J.-L. Colliot Thélène

Rationally chain connected varieties (RCC)

$k \subset \Omega$ alg closed fields X/k proj. integral.

Equivalent:

(a) exists T/k integral variety

$\mathcal{C} \to T$ $/k$ proper, fibres connected, all components rational curves

and $\varphi : \mathcal{C} \to X$ $/k$

$$\mathcal{C} \times_T \mathcal{C} \longrightarrow X \times X \quad \text{dominant}$$

(b) for any Ω, general pair of points in $X(\Omega)$ connected by chain of rational curves

(c) for any Ω, any two points in $X(\Omega)$ are connected by chain of rational curves

(d) there exists $\Omega \supset k$, Ω uncountable, a general pair of points in $X(\Omega)$ is connected by chain of rational curves

Example: Conic bundle over $\mathbb{P}^4_{\mathbb{C}}$.

Rationally connected varieties (RC)

$k \subset \Omega$ alg closed fields X/k proj integral

Equivalent:

(a) exists T/k integral variety
$f: \mathbb{P}^1 \times_k T \longrightarrow X$ such that

$$\mathbb{P}^1 \times \mathbb{P}^1 \times T \longrightarrow X \times X \quad \text{dominant}$$
$$(t, t', z) \longmapsto (f(t,z), f(t',z))$$

(b) exists Y/k integral, $x_0 \in X(k)$

$$F: \mathbb{P}^1 \times Y \longrightarrow X \quad \text{dominant}$$
$$0 \times Y \longmapsto x_0$$

(c) Exists Ω uncountable such that a general pair of Ω-points is connected by <u>one</u> $\mathbb{P}^1$

Ex: Unirational variety

Very free morphisms

E vector bundle on $\mathbb{P}^1_k$ $\quad k$ any field

$\Rightarrow \quad E \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$

Equivalent:
- all $a_i \geq 1$
- $H^1(\mathbb{P}^1, E(-2)) = 0$
- $H^1(\mathbb{P}^1, E(-s)) = 0$ all $s \leq 2$

such E called ample.

X/k smooth projective

$f : \mathbb{P}^1_k \longrightarrow X$ over k

f called very free

if $f^* T_X$ is ample on $\mathbb{P}^1_k$.

Differential calculus on Mor-schemes

k any field $\qquad X/k$ smooth projective

$x_0 \in X(k)$

$$\mathbb{P}^1_k \times \mathrm{Mor}_k(\mathbb{P}^1, X, 0 \mapsto x_0) \xrightarrow{ev} X$$
$$(t, \varphi) \longmapsto \varphi(t)$$

<u>Prop.</u> If $f : \mathbb{P}^1_k \to X \quad 0 \mapsto x_0$

very free, then

$[f]$ smooth point of $\mathrm{Mor}_k(\mathbb{P}^1, X, 0 \mapsto x_0)$

and ev is smooth in nbhd

of $\mathbb{P}^1 \setminus 0 \times [f]$

<u>Prop.</u> If $F : \mathbb{P}^1 \times \overset{\text{integral}}{Y} \longrightarrow X$ generically smooth

$0 \times Y \longrightarrow x_0$

then $\exists_{\emptyset \neq} U \subset Y$

$\forall u \in U \qquad F_u : \mathbb{P}^1_{k(u)} \to X_{k(u)}$ very free.

Separably rationally connected varieties (SRC)

X/k smooth projective (connected)
k alg closed.

Equivalent:

(a) Exists one $f : \mathbb{P}^1_k \longrightarrow X$ very free

(b) Exists T/k variety, $\mathbb{P}^1 \times T \xrightarrow{f} X$

such that

$$\mathbb{P}^1 \times \mathbb{P}^1 \times T \longrightarrow X \times X$$
$$(t, t', z) \longmapsto f(t,z), f(t',z)$$

generically ~~sm~~ smooth

(c) Exists Y/k variety (integral)
$x_0 \in X(k)$

$F : \mathbb{P}^1 \times Y \longrightarrow X$ generically smooth

$0 \times Y \longrightarrow x_0$

"A very general rational curve is free"

X/k smooth proj. char $k = 0$

$x_0 \in X(k)$

There exists $Z = Z(x_0)$

$$Z = \bigcup_{i=1}^{\infty} Z_i \qquad Z_i \subset X \text{ closed, proper}$$

if $f: \mathbb{P}^1 \longrightarrow X$

$0 \longmapsto x_0$

$$f(\mathbb{P}^1) \not\subset \bigcup Z_i$$

then f is very free.

Thm. k alg closed

SRC $\Longrightarrow$ RC $\Longrightarrow$ RCC (easy)

if char $k=0$ converse holds (hard)

(proof uses deformations of non free curves)

Thm. k alg closed

$X \subset \mathbb{P}^n$ smooth Fano (ω^{-1} ample)

then X is RCC (hard)

Campana, Kollár-Miyaoka-Mori

if char $k=0$ $\Longrightarrow$ X is SRC

Deformation theory

$(S, 0)$ pointed scheme $\qquad$ $\Bbbk =$ residue field at 0

Data:

$$\begin{tikzcd}
B_0 \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\square"] & Y_0 \arrow[rr, "f_0"] \arrow[d, phantom, "\cap"] & & \mathcal{X}_0 \arrow[d, phantom, "\cap"] \\
B \arrow[r, phantom, "\subset"] \arrow[rrr, bend left, "g"] & Y \arrow[dr, "\text{proper}"'] & & \mathcal{X} \arrow[dl, "\text{smooth}"] \\
& & S &
\end{tikzcd}$$

Y_0 no embedded components

$f_0 = g_0$

Assume

$$H^1(Y_0, f_0^* T_{\mathcal{X}_0} \otimes \mathcal{I}_{B_0}) = 0$$

then the S-scheme

$\underline{\mathrm{Mor}}_S(Y, \mathcal{X}, g)$ is smooth at the point $[f_0]$.

Hence:

there exists

$$\begin{array}{ccc} S_1 & \longrightarrow & S \\ \Psi & & \Psi \\ o_1 & \longrightarrow & o \end{array} \qquad \begin{array}{l} \text{étale} \\ \\ \mathbb{k} \xrightarrow{\sim} \mathbb{k}_1 \end{array}$$

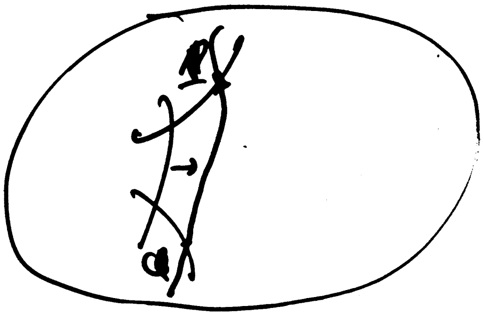
s.t. diagram may be cycled $/S_1$

$$\begin{tikzcd} \mathcal{Y} \times_S S_1 \arrow[rr] \arrow[dr] & & \mathcal{X} \times_S S_1 \arrow[dl] \\ & S_1 & \end{tikzcd}$$

respecting g and f_0.

Applications:

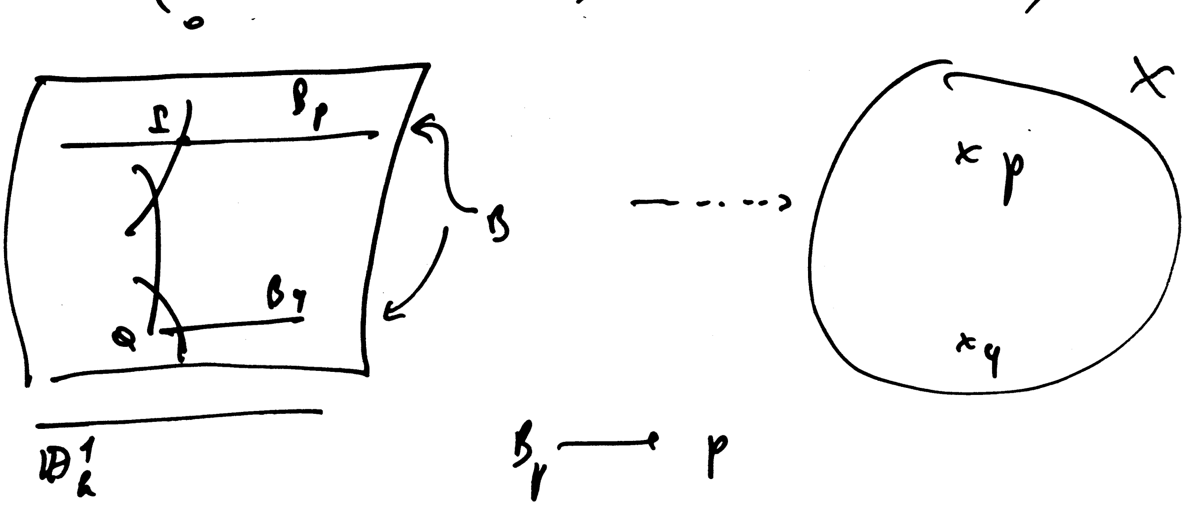
deformations, smoothing of trees of $\mathbb{P}^1$'s keeping some points fixed



X/k

$S = \mathbb{A}^1_k$ $\qquad \mathcal{Y} = \mathbb{P}^1_k \times \mathbb{A}^1_k$ $\qquad \mathcal{X} = X \times_k \mathbb{A}^1_k$

blown up a certain number of times above $0 \in \mathbb{A}^1(k)$



$B_p \longrightarrow p$

$B_q \longrightarrow q$

If $H^1(C_0, f^* T_X(-p-q)) = 0$ then after replacing $\mathbb{A}^1_k$ by $S_1 \xrightarrow{\text{étale}} \mathbb{A}^1_k$ ...

$0_1 \longrightarrow 0$

A basic proposition

$/k$ field $\qquad C/k \qquad C = \bigcup_{i=0}^{n} C_i$

each $C_i \cong \mathbb{P}^1_k$

$C^r = \bigcup_{i=0}^{r} C_i \qquad C_{r+1}$ cuts C^r transversally in one point p_{r+1}

$C^r \subset C^{r+1}$

exact sequence:

$$0 \to \mathcal{O}_{C_{r+1}}(-p_{r+1}) \longrightarrow \mathcal{O}_{C^{r+1}} \longrightarrow \mathcal{O}_{C^r} \to 0$$

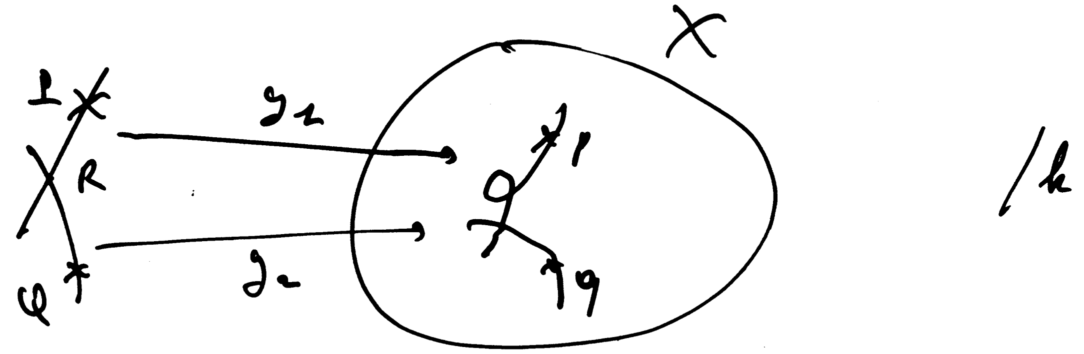
Proposition. E/C vector bundle.

If $H^1(C_0, E_{|C_0}) = 0$

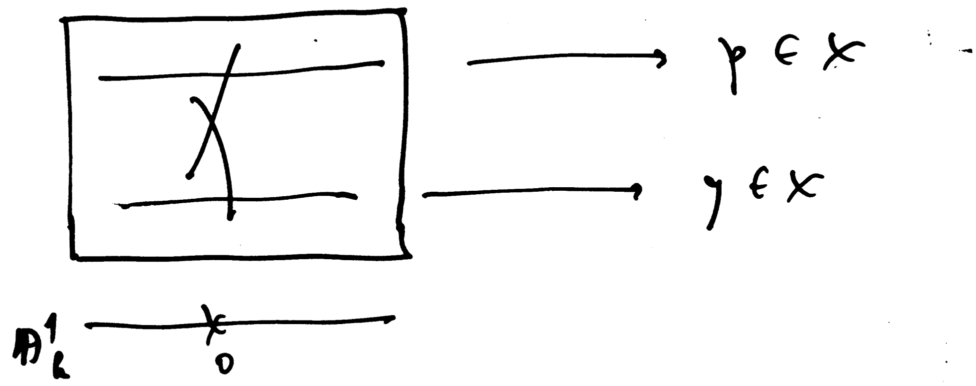
and $H^1(C_i, E_{|C_i}(-1)) = 0$ all $i \geq 1$

then $H^1(C, E) = 0$

[Smoothing the union of two very free $\mathbb{P}^1$'s which meet.



g_1, g_2 very free



$$f_0 : \; C_1 \cup C_2 \longrightarrow X \qquad C_0 = C_1 \cup C_2$$

$$0 \to \mathcal{O}_{C_2}(-R) \to \mathcal{O}_C \to \mathcal{O}_{C_1} \to 0$$

$$\otimes \mathcal{O}_C(-P-Q)$$

$$0 \to \mathcal{O}_{C_2}(-R-Q) \to \mathcal{O}_C(-P-Q) \to \mathcal{O}_{C_1}(-P) \to 0$$

$$\otimes f^* T_X$$

$$0 \to \underset{H^1=0}{f_2^* T_X(-2)} \to f^* T_X(-P-Q) \to \underset{H^1=0}{f_1^* T_X(-1)} \to 0$$

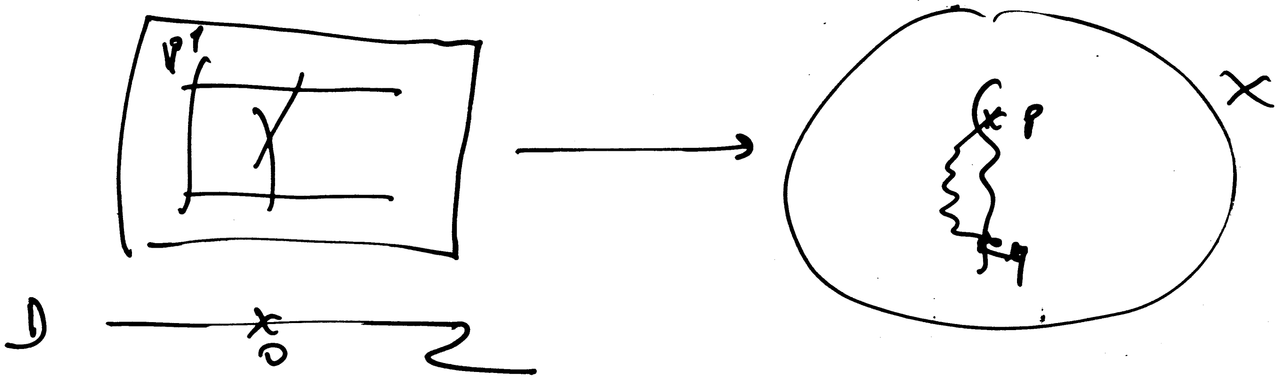
$$\Rightarrow H^1(C_0, f^* T_X(-P-Q)) = 0$$

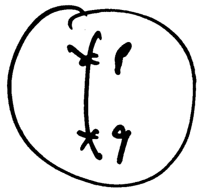
$\rightsquigarrow$ can deform.

get curve $D/_k$ with a k-point 0

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \\ D & \end{tikzcd}$$

$Z_{|D\setminus 0} \simeq \mathbb{P}^1 \times D\setminus 0$



If $(D\setminus 0)(k) \neq \emptyset$ find very free. $\mathbb{P}^1_k \longrightarrow$ 

If k is a large field then $(D\setminus 0)(k) \neq \emptyset$

(e.g. local field, p-adic, real)

$\rightsquigarrow$ classical case k alg. closed ..

k alg closed:

getting a $\mathbb{P}^1_k$ through several points

Prop. k alg closed. X/k smooth proj.

Assume for all $x \in X(k)$ there is a very free $\mathbb{P}^1$ through x.

Then $\forall\ x_1, \dots, x_n \in X(k)$ there is a very free $\mathbb{P}^1$ through $x_1, \dots, x_n$ simultaneously

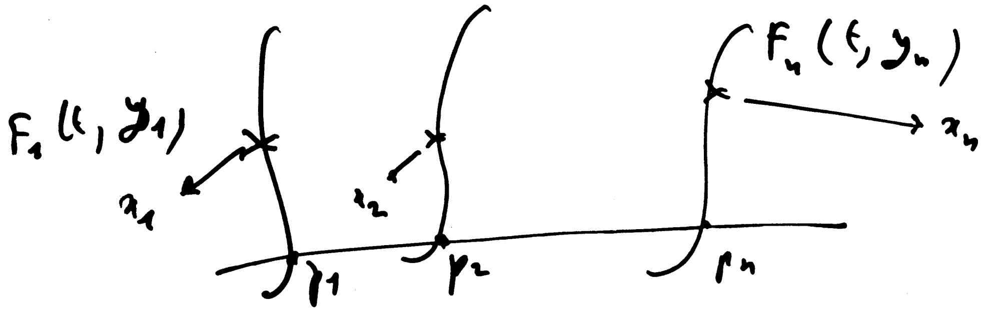
Proof: $F_i : \mathbb{P}^1 \times Y_i \longrightarrow X$ generically smooth

$0 \times Y_i \longrightarrow x_i$

all $F_i(t, y_i)$ very free

$\emptyset \neq \underset{\text{open}}{U} \subseteq \bigcap$ Im F_i

$g_0 : \mathbb{P}^1 \to X$ very free $g_0(\mathbb{P}^1) \cap U \neq \emptyset$



$\to$ may deform keeping the x_i.

$\to$ very free $\mathbb{P}^1 \longrightarrow X$ through the x_i.

Thm (Kollár 1999)

k field X/k smooth proj. abs. irred.
geometrically SRC $x_0 \in X(k)$.

There exists a diagram of k-morphisms

$$\begin{tikzcd} Z \arrow[r, "F"] \arrow[d, shift right] & X \\ T \arrow[u, shift right, "\sigma"'] & \end{tikzcd}$$

T/k smooth connected curve, $0 \in T(k)$

$Z_{|T\setminus 0} \cong \mathbb{P}^1 \times T \setminus 0$

$\forall u \in T$ $F_u : \underset{\mathbb{P}^1_{k(u)}}{\overset{\wr}{Z_u}} \longrightarrow X_{k(u)}$ very free

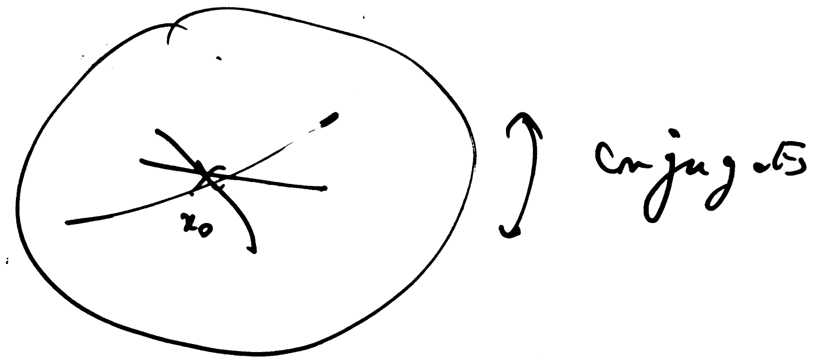
$F \circ \sigma = T \longrightarrow x_0 \in X$

Corollary: If k large field,
then $\exists$

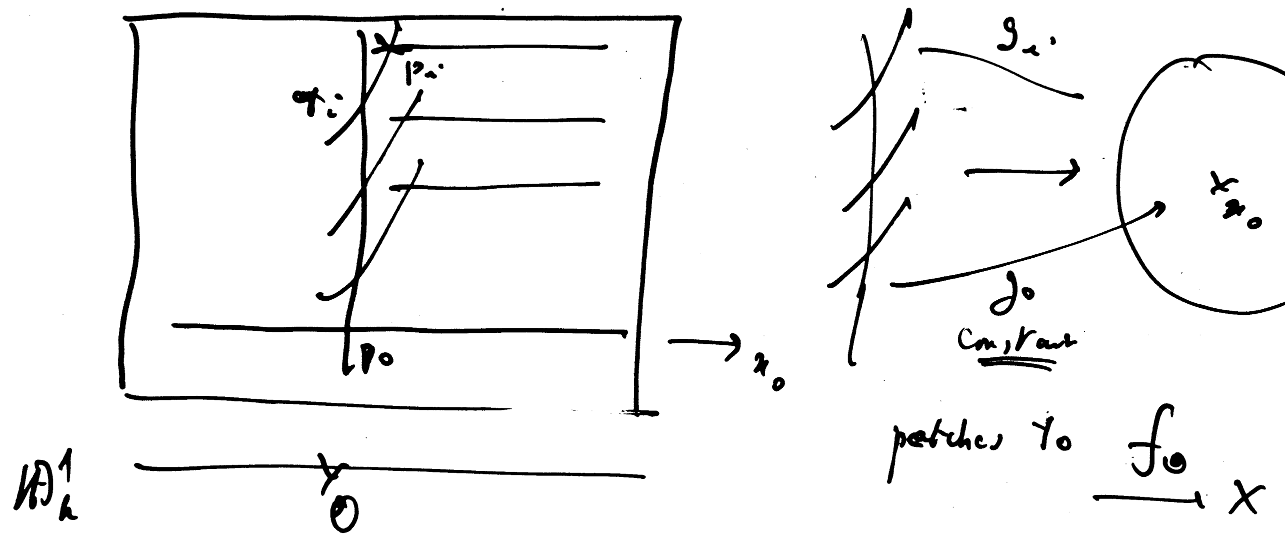
$$\begin{array}{ccc} \mathbb{P}^1_k & \xrightarrow{f} & X \\ 0 & \longrightarrow & x_0 \end{array}$$

very free

Proof



deformation problems



$$H^1(C_0, f_0^* T_X(-p_0 - p_1 - \dots - p_n)) = 0.$$

$\rightsquigarrow$ deforms, and to very free morphisms

Corollary (Kollár 1999)

k local (p-adic or real)

X/k smooth proj. $\underset{\text{geom.}}{\widehat{\ }}$ SRC

R-equivalence on $X(k)$ is open, hence finite

($X(k)$ is compact)

Proof: $P \in X(k)$ $\quad f : \mathbb{P}^1_k \to X$ very free

$0 \mapsto P$

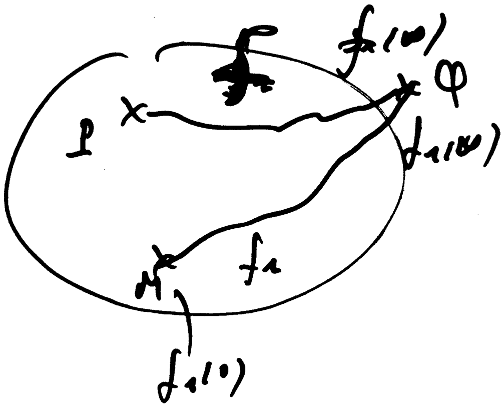
$\infty \mapsto Q$

$$\mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X, \infty \mapsto Q) \xrightarrow{ev} X$$

$$\left((\mathbb{P}^1 \setminus \infty) \times [f] \right) \subset \overset{\text{open}}{U} \xrightarrow[\text{smooth}]{\varphi} X$$

$\varphi(U(k))$ open in $X(k)$ (implicit function theorem)

R-equivalence $\Leftarrow$ open



f_1, f_2 very free

(in fact $\exists\, g : \mathbb{P}^1 \xrightarrow{\text{very free}} X$

$0 \mapsto P$

$1 \mapsto M$.)