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How to decide whether a field has u-invariant 4?

(I) u-invariant of a field (nonreal / real)

(II) Pfister's and Lang's conjecture

(III) Function fields of curves over $\mathbb{R}((t))$

(IV) Main theorem

(V) Generalisations

(I) F field, char $\neq 2$

We consider (regular) quadratic forms $/F$

Pb. of isotropy $\rightsquigarrow$ sufficient conditions in terms of invariants $\rightsquigarrow$ <u>dimension</u>
$\rightsquigarrow$ def. of:

- u-invariant for <u>F nonreal</u> $(-1 \in \sum F^{\times 2})$

<u>Kaplansky ('53):</u> $u(F) = \sup\{\dim \varphi \mid \varphi \text{ anis. qu. form} / F\}$

- <u>Examples:</u>

  1) $F = \mathbb{C}(X_1, \dots, X_n) \Rightarrow u(F) = 2^n$ (more gen. for $F/\mathbb{C}$ f. gen. $n = \mathrm{trdeg}_{\mathbb{C}} F$)

  <u>Note:</u> $\langle\langle X_1, \dots, X_n \rangle\rangle$ anis. $/F$ of dim $= 2^n$

  2) $F/\mathbb{F}_p$ $(p \neq 2)$ fin. gen. $\Rightarrow u(F) = 2^{\mathrm{trd}_{\mathbb{F}_p} F + 1}$

  <u>In both cases</u>: upper bound by Tsen-Lang theory.

- Above definition not interesting <u>for real fields</u> F $(-1 \notin \sum F^{\times 2})$:

  then $m \times \langle 1 \rangle$ anis. $\forall m \geq 1$ "$\Rightarrow u(F) = \infty$"

Generally accepted definition:

- Elman-Lam ('73):

$$u(F) = \sup\{\dim \varphi \mid \varphi \text{ anis. torsion form}/F\}$$

- φ is torsion iff $m \times \varphi$ hyperbolic f.s. $m \geq 1$ (iff $[\varphi] \in W_t F$)

$\therefore$ any qu. form/F is torsion if F is nonreal, thus $u(F)$ the same for those F

- Examples:

3) F local or global field $\Rightarrow u(F) = 4$

4) $u(F) = 0$ iff F real pythagorean ($\sum F^{\times 2} = F^{\times 2} \not\ni -1$); e.g. $F = \mathbb{R}$

5) $u(F((t))) = 2 \cdot u(F)$

6) $F/\mathbb{R}$ f. gen. $\operatorname{trd}_{\mathbb{R}} F = 1 \Rightarrow u(F) = 2$

(II) Open Problem (Lang, Pfister):

Let $F/\mathbb{R}$ f. gen., $n = \operatorname{trd}_{\mathbb{R}} F$. Is $u(F) = 2^n$?

- Certainly $u(F) \geq 2^n$. If $n \geq 2$, then $u(F) = 2^n$ only known if F nonreal & $-1 \in F^{\times 2}$.

"Simplest" open case: $F = \mathbb{R}(X, Y)$

$$\begin{array}{ccc} F(i) & & u = 4 \\ | & & \Downarrow \\ F & & u = 4 \text{ or } 6 \end{array}$$

Which??

(III) Idea: look at similar, easier fields, e.g. $F = \underbrace{\mathbb{R}((t))}_{k}(C)$

k hereditarily pythagorean

$k(i)$ is C_1-field, in part. $u(k(i)) = 2$.

" has abelian Galois group $\hat{\mathbb{Z}}$

C curve over k (assume with good reduction)

$F = k(C) \rightsquigarrow \operatorname{trd}_k F = 1$, e.g. $F = k(X)$

- Crucial observation: $p(F)=2$ for these F

Pythagoras number:

$$p(F) = \sup\{m\in\mathbb{N} \mid \exists a\in\Sigma F^{\times 2},\ a \text{ not sum of } < m \text{ squares in } F\}$$

$\therefore\ p(F)=1 \iff F$ pythagorean $(\Sigma F^{\times 2} = F^{\times 2})$

Proposition k hered. pythagorean iff $p(k(X))=2$.

Ex.: $k=\mathbb{R}((t))$; (idea of proof: show with $a\in\Sigma k(X)^{\times 2}$, $\langle 1,1,-a,-a\rangle$ is unramified w.r.t. any k-valuation of $k(X)$)

IV Main ~~theorem~~ result

Theorem (2004): Assume F s.t. $p(F)=2$, $u(F(i))=4$. Then $u(F)\leq 4$.

Proof: $u(F(i)) = 4 \Rightarrow u(F)\leq 6$ (recall case $F=\mathbb{R}(X,Y)$!)

But $u(F)\neq 5$. Assume now $u(F)=6$.

$\Rightarrow \exists\, \varphi$ anis. torsion $/F$, $\dim\varphi=6$

$u(F(i))=4 \Rightarrow \varphi_{F(i)}$ isotr. $\overset{\text{may ass.}}{\Longrightarrow} \varphi\cong\langle 1,1\rangle\perp\psi$, $\dim\psi=4$.

φ torsion $\underset{p=2}{\Longrightarrow} 2\times\varphi$ hyp. $\Rightarrow 2\times\psi$ isotr. $\Rightarrow \psi\simeq\sigma\perp\tau$ where $\dim\sigma=\dim\tau=2$ and $2\times\tau$ hyp.

$$\Longrightarrow \boxed{\varphi\simeq\langle 1,1\rangle\perp\sigma\perp\tau}$$

φ,τ torsion $\Rightarrow \langle 1,1\rangle\perp\sigma$ torsion $\Rightarrow \sigma\simeq\langle -a,-b\rangle$, $a,b\in\Sigma F^{\times 2}$

$p(F)=2 \Rightarrow \langle 1,1,-a\rangle$ isotr. $\Rightarrow \varphi$ isotr. $\lightning$. $\square$

Applications / Examples: $u(F)=4$ in the following cases

1) $F=k(X)$ where $k=\mathbb{R}((t))$, more gen. $F=k(C)$, C hyp. ell. curve w.g.red. to $\mathbb{R}$
2) $F=\mathbb{R}((X,Y))$ since $p(F)=2$ by Choi-Dai-Lam-Resnik ($\approx$'82)

- Note for 1) that $\langle 1, -(1+X^2), t, -t(1+X^2)\rangle$ torsion of dim $=4$ $/F$.
- It remains still open if $u(F)=\cancel{4}$ or 6 for $F=\mathbb{R}(X,Y)$.

  Note that $p(F)=4$!

(V) <u>Theorem (2004)</u>: Assume $u(F(i))=4$. Then

$$u(F)\leq 4 \iff \langle 1,1,-a,-ab\rangle \text{ universal } \forall a\in D_F(3), b\in D_F(2)$$

* generalizes the previous thm
* ~~generalizes a cri~~ strengthens a criterion ~~by~~ due to Pfister

<u>Proposition</u>: $F=k(X)$, k her. pyth.

Then $p(F)=2$, $u(F)\leq u(F(i))$

<u>Example</u>: $F=\mathbb{R}((t_1))\ldots((t_{n-1}))(X) \Rightarrow u(F)=u(F(i))=2^n$.

- Back to case $k=\mathbb{R}((t))$, $F=k(C)$, C ~~curve~~ curve

  Assume now F nonreal

  If $i\in F$ then $u(F)=4$

  if $i\notin F$, then $u(F(i))=4$, $p(F)=3$

<u>Ex.</u> $F=k(X)(\sqrt{-(1+X^2)})$. Then $p(F)=3$ & $|F^\times/D_F(2)|=2$

<u>Theorem (2004)</u> Assume F nonreal, $u(F(i))=4$, $|F^\times/D_F(2)|=2$.
Then $u(F)=4$.