$SL_1(A)$ and essential dimension

joint work with G. Favi

k, A central simple alg. / k

$$K/k \quad 1 \to \underbrace{SL_1(A)(K)}_{\{x \in (A\otimes_k K)^\times \mid \mathrm{Nrd}(x)=1\}} \longrightarrow \underbrace{GL_1(A)(K)}_{(A\otimes_k K)^\times} \xrightarrow{\mathrm{Nrd}} \underbrace{\mathbb{G}_m(K)}_{K^\times}$$

$$1 \to SL_1(A) \to GL_1(A) \to \mathbb{G}_m \to 1 \quad \text{exact seq. of alg. groups}$$

$F: \mathcal{C}_k \to \mathcal{S}ets$ cov. functor

K/k, $a \in F(K)$, $n \in \mathbb{N}$

- $ed(a) \leq n$ if there exist $K/L/k$ s.t. $\begin{cases} \operatorname{trdeg} L/k \leq n \\ a \in \operatorname{im}(F(L) \to F(K)) \end{cases}$
- $ed(a) = n$ if $ed(a) \leq n$ and $ed(a) \not\leq n-1$

$$ed_k(F) = \sup_{K/k,\ a \in F(K)} ed(a)$$

G alg. group / k $\quad ed_k(G) = ed_k(H^1(\cdot, G))$

$$K/k \quad GL_1(A)(K) \to \mathbb{G}_m(K) \to H^1(K, SL_1(A)) \to \underbrace{H^1(K, GL_1(A))}_{1}$$

$$\mathbb{G}_m(K) \twoheadrightarrow H^1(K, SL_1(A)) \text{ surjection } \Rightarrow ed_k(SL_1(A)) \leq 1$$

$$\boxed{K^\times / \mathrm{Nrd}((A\otimes_k K)^\times) \xrightarrow{\cong} H^1(K, SL_1(A))} \quad \text{bijection}$$

Thm A The foll. cond. are equiv.

1) A is split
2) $ed_k(SL_1(A)) = 0$
3) $ed(\bar{t}) = 0 \quad \bar{t} \in H^1(k(t), SL_1(A))$

Thm B $n \in \mathbb{N} \setminus \{0\}$

$$\mathrm{ed}\,k\Big(\underbrace{SL_1(A) \times \dots \times SL_1(A)}_{n}\Big) = n\,\mathrm{ed}\,k\big(SL_1(A)\big)$$

Description of $H^1\big(k(t), SL_1(A)\big)$

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degree d cohomological invariant

Proof of Thm A

$1) \Rightarrow 2) \Rightarrow 3)$

$3) \Rightarrow 1)$ $\mathrm{ed}(E) = 0 \Rightarrow \exists \lambda \in k^\times,\ x \in \big(A \otimes_k k(t)\big)^\times \,/\, t = \lambda \,\mathrm{Nrd}(x)$

$\Downarrow$

$\big($ $\mathrm{ind}(A) = 1$, A split $\big($

Prop

$$\begin{tikzcd} \big(A \otimes_k k(t)\big)^\times \arrow[r, "\mathrm{Nrd}"] \arrow[rr, bend right, "\theta"'] & k(t)^\times \arrow[r, "\deg"] & \mathbb{Z} \end{tikzcd}$$

Then $\mathrm{im}(\theta) = \mathrm{ind}(A)\,\mathbb{Z}$

Proof we may suppose that A is a div. alg.

$\boxed{\supseteq}$ $\mathrm{Nrd}(1 \otimes t) = t^{\mathrm{ind}(A)}$

$\boxed{\subseteq}$ $a \in \big(A \otimes_k k(t)\big)^\times$; $a = \dfrac{b}{c}$, $c \in k[t]$, $b \in A \otimes_k k[t]$

$$b = b_r \otimes t^r + \dots + b_0 \otimes 1 \ , \quad b_i \in A \ , \ b_r \neq 0$$

$$\mathrm{Nrd}(b) = \underbrace{\mathrm{Nrd}(b_r)}_{\neq 0}\, t^{r\,\mathrm{ind}(A)} + \dots$$

Proof of Thm B $n \in \mathbb{N} \setminus \{0\}$, A non-split

$\mathrm{ed}\,k(G \times H)$?

$\mathrm{ed}\,k(G \times G)$?

we may suppose k infinite

$K_n = k(t_1, \dots, t_n)$

$F = H^1\big(\cdot, SL_1(A)\big)$

$F_n = H^1\big(\cdot, \underbrace{SL_1(A) \times \dots \times SL_1(A)}_{n}\big) \cong \underbrace{F \times \dots \times F}_{n}$

$a_n = (\overline{t}_1, \dots, \overline{t}_n) \in F_n(K_n)$

$$n\,\mathrm{ed}\,k\big(SL_1(A)\big) \ \| \ \mathrm{ed}\,k\Big(\underbrace{SL_1(A) \times \dots \times SL_1(A)}_{n}\Big) = \mathrm{ed}\,k(F_n) \leq n\,\mathrm{ed}\,k(F)$$

$\mathrm{ed}(a_n) = n$?

$\underline{n=1}$ Thm A $\Rightarrow$ ~~ed k~~ $\mathrm{ed}_{k_n}(a_1) = 1$

$\underline{n \geq 2}$ suppose t. $\mathrm{ed}(a_n) < n$

$\Rightarrow$ there exist $\begin{array}{c} L \\ | \\ k \end{array}$ $\mathrm{trdeg}\, L/k < n$, $b \in F(L)$ / $F(L) \to F(k_n)$, $b \mapsto a_n$, $b = (b_1, \dots, b_n)$

$$\begin{cases} t_1 = b_1 \,\mathrm{Nrd}(x_1) \\ \vdots \\ t_{n-1} = b_{n-1} \,\mathrm{Nrd}(x_{n-1}) \\ t_n = b_n \,\mathrm{Nrd}(x_n) \end{cases} \qquad x_i \in (A \otimes_k k_n)^\times$$

$\lambda \in k$ $\quad v$ $(t_n - \lambda)$-valuation on k_n

choose λ s.t. $b_i \in \mathcal{O}_v$, $x_i, x_i^{-1} \in A \otimes_k \mathcal{O}_v$

method inspired by method developped by M. Rost

$$\mathcal{O}_v \twoheadrightarrow k_v = k(t_1, \dots, t_{n-1})$$

$v' = v|_L$ $\qquad \begin{array}{c} k_v \\ | \\ k_{v'} \\ | \\ k \end{array}$

$$\left.\begin{cases} t_1 = b_1(\lambda) \,\mathrm{Nrd}(x_1(\lambda)) \\ \vdots \\ t_{n-1} = b_{n-1}(\lambda) \,\mathrm{Nrd}(x_{n-1}(\lambda)) \end{cases}\right\} \mathrm{ed}(a_{n-1}) = n-1 \Rightarrow \mathrm{trdeg}\, \cancel{k_{n-1}}\, k_{v'} = n-1$$

$\Downarrow$ v' ~~is~~ trivial

$$\lambda = b_n(\lambda) \,\mathrm{Nrd}(x_n(\lambda)) = b_n \,\mathrm{Nrd}(x_n(\lambda)) \Rightarrow b_n \in k_{n-1}$$

$t_n = b_n \,\mathrm{Nrd}(x_n) \Rightarrow A \otimes_k k_{n-1}$ split $\Rightarrow A$ split

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Description of $H^1(k(t), SL_1(A))$

X set of irred. monic polynomials in $k[t]$

$x \in X$, v_x x-adic valuation on $k[t]$

$$x \in X, \quad \begin{tikzcd} (A \otimes_k k(t))^\times \arrow[r, "\mathrm{Nrd}"] \arrow[rr, bend right, "\eta_x"'] & k(t)^\times \arrow[r, "v_x"] & \mathbb{Z} \end{tikzcd} \qquad \mathrm{im}(\eta_x) = \mathrm{ind}(A)\mathbb{Z}$$

$\partial_x : H^1(k(t), SL_1(A)) \longrightarrow \mathbb{Z}/\mathrm{ind}(A)$ ~~can~~ hom. induced by v_x

Thm

$$1 \to H^1(k, SL_1(A)) \to H^1(k(t), SL_1(A)) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in X} \mathbb{Z}/\mathrm{ind}(A)\mathbb{Z} \to 1$$

is a split exact sequence.

Corollary

The foll. cond. are equiv.

1) A is split
2) $H^1(k(t), SL_1(A)) = 1$
3) $H^1(k(t), SL_1(A))$ is finite