INTRODUCTION:

MODULI PROBLEMS

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"DEFINITION"

A <u>moduli problem</u> is a class of geometric objects which one tries to view as another geometric object.

OUR BASIC EXAMPLE will be the

"moduli problem of smooth n-pointed curves of genus g":

For every scheme S, we put

$\mathcal{M}_{g,n}(S)$ = category with

- objects: $(X \xrightarrow{f} S, x_1, \ldots, x_n)$ where
  $f: X \to S$ is smooth, proper, with geometric fibres 1-dimensional, connected, of genus g, and
  $x_1, \ldots, x_n : S \to X$ are <u>disjoint sections</u>
- morphisms: S-isomorphisms respecting the sections.

$\underline{M}_{g,n}(S)$ = the set of isomorphism classes of objects of $\mathcal{M}_{g,n}(S)$.

The latter is just a set, while by construction $\mathcal{M}_{g,n}(S)$ is a groupoid (= category with all maps invertible).

For every morphism $S' \to S$, we have

base change functors

$$\mathcal{M}_{g,n}(S) \longrightarrow \mathcal{M}_{g,n}(S')$$

whence natural maps

$$\underline{M}_{g,n}(S) \longrightarrow \underline{M}_{g,n}(S')$$

making each $\underline{M}_{g,n}$ into a functor

$$\underline{M}_{g,n} : (\text{Schemes})^{\circ} \longrightarrow (\text{Sets})$$

In general, $\underline{M}_{g,n}$ is NOT (representable by) a scheme.

(In fact, it is a scheme iff $n > 2g+2$).

For instance, one can find two curves

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[dl] \\ & S & \end{tikzcd}$$

which are not isomorphic, but locally isomorphic over S (for the étale topology, or even the Zariski topology).

Thus, $\underline{M}_{g,0}$ (any g) is not a sheaf for these topologies, hence not representable.

(And, of course, if we take the associated sheaf we lose even more information)

MAIN IDEA:

$\mathcal{M}_{g,n}$ is a BETTER OBJECT to look at than $\underline{M}_{g,n}$:

- it is the natural object one tries to study (no loss of information)
- it has good local-to-global / sheaf-theoretic / descent properties
- it has good approximations by schemes

Of course, in a sense it is a more complicated object ($\mathcal{M}_{g,n}(S)$ is a groupoid, not a set).

FIBERED GROUPOIDS

let C be a category
(typically: the category of schemes, possibly over a fixed "base scheme")

A fibered groupoid $\mathcal{M}$ over C (C-groupoid) consists of the following data:

- for each $U \in \mathrm{ob}\, C$, a groupoid $\mathcal{M}(U)$
- for each map $V \xrightarrow{f} U$ in C, a functor

$$f^*: \mathcal{M}(U) \longrightarrow \mathcal{M}(V)$$

- for each composite map $W \xrightarrow{g} V \xrightarrow{f} U$, an isomorphism

$$g^* f^* \xrightarrow{\sim} (fg)^*$$

of functors $\mathcal{M}(U) \longrightarrow \mathcal{M}(W)$
+ compatibility with the associativity of composition in C.

<u>Examples</u>:

- $C = (\text{Schemes})$:
  - $U \longmapsto \mathcal{M}_{g,n}(U)$
  - $U \longmapsto$ cat. of all U-schemes + U-isomorphisms
  - $U \longmapsto QCOH(U) :=$ cat. of quasi-coherent $\mathcal{O}_U$-Modules (+isomorphisms)
  - $U \longmapsto BUN_n(U) :=$ cat. of locally free $\mathcal{O}_U$-modules of rank n (+ isomorphisms)
    $(n \in \mathbb{N})$

- For any C, any presheaf on C, i.e. any functor

$$F : C^\circ \longrightarrow (\text{Sets})$$

defines a C-groupoid (denoted by F):

$F(U) :=$ the discrete category $F(U)$:

set of objects $= F(U)$

maps $=$ identities.

MORPHISMS OF GROUPOIDS

If $\mathcal{M}$, $\mathcal{N}$ are C-groupoids, a morphism
$$\Phi : \mathcal{M} \to \mathcal{N}$$
consists of the following data:

- For each $U \in \mathrm{ob}\, C$, a functor
$$\Phi(U) : \mathcal{M}(U) \to \mathcal{N}(U)$$
- For each $V \xrightarrow{f} U$ in C, consider the diagram

of functors:
$$\begin{array}{ccc} \mathcal{M}(U) & \xrightarrow{\Phi(U)} & \mathcal{N}(U) \\ \downarrow f^* & & \downarrow f^* \\ \mathcal{M}(V) & \xrightarrow{\Phi(V)} & \mathcal{N}(V) \end{array}$$

We require (as part of the data) an isomorphism
$$f^* \circ \Phi(U) \xrightarrow{\sim} \Phi(V) \circ f^*$$
of functors $\mathcal{M}(U) \to \mathcal{N}(V)$.
(+ compatibility with the associativity data).

Examples:

- $\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,n-1} \quad (n>0)$

  "forget the n-th marked point"

- $\mathcal{M}_{1,1} \xrightarrow{j} \mathbb{A}^1$ (viewed as a presheaf)

$$\underset{\text{elliptic curve}}{(X \rightleftarrows S)} \longmapsto j(X/S) \in \Gamma(S, \mathcal{O}_S) = \mathbb{A}^1(S)$$

- Let us define two morphisms

$$\Phi, \Psi : \mathcal{M}_{1,1} \longrightarrow BUN_1$$

by
$$\Phi(X \underset{\varepsilon}{\overset{p}{\rightleftarrows}} S) := p_* \Omega^1_{X/S}$$
$$\Psi(X \underset{\varepsilon}{\overset{p}{\rightleftarrows}} S) := \varepsilon^* \Omega^1_{X/S}$$

These are different morphisms, but $\Phi(X)$ and $\Psi(X)$ are known to be canonically isomorphic.

So there should be a notion of (iso)morphism between morphisms!

Easy exercise: we obtain in this way an equivalence of categories

$$\{1\text{-morphisms } U \to \mathcal{N}\} \xrightarrow{\simeq} \mathcal{N}(U).$$

For instance: if S is a scheme, a 1-morphism

$$S \longrightarrow \mathcal{M}_{g,n}$$

is "the same thing" as an object of $\mathcal{M}_{g,n}(S)$.

PRODUCTS

Given a diagram of C-groupoids

$$\begin{tikzcd} & \mathcal{N} \arrow[d, "\Psi"] \\ \mathcal{M} \arrow[r, "\Phi"] & \mathcal{P} \end{tikzcd}$$

there is a "fibre product" groupoid, assigning to each $U \in \mathrm{ob}\, C$ the fibre product category

$$\mathcal{M}(U) \times_{\mathcal{P}(U)} \mathcal{N}(U)$$

whose objects are triples

$$(X, Y, \alpha)$$

with $\begin{cases} X \in \mathrm{ob}\, \mathcal{M}(U) \\ Y \in \mathrm{ob}\, \mathcal{N}(U) \\ \alpha : \Phi(X) \xrightarrow{\simeq} \Psi(Y) \end{cases}$ (isomorphism in $\mathcal{P}(U)$)

Example:

Assume: $\mathcal{M}$ is a C-groupoid,

U_1, U_2 objects of C

$X_i \in \mathrm{ob}\, \mathcal{M}(U_i) \quad (i=1,2)$

Viewing X_i as a 1-morphism $U_i \to \mathcal{M}$, we get a diagram:

$$\begin{tikzcd} & U_2 \arrow[d, "X_2"] \\ U_1 \arrow[r, "X_1"] & \mathcal{M} \end{tikzcd}$$

What is $U_1 \times_{X_1, \mathcal{M}, X_2} U_2$?

Answer: it is the presheaf on C given by

$$T \mapsto \left\{ (u_1, u_2, \alpha) : \begin{array}{l} u_1 = \text{a morphism } T \to U_1, \\ u_2 = \text{a morphism } T \to U_2, \\ \alpha : u_1^* X_1 \xrightarrow{\sim} u_2^* X_2 \text{ in } \mathcal{M}(T) \end{array} \right\}$$

or, in standard notations,

$$\underline{\mathrm{Isom}}_{U_1 \times U_2} \left(pr_1^* X_1, pr_2^* X_2 \right)$$

Example:

Consider the 1-morphism "forget the marked point"

$$\Phi : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,0}$$

If S is a scheme and X is an S-curve of genus g, then the fibre product $S \times_{X, \mathcal{M}_{g,0}, \Phi} \mathcal{M}_{g,1}$ is the S-curve X! In other words, we have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{M}_{g,1} \arrow[d, "\Phi"] \\ S \arrow[r, "X"] & \mathcal{M}_{g,0} \end{tikzcd}$$

which shows that $\mathcal{M}_{g,1}$ can be seen as the universal curve over $\mathcal{M}_{g,0}$.

REPRESENTABLE MORPHISMS

(For safety, assume C has fibre products)

A 1-morphism $\Phi : \mathcal{M} \longrightarrow \mathcal{N}$

is representable if for each $U \in \mathrm{ob}\, C$ and $X : U \longrightarrow \mathcal{N}$ (i.e. object of $\mathcal{N}(U)$) the fibre product $U \times_{X,\mathcal{N},\Phi} \mathcal{M}$ is a presheaf, representable by an object of C.

For instance, the "forgetful" morphism

$$\mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,0}$$

is representable.

Another example: for $g \geq 2$, consider

$$3K : \mathcal{M}_{g,0} \longrightarrow BUN_r \qquad (r = 5g-5)$$

$$(X \xrightarrow{f} U) \longmapsto f_* \omega_{X/U}^{\otimes 3}$$

I claim that $3K$ is representable:

Pick a scheme U and a 1-morphism $U \longrightarrow BUN_r$ (that is, a locally free sheaf $\mathcal{E}$ on U, of rank r):

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{M}_{g,0} \arrow[d, "3K"] \\ U \arrow[r, "\mathcal{E}"] & BUN_r \end{tikzcd}$$

Then, for a U-scheme T, we have

$\mathcal{N}(T)$ = cat. of curves $X \xrightarrow{f} T$ of genus g, <u>plus</u> isomorphism $f_* \omega_{X/T}^{\otimes 3} \xrightarrow{\sim} \mathcal{E}_T$.

Such a curve is naturally (3-canonically) embedded in $\mathbb{P}(\mathcal{E}_T)$. Putting $P = \mathbb{P}(\mathcal{E})$, we obtain an equivalence:

$\mathcal{N}(T) \xrightarrow{\sim}$ cat. of embedded smooth curves of genus g: $X \hookrightarrow P \times_U T$, plus isomorphism $\mathcal{O}(1)|_X \simeq \omega_{X/T}^{\otimes 3}$

The representability then follows from Hilbert scheme theory.

PROPERTIES OF REPRESENTABLE MORPHISMS

If $\mathcal{P}$ is a property (i.e. a class) of morphisms of C, which is stable by base change, it makes sense to say that a representable 1-morphism $\Phi : \mathcal{M} \to \mathcal{N}$ has property $\mathcal{P}$.

For instance, in the above examples,

$\Phi : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,0}$ is proper and smooth.

$3K : \mathcal{M}_{g,0} \longrightarrow BUN_{5g-5}$ is surjective and smooth.

USING THE DIAGONAL

(We assume C has fibre products)

<u>Proposition</u> For a C-groupoid $\mathcal{M}$, the following conditions are equivalent:

(i) The diagonal 1-morphism

$$\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$$
$$X \longmapsto (X, X)$$

is representable.

(ii) For all $U \in \mathrm{ob}\, C$ and $X, Y \in \mathrm{ob}\, \mathcal{M}(U)$, the presheaf $\underline{\mathrm{Isom}}_{\mathcal{M}}(X, Y)$ is representable (by an object of C/U).

(iii) For each $U \in \mathrm{ob}\, C$, every 1-morphism $U \longrightarrow \mathcal{M}$ is representable.

WARNING: $\Delta_{\mathcal{M}}$ is NOT in general a 'monomorphism', i.e. a fully faithful functor.

In fact:

$\Delta_{\mathcal{M}}(U) : \mathcal{M}(U) \longrightarrow (\mathcal{M} \times \mathcal{M})(U)$ is fully faithful for each U

$$\Updownarrow$$

$\mathcal{M}$ is (associated to) a presheaf on C

<u>Note</u>: The properties in the above Proposition are satisfied for $\mathcal{M} = \mathcal{M}_{g,n}$.

In fact,

$$\Delta_{\mathcal{M}_{g,n}} : \mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,n} \times \mathcal{M}_{g,n}$$

is representable, separated, of finite type.

If $2g-2+n>0$, it is finite unramified (objects of $\mathcal{M}_{g,n}$ have no infinitesimal automorphisms)

If $n > 2g+2$ then it is a monomorphism, in fact a closed immersion (objects of $\mathcal{M}_{g,n}$ have no nontrivial automorphisms, and $\mathcal{M}_{g,n}$ is a presheaf in this case)

But for instance, for $g=n=0$, consider $X : \operatorname{Spec}\mathbb{Z} \to \mathcal{M}_{0,0}$ defined by $\mathbb{P}^1$: then

Then $\operatorname{Spec}\mathbb{Z} \times_{X,\mathcal{M}_{0,0},X} \operatorname{Spec}\mathbb{Z} \simeq \underline{PGL}_{2,\mathbb{Z}}$