# $F$ a P.S.T. $\rightsquigarrow$ $C_*F$ a complex of PST's

$\Delta^\bullet$: $pt \rightrightarrows \mathbb{A}^1 \;\substack{\to\\\to\\\to}\; \mathbb{A}^2 \;\substack{\to\\\to\\\to\\\to}\; \mathbb{A}^3$ cosimplicial scheme

$R \leftleftarrows R[t_1] \leftleftarrows R[t_1, t_2] \leftleftarrows$ simplicial ring

$t_i = 0$

$\partial(t_{n+1}) = 1 - t_0 - \cdots - t_n$

$F(X \times \Delta^\bullet)$: $F(X) \leftleftarrows F(X \times \mathbb{A}^1) \leftleftarrows F(X \times \mathbb{A}^2) \leftleftarrows$

$(C_*F)\, X: 0 \leftarrow F(X) \leftarrow F(X \times \mathbb{A}^1) \leftarrow \cdots \leftarrow F(X \times \mathbb{A}^n) \leftarrow \cdots$

easy to see: $H_n(C_*F)$ are h.i. PST's

If $F$ is h.i. then $F(X) \xrightarrow{\sim} C_*F(X)$

**Example**: $F = \mathcal{O}$ is bad because $\boxed{C_*\mathcal{O} \simeq 0}$

**Proof**: If $f(t_1, \dots, t_n) \in R[t_1, \dots t_n]$ has $\partial_i f = 0$ all $i$ then $f$ is divisible by $t_1 \cdots t_n (1 - \Sigma t_n)$. Take $\underbrace{g = \frac{t_{n+1}}{1 - \Sigma t_i} f}_{R[t_1, \dots, t_{n+1}]}$

**Application** $\mathbb{G}_m \oplus pt = \mathbb{A}^1 - \{0\}$ in $\widehat{\underline{Cor}}$

$\mathbb{G}_m^{\wedge 2} \oplus \mathbb{G}_m \times pt \oplus pt \times \mathbb{G}_m \oplus pt = (\mathbb{A}^1 - \{0\})^2$

define $\mathbb{G}_m^{\wedge q}$ as summand of $(\mathbb{A}^1 - \{0\})^q$

$\cong \bigoplus \binom{q}{i} \mathbb{G}_m^{\wedge i}$

**definition** $\mathbb{Z}(q) = C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]$ cochain cx.

$\cdots \longrightarrow \mathbb{Z}_{tr}\mathbb{G}_m^{\wedge q}(\mathbb{A}^q_X) \cdots \longrightarrow \mathbb{Z}_{tr}\mathbb{G}_m^{\wedge q}(\mathbb{A}^1_X) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})X \to 0$

(degrees: $0$, $q-1$, $q$)

defn $\boxed{H^p(X, \mathbb{Z}(q)) = \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q))} = \mathbb{H}^{p-q}(X, C_*\mathbb{G}_m^{\wedge q})$

(or nis)

= Motivic Cohomology

# Nisnevich Topology

elementary square

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & Y \arrow[d, "\text{étale map}"] \\ U \arrow[r, hook, "\text{open}"'] & X \end{tikzcd} \qquad \begin{matrix} Y-V \\ \downarrow \cong \\ X-U \end{matrix}$$

Nisnevich lifting property:

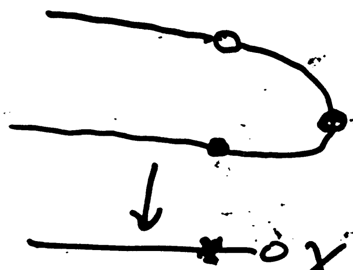
$(\forall x \in X)(\exists y \in Y \cup U)$ $y \mapsto x$

$k(y) \xleftarrow{\cong} k(x)$

$X_{et} \longrightarrow X_{nis} \longrightarrow X_{zar}$

$H^*_{et} \leftarrow H^*_{nis} \leftarrow H^*_{zar}$

- "local" = hensel local ring
- $cd_{nis}(X) = \dim(X)$



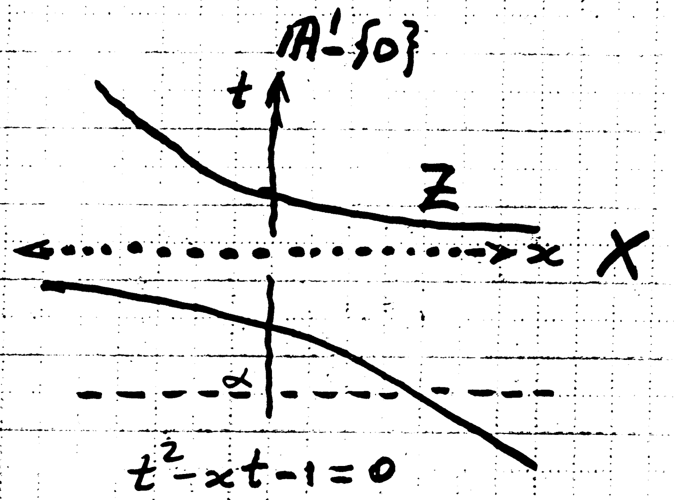
$x = ay^2$

$X = \mathbb{A}^1 - \{0\}$

$U = \mathbb{A}^1 - \{0, (x = a)\}$

Nisnevich cover $\Longleftrightarrow$ $\sqrt{a} \in k$

$\underline{\mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\})} \to \underline{\mathbb{Z} \times \mathcal{O}^*}$




$\mathrm{Cor}(X, \mathbb{A}^1 - \{0\}) \to \mathbb{Z} \times \mathcal{O}^*(X)$

$[Z] \mapsto (n, a_0)$

<u>elementary</u> <u>correspondence</u>

- irred. proper over $X$
- divisor in $X \times \mathbb{P}^1$ missing $X \times \{0, \infty\}$
- prime ideal generated by $t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ (unit) in $R[t, t^{-1}]$

finite over $X$

$\mathrm{Spec}(R) \subset X$

$t^2 - xt - 1 = 0$

$(t - \alpha)^2 = 0$

$n \in \mathbb{Z}$

$a_0 \in \mathcal{O}_X^*$

$\mathcal{M}$ = Kernel is nonzero, homotopic to 0

$$[t^2 - f(x)t - \alpha] - 2[t^2 - \alpha]$$

$-s f(x) t$

$X \times \mathrm{Spec}\, k[s]$

$R[s]$ locally

<u>Theorem</u> $\boxed{\mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\})(- \times \Delta^\cdot) \xrightarrow{\simeq} \mathbb{Z} \times \mathcal{O}^*}$ in PST

The $\mathbb{Z} = \mathbb{Z}_{tr}(pt)$ is split by $pt \xrightarrow{t=1} \mathbb{A}^1 - \{0\} \to pt$

$\mathbb{G}_m$ = complement

<u>defn</u> $\mathbb{Z}(1)$ is the cochain complex $\mathbb{G}_m(- \times \Delta^\cdot)[-1]$

$$\underset{-2}{\cdots} \to \underset{-1}{\mathbb{G}_m(X[s_1, s_2])} \to \underset{0}{\mathbb{G}_m(X[s])} \to \underset{+1}{\mathbb{G}_m(X)} \to 0 \quad \simeq \mathcal{O}_X^*[-1]$$

<u>Cor</u> $H^n_{Zar}(X, \mathbb{Z}(1)) \cong H^{n+1}(X, \mathcal{O}_X^*) = \begin{cases} \mathcal{O}^*(X) & n = 1 \\ \mathrm{Pic}(X) & n = 2 \\ 0 & \text{else} \end{cases}$

**Theorem** $H^n(k, \mathbb{Z}(n)) \xrightarrow[\Theta]{\cong} K_n^M(k)$

$n=0 \quad H^0(k,\mathbb{Z}) = \mathbb{Z}$

$n=1 \quad H^1(k,\mathbb{Z}(1)) = H^0(k,\mathcal{O}_X^*) = k^*$

$$\mathbb{Z}_{tr}(\mathbb{A}^1-\{0\})^n(k[t]) \xrightarrow{\partial_0 - \partial_1} \frac{\mathbb{Z}_{tr}(\mathbb{A}^1-\{0\})^n(k)}{\oplus \underbrace{\mathbb{Z}_{tr}(\mathbb{A}^1-\{0\})^{n-1}(k)}_{i^{th}\text{ coord}=1}} \longrightarrow H^n(k,\mathbb{Z}(n)) \longrightarrow 0$$

$$\begin{array}{c} \| \\ \text{free on } (k-\{0\})^n = n\text{-tuples of units} \end{array} \longrightarrow K_n^M(k)$$

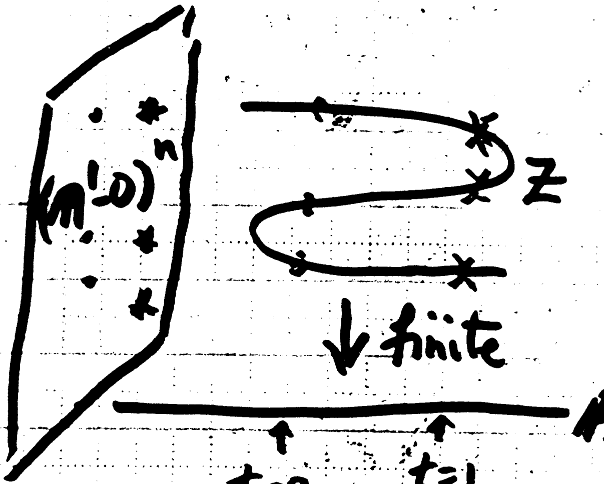
$$\{x_1,\dots,1,\dots,x_n\} = 0$$

$x \in (\mathbb{A}^1-\{0\})^n(k)$ maximal ideal of $k[x_1,\frac{1}{x_1},\dots,x_n,\frac{1}{x_n}]$

If $k(x) \neq k$, get $\{x_1,\dots,x_n\} \in K_n^M(k(x))$

$$\downarrow \text{Norm}$$

$$K_n^M(k)$$




**Weil Reciprocity** in $K_n^M(k)$

$\partial_0\{f_1,\dots,f_n\} = \partial_1\{f_1,\dots,f_n\}$

$f_i$ units in $\mathcal{O}(Z)$

Nesterenko-Suslin, Izvestia CCCP (1989)

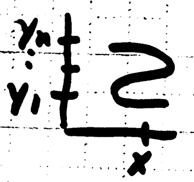
# Homological Algebra

nisnevich/étale topology

**Lemma** ① If $F$ is P.S.T., so is $F_{nis}$, $F_{et}$

② The canonical flasque resolution is by P.S.T.'s

$$0 \to F_{et} \to E^0 \to E^1 \to \cdots \qquad E^0(X) = \prod_{x \in X(\bar{k})} F_x$$



$$\begin{array}{ccc} E^0(Y) & \to & E^0(X) \\ \downarrow & & \downarrow \\ \prod_{y_i} F_{y_i} & \to & F_x \end{array}$$

③ $\left.\begin{array}{l} H^n_{nis}(X,F) \\ H^n_{et}(X,F) \end{array}\right\}$ are PST's

$$\boxed{Br(X) = H^2_{et}(X, \mathcal{O}_X^*)_{tors} \text{ is a PST}}$$

**$D^-$ (sheaves with transfer)** $= \boxed{DM^-_{eff}}$

$J$ = thick subcat closed under $\oplus_\alpha$ generated by $[X \times \mathbb{A}^1 \to X]$ "$\mathbb{A}^1$-weak equivalences"

variants with coeffs eg $DM^-_{eff}(k; \mathbb{Z}/n)$

$\mathcal{H}$ = abelian category of h.i. PST's "Voevodsky modules"

$D^-_{\mathcal{H}} \subset D^-$ complexes with cohom. in $\mathcal{H}$.

**Lemma**: $C_*F \simeq F$ in $DM^-_{eff}$ and $C_*F$ in $D^-_{\mathcal{H}}$

**Theorem** $\boxed{D^-_{\mathcal{H}} \cong DM^-_{eff}}$

**Thm** étale version $\quad \boxed{DM^-_{et,eff}(k, \mathbb{Z}/n) \cong D^-(G, \mathbb{Z}/n)}$

$cd_n(k) < \infty$

$G = Gal(\bar{k}/k)$ Galois modules "Artin motives"

$$DM^{eff}_{gm} \xrightarrow[\text{faithful}]{\text{full}} DM^-_{eff}$$