_S}(\mathbb{1}) = \mathbb{1}$

b) Ψ_{Id_S} is monoidal.

⋮

2) A Conservability Theorem for Ψ_{Id_S}

i.e. The motivic analogue of theorem 1. is true

Then we can proove the following result:

"Every object in $DM(k)$ wich is of finite type is schur finite."

In particular:

"Every object in $CHM(k)_{\mathbb{Q}}$ is Schur finite"

Proof Of This implication:

It suffices to proove that for every $f : X \to k$ smooth and projective, $[X]$ is S-finite.

By duality it suffices to do it for $f_* \mathbb{Z}_X$

We proceed by induction on $\dim X = n$.

for $n=0$, there is little to proove!...

So suppose $n \geqslant 1$.

<u>1st Case</u>: If X is birational to a <u>smooth</u> hypersurface of $\mathbb{P}^{n+1}$.

Of course we can suppose That $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree d.

If $d=1$, it's okey!

So suppose $\deg(X) = d \geq 2$

And argue by induction on d.

Choose $Y', H' \subseteq \mathbb{P}^{n+1}$ general hypersurfaces of degree $d-1$ and 1

$\rightsquigarrow$ $X \sim_{Rat} Y' + H'$

Then Blow up $X \cap Y'$ and then $X \cap H'$ $\rightsquigarrow$ E.

let Y and $H \subset E$ be the proper transforms.

So we have:

$$X \sim_{Rat} Y + H \quad \text{in } E$$

$$X \cap Y = X \cap H = \phi.$$

So there exists:

$$f : E \longrightarrow \mathbb{P}^1$$

s.t. $f^{-1}(0) = X$, $f^{-1}(1) = Y \cup H$

let η = generic pt of $\mathbb{P}^1$

$\Psi_0 : D\Pi(\eta) \longrightarrow D\Pi(k)$

$\Psi_1 : D\Pi(\eta) \xrightarrow{\;\;} D\Pi(k)$

$$\star \quad \begin{array}{ccc} \Psi_0[f_{\eta *}\mathbb{1}] & = & f_{\eta *}\mathbb{1} \\ \wr\wr & & \wr\wr \\ [E_\eta] & & [X] \end{array}$$

$\star$ $\Psi_1[f_{\eta *}\mathbb{1}]$ is built up from:

$[Y]$, $[H]$ and $[Y \cap H]$.

By induction $\Psi_1[f_{\eta *}\mathbb{1}]$ is Schur-finite

Now using the conservativity of Ψ_1 (and the fact that the functor is monoidal) we get:

$f_* \mathbb{1}$ is Schur-finite.

And finally: $T(X) \simeq$ [illegible] is also Schur-finite.

This settle the case where X is birational to a smooth hypersurface of $\mathbb{P}^{n+1}$.

<u>The General Case</u> (12)

It is the same trick.

So take X/k a smooth projective variety.

X is birational to some (possibly singular) hypersurface of degree d in $\mathbb{P}^{n+1}$. Call

$$Z' \subset \mathbb{P}^{n+1}$$

The hypersurface.

Blow up until singularities are gone:

$$Z \subset E, \quad \text{with } Z \text{ smooth.}$$

and: $Z + m_1 D_1 + \cdots + m_r D_r \sim_{Rat} Y$

with Y the pure transforms of a general hypersurface of deg d in $\mathbb{P}^{n+1}$.

By Blowing further

We can suppose that

$$Z \cap Y = D_1 \cap Y = \cdots = D_r \cap Y = \phi.$$

Then we get a map

$f : E \longrightarrow \mathbb{P}^1$

With $f^{-1}(0) = Z + n_1 D_1 + \cdots + n_r D_r$

$f^{-1}(1) = Y.$

By the first step we know

that $[Y]$ is Schur-finite

$(= \Psi_1(f_{y*} \mathbb{1}))$

The Conservativity theorem tells us

That $f_{y*} \mathbb{1}$ is already schur-finite

So finally, [illegible]

$\Psi_0(f_{\eta *}\mathbb{1})$ is Schur finite.

$$\| \\ f_{\eta *}(\Psi_f \mathbb{1})$$

But in this case we can describe $\Psi_f \mathbb{1}$ explicitly.

And we can see (under the induction hypothesis)

That [illegible]

$$\left(\begin{array}{c} \text{S-finite of} \\ f_{0*}(\Psi_f \mathbb{1}) \end{array} \right) \iff \begin{array}{c} \text{S-finitness of} \\ [Z], [D_1'], \ldots, [D_r'] \end{array}$$

with $D_i' \to D_i$ are certain finite covers

In particular we get the Schur finitness of $[Z]$.

Some ideas for the definition of the vanishing-cycles functors:

There is at least two possible definitions:

1. (Markus Spitzweck) his idea is to give a mathematical meaning to the formula

$$\Psi(A) = i^* j_* A \otimes \Psi$$

for ... $i^* j_* \Psi$

2. (AG). A direct definition:

Sketch of the definition:

take the diagram:

$$\mathbb{G}_m \xrightarrow{j} \mathbb{A}^1 \xleftarrow{i_0} 0$$

in $DM(\mathbb{G}_m)$ we have a special map:

$$\Theta : \mathbb{1} \longrightarrow \mathbb{1}(1)[1]$$

Given by

$$\begin{array}{ccc} \mathbb{1}_{\mathbb{G}_m} & \longrightarrow & (\mathbb{G}_m, 1)_{\mathbb{G}_m} \\ \| & & \| \\ [\mathbb{G}_m \xrightarrow{Id} \mathbb{G}_m] & \xrightarrow{\Delta} & \dfrac{[\mathbb{G}_m \times \mathbb{G}_m \xrightarrow{P_1} \mathbb{G}_m]}{\{1\} \times \mathbb{G}_m \to \mathbb{G}_m} \end{array}$$

As in the étale case $\Theta^2 = 0$.

So we set:

$$\Psi(A) = \operatorname{Tot}(A \to A(1)[1] \xrightarrow{\Theta} A(2)[2] \xrightarrow{\Theta} \cdots)$$

$i^*(\mathbb{G}_m) =$

Sketch of the "proof" of the Conservativity theorem:

⚠ The proof is not yet fully verified!

1st Step: the reduction to a statement on homotopy invariant sheaves with transfers. Basically it is the following statement:

"Let M_* be a homotopy module [cf Deglise], such that for $n \geq 0$, $j^* M_n = 0$"

Then M_* is zero.

2nd Step:

A description of the function "i^*j_*".

Basically, I prove that to compute "$i^*j_*(M)(s')$" with s'/s a finite extension, it suffices to compute "$i^*j_*(M_{\leq 1})$"

[$M_{\leq 1}$ is the maximal 1-motive in M.]

Morally: $M_{\leq 1}$ is not the quotient of a semi-abelian variety but rather of an Ind-semi-ab. variety.

3rd Step:

Use the above description of "$i^* j_* \Gamma_n$" to get information on the $(\Gamma_n)_{\leq 1}$.

here we have to use some diophantine Geometry...

4th Step:

We put all the information on the (Γ_n) $n \geq 0$, to prove that (Γ_*) is actually a quotient of Milnor K-theory spectrum.

Then it is easy to prove the vanishing of (Γ_*).