

# ESSENTIAL DIMENSION OF FINITE $p$ -GROUPS

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ABSTRACT. We prove that the essential dimension and  $p$ -dimension of a  $p$ -group  $G$  over a field  $F$  containing a primitive  $p$ -th root of unity is equal to the least dimension of a faithful representation of  $G$  over  $F$ .

The notion of the essential dimension  $\text{ed}(G)$  of a finite group  $G$  over a field  $F$  was introduced in [5]. The integer  $\text{ed}(G)$  is equal to the smallest number of algebraically independent parameters required to define a Galois  $G$ -algebra over any field extension of  $F$ . If  $V$  is a faithful linear representation of  $G$  over  $F$  then  $\text{ed}(G) \leq \dim(V)$  (cf. [2, Prop. 4.15]). The essential dimension of  $G$  can be smaller than  $\dim(V)$  for every faithful representation  $V$  of  $G$  over  $F$ . For example, we have  $\text{ed}(\mathbb{Z}/3\mathbb{Z}) = 1$  over  $\mathbb{Q}$  or any field  $F$  of characteristic 3 (cf. [2, Cor. 7.5]) and  $\text{ed}(S_3) = 1$  over  $\mathbb{C}$  (cf. [5, Th. 6.5]).

In this paper we prove that if  $G$  is a  $p$ -group and  $F$  is a field of characteristic different from  $p$  containing  $p$ -th roots of unity, then  $\text{ed}(G)$  coincides with the least dimension of a faithful representation of  $G$  over  $F$  (cf. Theorem 4.1).

We also compute the essential  $p$ -dimension of a  $p$ -group  $G$  introduced in [15]. We show that  $\text{ed}_p(G) = \text{ed}(G)$  over a field  $F$  containing  $p$ -th roots of unity.

In the paper the word “scheme” means a separated scheme of finite type over a field and “variety” an integral scheme.

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## 1. PRELIMINARIES

1.1. **Severi-Brauer varieties.** (cf. [1]) Let  $A$  be a central simple algebra of degree  $n$  over a field  $F$ . The *Severi-Brauer variety*  $P = \text{SB}(A)$  of  $A$  is the variety of right ideals in  $A$  of dimension  $n$ . For a field extension  $L/F$ , the algebra  $A$  is split over  $L$  if and only if  $P(L) \neq \emptyset$  if and only if  $P_L \simeq \mathbb{P}_L^{n-1}$ .

The change of field map  $\text{deg} : \text{Pic}(P) \rightarrow \text{Pic}(P_L) = \mathbb{Z}$  for a splitting field extension  $L/F$  identifies  $\text{Pic}(P)$  with  $e\mathbb{Z}$ , where  $e$  is the exponent (period) of  $A$ . In particular,  $P$  has divisors of degree  $e$ . The algebra  $A$  is split over  $L$  if and only if  $P_L$  has a prime divisor of degree 1 (a hyperplane).

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**1.2. Groupoids and gerbes.** (cf. [4]) Let  $\mathcal{X}$  be a groupoid over  $F$  in the sense of [19]. We assume that for any field extension  $L/F$ , the isomorphism classes of objects in the category  $\mathcal{X}(L)$  form a set which we denote by  $\widehat{\mathcal{X}}(L)$ . We can view  $\widehat{\mathcal{X}}$  as a functor from the category  $\mathbf{Fields}/F$  of field extensions of  $F$  to  $\mathbf{Sets}$ .

**Example 1.2.1.** If  $G$  is an algebraic group over  $F$ , then the groupoid  $BG$  is defined as the category of  $G$ -torsors over a scheme over  $F$ . Hence the functor  $\widehat{BG}$  takes a field extension  $L/F$  to the set of all isomorphism classes of  $G$ -torsors over  $L$ .

Special examples of groupoids are *gerbes banded by a commutative group scheme*  $C$  over  $F$ . There is a bijection between the set of isomorphism classes of gerbes banded by  $C$  and the Galois cohomology group  $H^2(F, C)$  (cf. [7, Ch. 4] and [13, Ch. 4, §2]). The split gerbe  $BC$  corresponds to the trivial element of  $H^2(F, C)$ .

**Example 1.2.2.** (Gerbes banded by  $\mu_n$ ) Let  $A$  be a central simple  $F$ -algebra and  $n$  an integer with  $[A] \in \mathrm{Br}_n(F) = H^2(F, \mu_n)$ . Let  $P$  be the Severi-Brauer variety of  $A$  and  $S$  a divisor on  $P$  of degree  $n$ . Denote by  $\mathcal{X}_A$  the gerbe banded by  $\mu_n$  corresponding to  $[A]$ . For a field extension  $L/F$ , the set  $\widehat{\mathcal{X}}_A(L)$  has the following explicit description (cf. [4]):  $\widehat{\mathcal{X}}_A(L)$  is nonempty if and only if  $P$  is split over  $L$ . In this case  $\widehat{\mathcal{X}}_A(L)$  is the set of equivalence classes of the set

$$\{f \in L(P)^\times : \mathrm{div}(f) = nH - S_L, \text{ where } H \text{ is a hyperplane in } P_L\},$$

and two functions  $f$  and  $f'$  are equivalent if  $f' = fh^n$  for some  $h \in L(P)^\times$ .

**1.3. Essential dimension.** Let  $T : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$  be a functor. For a field extension  $L/F$  and an element  $t \in T(L)$ , the *essential dimension of  $t$* , denoted  $\mathrm{ed}(t)$ , is the least  $\mathrm{tr. deg}_F(L')$  over all subfields  $L' \subset L$  over  $F$  such that  $t$  belongs to the image of the map  $T(L') \rightarrow T(L)$ . The *essential dimension of the functor  $T$*  is the supremum of  $\mathrm{ed}(t)$  over all  $t \in T(L)$  and field extensions  $L/F$ .

Let  $p$  be a prime integer and  $t \in T(L)$ . The *essential  $p$ -dimension of  $t$* , denoted  $\mathrm{ed}_p(t)$ , is the least  $\mathrm{tr. deg}_F(L')$  over all subfields  $L' \subset L$  over  $F$ , where  $L'$  is a finite field extension of  $L$  of degree prime to  $p$  such that the image of  $t$  in  $T(L')$  belongs to the image of the map  $T(L') \rightarrow T(L)$ . The *essential  $p$ -dimension of the functor  $T$*  is the supremum of  $\mathrm{ed}_p(t)$  over all  $t \in T(L)$  and field extensions  $L/F$ . Clearly,  $\mathrm{ed}(T) \geq \mathrm{ed}_p(T)$ .

Let  $G$  be an algebraic group over  $F$ . The *essential dimension of  $G$*  (respectively the *essential  $p$ -dimension of  $G$* ) is the essential dimension (respectively the essential  $p$ -dimension) of the functor taking a field extension  $L/F$  to the set of isomorphism classes of  $G$ -torsors over  $\mathrm{Spec} L$ .

If  $G$  is a finite group, we view  $G$  as a constant group over a field  $F$ . Every  $G$ -torsor over  $\mathrm{Spec} L$  has the form  $\mathrm{Spec} K$  where  $K$  is a Galois  $G$ -algebra over  $L$ . Therefore,  $\mathrm{ed}(G)$  is the essential dimension of the functor taking a field  $L$  to the set of isomorphism classes of Galois  $G$ -algebras over  $L$ .

**Example 1.3.1.** Let  $\mathcal{X}$  be a groupoid over  $F$ . The *essential dimension* of  $\mathcal{X}$ , denoted by  $\text{ed}(\mathcal{X})$ , is the essential dimension  $\text{ed}(\widehat{\mathcal{X}})$  of the functor  $\widehat{\mathcal{X}}$  defined in §1.2. The *essential  $p$ -dimension* of  $\text{ed}_p(\mathcal{X})$  is defined similarly. In particular,  $\text{ed}(BG) = \text{ed}(G)$  and  $\text{ed}_p(BG) = \text{ed}_p(G)$  for an algebraic group  $G$  over  $F$ .

**1.4. Canonical dimension.** (cf. [3], [11]) Let  $F$  be a field and  $\mathcal{C}$  a class of field extensions of  $F$ . A field  $E \in \mathcal{C}$  is called *generic* if for any  $L \in \mathcal{C}$  there is an  $F$ -place  $E \rightsquigarrow L$ .

The *canonical dimension*  $\text{cdim}(\mathcal{C})$  of the class  $\mathcal{C}$  is the minimum of the  $\text{tr. deg}_F E$  over all generic fields  $E \in \mathcal{C}$ .

Let  $p$  be a prime integer. A field  $E$  in a class  $\mathcal{C}$  is called  *$p$ -generic* if for any  $L \in \mathcal{C}$  there is a finite field extension  $L'$  of  $L$  of degree prime to  $p$  and an  $F$ -place  $E \rightsquigarrow L'$ . The *canonical  $p$ -dimension*  $\text{cdim}_p(\mathcal{C})$  of the class  $\mathcal{C}$  is the least  $\text{tr. deg}_F E$  over all  $p$ -generic fields  $E \in \mathcal{C}$ . Obviously,  $\text{cdim}(\mathcal{C}) \geq \text{cdim}_p(\mathcal{C})$ .

Let  $T : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$  be a functor. Denote by  $\mathcal{C}_T$  the class of *splitting fields of  $T$* , i.e., the class of field extensions  $L/F$  such that  $T(L) \neq \emptyset$ . The *canonical dimension ( $p$ -dimension) of  $T$* , denoted  $\text{cdim}(T)$  (respectively  $\text{cdim}_p(T)$ ), is the canonical dimension ( $p$ -dimension) of the class  $\mathcal{C}_T$ .

If  $X$  is a scheme over  $F$ , we write  $\text{cdim}(X)$  and  $\text{cdim}_p(X)$  for the canonical dimension and  $p$ -dimension of  $X$  viewed as a functor  $L \mapsto X(L) = \text{Mor}_F(\text{Spec } L, X)$ .

**Example 1.4.1.** Let  $\mathcal{X}$  be a groupoid over  $F$ . We define the *canonical dimension*  $\text{cdim}(\mathcal{X})$  and  *$p$ -dimension*  $\text{cdim}_p(\mathcal{X})$  of  $\mathcal{X}$  as the canonical dimension and  $p$ -dimension of the functor  $\widehat{\mathcal{X}}$ .

**Example 1.4.2.** If  $X$  is a regular and complete variety over  $F$  viewed as a functor then  $\text{cdim}(X)$  is equal to the smallest dimension of a closed subvariety  $Z \subset X$  such that there is a rational morphism  $X \dashrightarrow Z$  (cf. [11, Cor. 4.6]). If  $p$  is a prime integer then  $\text{cdim}_p(X)$  is equal to the smallest dimension of a closed subvariety  $Z \subset X$  such that there are dominant rational morphisms  $X' \dashrightarrow X$  of degree prime to  $p$  and  $X' \dashrightarrow Z$  for some variety  $X'$  (cf. [11, Prop. 4.10]).

**Remark 1.4.3.** (A relation between essential and canonical dimension) Let  $T : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$  be a functor. We define the “contraction” functor  $T^c : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$  as follows. For a field extension  $L/F$ , we have  $T^c(L) = \emptyset$  if  $T(L)$  is empty and  $T^c(L)$  is a one element set otherwise. If  $X$  is a regular and complete variety over  $F$  viewed as a functor then one can show that  $\text{ed}(X^c) = \text{cdim}(X)$  and  $\text{ed}_p(X^c) = \text{cdim}_p(X)$ .

**1.5. Valuations.** Let  $K/F$  be a regular field extension, i.e., for any field extension  $L/F$ , the ring  $K \otimes_F L$  is a domain. We write  $KL$  for the quotient field of  $K \otimes_F L$ .

Let  $v$  be a valuation on  $L$  over  $F$  with residue field  $R$ . Let  $O$  be the associated valuation ring and  $M$  its maximal ideal. As  $K \otimes_F R$  is a domain, the ideal  $\widetilde{M} := K \otimes_F M$  in the ring  $\widetilde{O} := K \otimes_F O$  is prime. The localization ring  $\widetilde{O}_{\widetilde{M}}$  is

a valuation ring in  $KL$  with residue field  $KR$ . The corresponding valuation  $\tilde{v}$  of  $KL$  is called the *canonical extension of  $v$  on  $KL$* . Note that the groups of values of  $v$  and  $\tilde{v}$  coincide.

We shall need the following lemma.

**Lemma 1.1** (cf. [11, Lemma 3.2]). *Let  $v$  be a discrete valuation (of rank 1) of a field  $L$  with residue field  $R$  and  $L'/L$  a finite field extension of degree prime to  $p$ . Then  $v$  extends to a discrete valuation of  $L'$  with residue field  $R'$  such that the ramification index and the degree  $[R' : R]$  are prime to  $p$ .*

*Proof.* If  $L'/L$  is separable and  $v_1, \dots, v_k$  are all the extensions of  $v$  on  $L'$  then  $[L' : L] = \sum e_i [R_i : R]$  where  $e_i$  is the ramification index and  $R_i$  is the residue field of  $v_i$  (cf. [20, Ch. VI, Th. 20 and p. 63]). It follows that the integer  $e_i [R_i : R]$  is prime to  $p$  for some  $i$ .

If  $L'/L$  is purely inseparable of degree  $q$  then the valuation  $v'$  of  $L'$  defined by  $v'(x) = v(x^q)$  satisfies the desired properties. The general case follows.  $\square$

## 2. CANONICAL DIMENSION OF A SUBGROUP OF $\text{Br}(F)$

Let  $F$  be an arbitrary field,  $p$  a prime integer and  $D$  a finite subgroup of  $\text{Br}_p(F)$  of dimension  $r$  over  $\mathbb{Z}/p\mathbb{Z}$ . In this section we determine the canonical dimension  $\text{cdim } D$  and the canonical  $p$ -dimension  $\text{cdim}_p D$  of the class of common splitting fields of all elements of  $D$ . We say that a basis  $\{a_1, a_2, \dots, a_r\}$  of  $D$  is *minimal* if for any  $i = 1, \dots, r$  and any element  $d \in D$  outside of the subgroup generated by  $a_1, \dots, a_{i-1}$ , we have  $\text{ind } d \geq \text{ind } a_i$ .

One can construct a minimal basis of  $D$  by induction as follows. Let  $a_1$  be a nonzero element of  $D$  of minimal index. If the elements  $a_1, \dots, a_{i-1}$  are already chosen for some  $i \leq r$ , we take for the  $a_i$  an element of  $D$  of the minimal index among the elements outside of the subgroup generated by  $a_1, \dots, a_{i-1}$ .

In this section we prove the following

**Theorem 2.1.** *Let  $F$  be an arbitrary field,  $p$  a prime integer,  $D \subset \text{Br}_p(F)$  a subgroup of dimension  $r$  and  $\{a_1, a_2, \dots, a_r\}$  a minimal basis of  $D$ . Then*

$$\text{cdim}_p(D) = \text{cdim}(D) = \left( \sum_{i=1}^r \text{ind } a_i \right) - r.$$

We prove Theorem 2.1 in several steps.

Let  $\{a_1, a_2, \dots, a_r\}$  be a minimal basis of  $D$ . For every  $i = 1, 2, \dots, r$ , let  $P_i$  be the Severi-Brauer variety of a central division  $F$ -algebra  $A_i$  representing the element  $a_i \in \text{Br}_p F$ . We write  $P$  for the product  $P_1 \times P_2 \times \dots \times P_r$ . We have

$$\dim P = \sum_{i=1}^r \dim P_i = \left( \sum_{i=1}^r \text{ind } a_i \right) - r.$$

Moreover, the classes of splitting fields of  $P$  and  $D$  coincide, hence  $\text{cdim}(D) = \text{cdim}(P)$  and  $\text{cdim}_p(D) = \text{cdim}_p(P)$ . Thus, the statement of Theorem 2.1 is equivalent to the equality  $\text{cdim}_p(P) = \text{cdim}(P) = \dim(P)$ .

Let  $r \geq 1$  and  $0 \leq n_1 \leq n_2 \leq \dots \leq n_r$  be integers and  $K = K(n_1, \dots, n_r)$  the subgroup of the polynomial ring  $\mathbb{Z}[x]$  in  $r$  variables  $x = (x_1, \dots, x_r)$  generated by the monomials  $p^{e(j_1, \dots, j_r)} x_1^{j_1} \dots x_r^{j_r}$  for all  $j_1, \dots, j_r \geq 0$ , where the exponent  $e(j_1, \dots, j_r)$  is 0 if all the  $j_1, \dots, j_r$  are divisible by  $p$ , otherwise  $e(j_1, \dots, j_r) = n_k$  with the maximum  $k$  such that  $j_k$  is not divisible by  $p$ . In fact,  $K$  is a subring of  $\mathbb{Z}[x]$ .

**Remark 2.2.** Let  $A_1, \dots, A_r$  be central division algebras over some field such that for any non-negative integers  $j_1, \dots, j_r$ , the index of the tensor product  $A_1^{\otimes j_1} \otimes \dots \otimes A_r^{\otimes j_r}$  is equal to  $p^{e(j_1, \dots, j_r)}$ . The group  $K$  can be interpreted as the colimit of the Grothendieck groups of the product over  $i = 1, \dots, r$  of the Severi-Brauer varieties of the matrix algebras  $M_{l_i}(A_i)$  over all positive integers  $l_1, \dots, l_r$ .

We set  $h = (h_1, \dots, h_r)$  with  $h_i = 1 - x_i \in \mathbb{Z}[x]$ .

**Proposition 2.3.** *Let  $bh_1^{i_1} \dots h_r^{i_r}$  be a monomial of the lowest total degree of a polynomial  $f$  in the variables  $h$  lying in  $K$ . Assume that the integer  $b$  is not divisible by  $p$ . Then  $p^{n_1} \mid i_1, \dots, p^{n_r} \mid i_r$ .*

*Proof.* We recast the proof for  $r = 1$  given in [8, Lemma 2.1.2] to the case of arbitrary  $r$ .

We proceed by induction on  $m = r + n_1 + \dots + n_r$ . The case  $m = 1$  is trivial. If  $m > 1$  and  $n_1 = 0$ , then  $K = K(n_2, \dots, n_r)[x_1]$  and we are done by induction applied to  $K(n_2, \dots, n_r)$ . In what follows we assume that  $n_1 \geq 1$ .

Since  $K(n_1, n_2, \dots, n_r) \subset K(n_1 - 1, n_2, \dots, n_r)$ , by the induction hypothesis  $p^{n_1 - 1} \mid i_1, p^{n_2} \mid i_2, \dots, p^{n_r} \mid i_r$ . It remains to show that  $i_1$  is divisible by  $p^{n_1}$ .

Consider the additive operation  $\varphi: \mathbb{Z}[x] \rightarrow \mathbb{Q}[x]$  which takes a polynomial  $g \in \mathbb{Z}[x]$  to the polynomial  $p^{-1}x_1 \cdot g'$ , where  $g'$  is the partial derivative of  $g$  with respect to  $x_1$ . We have

$$\varphi(K) \subset K(n_1 - 1, n_2 - 1, \dots, n_r - 1) \subset K(n_1 - 1)[x_2, \dots, x_r]$$

and

$$\varphi(h_1^{j_1} h_2^{j_2} \dots h_r^{j_r}) = -p^{-1} j_1 h_1^{j_1 - 1} h_2^{j_2} \dots h_r^{j_r} + p^{-1} j_1 h_1^{j_1} h_2^{j_2} \dots h_r^{j_r}.$$

Since  $bh_1^{i_1} \dots h_r^{i_r}$  is a monomial of the lowest total degree of the polynomial  $f$ , it follows that  $-bp^{-1}i_1 h_1^{i_1 - 1} h_2^{i_2} \dots h_r^{i_r}$  is a monomial of  $\varphi(f)$  considered as a polynomial in  $h$ . As

$$\varphi(f) \in K(n_1 - 1)[x_2, \dots, x_r],$$

we see that  $-bp^{-1}i_1 h_1^{i_1 - 1}$  is a monomial of a polynomial from  $K(n_1 - 1)$ . It follows that  $p^{-1}i_1$  is an integer and by Lemma 2.4 below, this integer is divisible by  $p^{n_1 - 1}$ . Therefore  $p^{n_1} \mid i_1$ .  $\square$

**Lemma 2.4.** *Let  $g$  be a polynomial in  $h_1$  lying in  $K(m)$  for some  $m \geq 0$ . Let  $bh_1^{i-1}$  be a monomial of  $g$  such that  $i$  is divisible by  $p^m$ . Then  $b$  is divisible by  $p^m$ .*

*Proof.* We write  $h$  for  $h_1$  and  $x$  for  $x_1$ . Note that  $h^i \in K(m)$  since  $i$  is divisible by  $p^m$ . Moreover, the quotient ring  $K(m)/(h^i)$  is additively generated by  $p^{e(j)}x^j$  with  $j < i$ . Indeed, the polynomial  $x^i - (-h)^i = x^i - (x-1)^i$  is a linear combination with integer coefficients of  $p^{e(j)}x^j$  with  $j < i$ . Consequently, for any  $k \geq 0$ , multiplying by  $p^{e(k)}x^k$ , we see that the polynomial  $p^{e(i+k)}x^{i+k} = p^{e(k)}x^{i+k}$  modulo the ideal  $(h^i)$  is a linear combination with integer coefficients of the  $p^{e(j)}x^j$  with  $j < i+k$ .

Thus,  $K(m)/(h^i)$  is additively generated by  $p^{e(j)}(1-h)^j$  with  $j < i$ . Only the generator  $p^{e(i-1)}(1-h)^{i-1} = p^m(1-h)^{i-1}$  has a nonzero  $h^{i-1}$ -coefficient and that coefficient is divisible by  $p^m$ .  $\square$

Let  $Y$  be a scheme over the field  $F$ . We write  $\text{CH}(Y)$  for the Chow group of  $Y$  and set  $\text{Ch}(Y) = \text{CH}(Y)/p\text{CH}(Y)$ . We define  $\text{Ch}(\overline{Y})$  as the colimit of  $\text{Ch}(Y_L)$  where  $L$  runs over all field extensions of  $F$ . Thus for any field extension  $L/F$ , we have a canonical homomorphism  $\text{Ch}(Y_L) \rightarrow \text{Ch}(\overline{Y})$ . This homomorphism is an isomorphism if  $Y = P$ , the variety defined above, and  $L$  is a splitting field of  $P$ .

We define  $\overline{\text{Ch}}(Y)$  to be the image of the homomorphism  $\text{Ch}(Y) \rightarrow \text{Ch}(\overline{Y})$ .

**Proposition 2.5.** *We have  $\overline{\text{Ch}}^j(P) = 0$  for any  $j > 0$ .*

*Proof.* Let  $K_0(P)$  be the Grothendieck group of  $P$ . We write  $K_0(\overline{P})$  for the colimit of  $K_0(P_L)$  taken over all field extensions  $L/F$ . The group  $K_0(\overline{P})$  is canonically isomorphic to  $K_0(P_L)$  for any splitting field  $L$  of  $P$ . Each of the groups  $K_0(P)$  and  $K_0(\overline{P})$  is endowed with the topological filtration. The subsequent factor groups  $G^j K_0(P)$  and  $G^j K_0(\overline{P})$  of these filtrations fit into the commutative square

$$\begin{array}{ccc} \text{CH}^j(\overline{P}) & \longrightarrow & G^j K_0(\overline{P}) \\ \uparrow & & \uparrow \\ \text{CH}^j(P) & \longrightarrow & G^j K_0(P) \end{array}$$

where the top map is an isomorphism. Therefore it suffices to show that the image of the homomorphism  $G^j K_0(P) \rightarrow G^j K_0(\overline{P})$  is divisible by  $p$  for any  $j > 0$ .

The ring  $K_0(\overline{P})$  is identified with the quotient of the polynomial ring  $\mathbb{Z}[h]$  by the ideal generated by  $h_1^{\text{ind } a_1}, \dots, h_r^{\text{ind } a_r}$ . Under this identification, the element  $h_i$  is the pull-back to  $P$  of the class of a hyperplane in  $P_i$  over a splitting field and the  $j$ -th term  $K_0(\overline{P})^{(j)}$  of the filtration is generated by the classes of monomials of degree at least  $j$ . The group  $G^j K_0(\overline{P})$  is identified with the group of all homogeneous polynomials of degree  $j$ .

The group  $K_0(P)$  is isomorphic to the direct sum of  $K_0(B)$ , where  $B = A_1^{\otimes j_1} \otimes \dots \otimes A_r^{\otimes j_r}$ , over all  $j_i$  with  $0 \leq j_i < \text{ind } a_i$  (cf. [14, §9]). The image of the natural map  $K_0(B) \rightarrow K_0(B_L) = \mathbb{Z}$ , where  $L$  is a splitting field of  $B$ , is equal to  $\text{ind}(a_1^{j_1} \dots a_r^{j_r})\mathbb{Z}$ . The image of the homomorphism  $K_0(P) \rightarrow K_0(\overline{P})$

(which is in fact an injection) is generated by

$$\text{ind}(a_1^{j_1} \cdots a_r^{j_r})(1 - h_1)^{j_1} \cdots (1 - h_r)^{j_r}$$

over all  $j_1, \dots, j_r \geq 0$ .

We embed  $K_0(\overline{P})$  into the polynomial ring  $\mathbb{Z}[x] = \mathbb{Z}[x_1, \dots, x_r]$  as a subgroup by identifying a monomial  $h_1^{j_1} \cdots h_r^{j_r}$  where  $0 \leq j_i < \text{ind } a_i$  with the polynomial  $(1 - x_1)^{j_1} \cdots (1 - x_r)^{j_r}$ . As the elements  $a_1, \dots, a_r$  form a minimal basis of  $D$ , the index  $\text{ind}(a_1^{j_1} \cdots a_r^{j_r})$  is a power of  $p$  with the exponent at least  $e(\log_p \text{ind } a_1, \dots, \log_p \text{ind } a_r)$ . Therefore,

$$K_0(P) \subset K(\log_p \text{ind } a_1, \dots, \log_p \text{ind } a_r) \subset \mathbb{Z}[x].$$

An element of  $K_0(P)^{(j)}$  with  $j > 0$  is a polynomial  $f$  in  $h$  of degree at least  $j$ . The image of  $f$  in  $G^j K_0(\overline{P})$  is the  $j$ -th homogeneous part  $f_j$  of  $f$ . As the degree of  $f$  with respect to  $h_i$  is less than  $\text{ind } a_i$ , it follows from Proposition 2.3 that all the coefficients of  $f_j$  are divisible by  $p$ .  $\square$

Let  $d = \dim P$  and  $\alpha \in \text{CH}^d(P \times P)$ . The *first multiplicity*  $\text{mult}_1(\alpha)$  of  $\alpha$  is the image of  $\alpha$  under the push-forward map  $\text{CH}^d(P \times P) \rightarrow \text{CH}^0(P) = \mathbb{Z}$  given by the first projection  $P \times P \rightarrow P$  (cf. [10]). Similarly, we define the *second multiplicity*  $\text{mult}_2(\alpha)$ .

**Corollary 2.6.** *For any element  $\alpha \in \text{CH}^d(P \times P)$ , we have*

$$\text{mult}_1(\alpha) \equiv \text{mult}_2(\alpha) \pmod{p}.$$

*Proof.* We follow the proof of [9, Th. 2.1]. The homomorphism

$$f: \text{CH}^d(P \times P) \rightarrow (\mathbb{Z}/p\mathbb{Z})^2,$$

taking an  $\alpha \in \text{CH}^d(P \times P)$  to  $(\text{mult}_1(\alpha), \text{mult}_2(\alpha))$  modulo  $p$ , factors through the group  $\overline{\text{Ch}}^d(P \times P)$ . Since for any  $i$ , any projection  $P_i \times P_i \rightarrow P_i$  is a projective bundle, the Chow group  $\overline{\text{Ch}}^d(P \times P)$  is a direct sum of several copies of  $\overline{\text{Ch}}^i(P)$  for some  $i$ 's and the value  $i = 0$  appears once. By Proposition 2.5, the dimension over  $\mathbb{Z}/p\mathbb{Z}$  of the vector space  $\overline{\text{Ch}}^d(P \times P)$  is equal to 1 and consequently the dimension of the image of  $f$  is at most 1. Since the image of the diagonal class under  $f$  is  $(1, 1)$ , the image of  $f$  is generated by  $(1, 1)$ .  $\square$

**Corollary 2.7.** *Any rational map  $P \dashrightarrow P$  is dominant.*

*Proof.* Let  $\alpha \in \text{CH}^d(P \times P)$  be the class of the closure of the graph of a rational map  $P \dashrightarrow P$ . We have  $\text{mult}_1(\alpha) = 1$ . Therefore, by Corollary 2.6,  $\text{mult}_2(\alpha) \neq 0$ , and it follows that the rational map is dominant.  $\square$

**Corollary 2.8.**  $\text{cdim}_p P = \text{cdim } P = \dim P$ .

*Proof.* As  $\text{cdim}_p P \leq \text{cdim } P \leq \dim P$ , it suffices to show that  $\text{cdim}_p P = \dim P$ . Let  $Z \subset P$  be a closed subvariety and  $f: P' \dashrightarrow P$  and  $g: P' \dashrightarrow Z$  dominant rational morphisms such that  $\deg f$  is prime to  $p$ . Let  $\alpha$  be the class in  $\text{CH}^d(P \times P)$  of the closure in  $P \times P$  of the image of  $f \times g: P' \dashrightarrow P \times Z$ .

As  $\text{mult}_1(\alpha) = \deg f$  is prime to  $p$ , by Corollary 2.6, we have  $\text{mult}_2(\alpha) \neq 0$ , i.e.,  $Z = P$ . By Example 1.4.2,  $\text{cdim}_p P = \dim P$ .  $\square$

The corollary completes the proof of Theorem 2.1.

**Remark 2.9.** Theorem 2.1 can be generalized to the case of any finite subgroup  $D \subset \text{Br}(F)$  consisting of elements of  $p$ -primary orders. Let  $\{a_1, a_2, \dots, a_r\}$  be elements of  $D$  such that their images  $\{a'_1, a'_2, \dots, a'_r\}$  in  $D/D^p$  form a minimal basis, i.e., for any  $i = 1, \dots, r$  and any element  $d \in D$  with the class in  $D/D^p$  outside of the subgroup generated by  $a'_1, \dots, a'_{i-1}$ , the inequality  $\text{ind } d \geq \text{ind } a_i$  holds. In particular,  $\{a_1, a_2, \dots, a_r\}$  generate  $D$ . Then, as in Theorem 2.1, we have

$$\text{cdim}_p(D) = \text{cdim}(D) = \left( \sum_{i=1}^r \text{ind } a_i \right) - r .$$

Indeed, the group  $D$  and the variety  $P = P_1 \times \dots \times P_r$ , where  $P_i$  for every  $i = 1, \dots, r$  is the Severi-Brauer variety of a central division algebra representing the element  $a_i$ , have the same splitting fields. Therefore,  $\text{cdim}(D) = \text{cdim}(P)$  and  $\text{cdim}_p(D) = \text{cdim}_p(P)$ . Corollaries 2.6, 2.7 and 2.8 hold for  $P$  since  $K_0(P) \subset K(\log_p \text{ind } a_1, \dots, \log_p \text{ind } a_r)$ .

**Remark 2.10.** One can compute the canonical  $p$ -dimension of an arbitrary finite subgroup of  $D \subset \text{Br}(F)$  as follows. Let  $D'$  be the Sylow  $p$ -subgroup of  $D$ . Write  $D = D' \oplus D''$  for a subgroup  $D'' \subset D$  and let  $L/F$  be a finite field extension of degree prime to  $p$  such that  $D''$  is split over  $L$ . Then  $D_L = D'_L$  and  $\text{cdim}_p(D) = \text{cdim}_p(D_L) = \text{cdim}_p(D'_L) = \text{cdim}_p(D') = \text{cdim}(D')$ .

### 3. ESSENTIAL AND CANONICAL DIMENSION OF GERBES BANDED BY $(\mu_p)^s$

In this section we relate the essential and canonical ( $p$ -)dimensions of gerbes banded by  $(\mu_p)^s$  where  $s \geq 0$ . The following statement is a generalization of [4, Th. 7.1].

**Theorem 3.1.** *Let  $p$  be a prime integer and  $\mathcal{X}$  a gerbe banded by  $(\mu_p)^s$  over an arbitrary field  $F$ . Then*

$$\text{ed}(\mathcal{X}) = \text{ed}_p(\mathcal{X}) = \text{cdim}_p(\mathcal{X}) + s = \text{cdim}(\mathcal{X}) + s .$$

*Proof.* The gerbe  $\mathcal{X}$  is given by an element in  $H^2(F, (\mu_p)^s) = \text{Br}_p(F)^s$ , i.e., by an  $s$ -tuple of central simple algebras  $A_1, A_2, \dots, A_s$  with  $[A_i] \in \text{Br}_p(F)$ . Let  $P$  be the product of the Severi-Brauer varieties  $P_i := \text{SB}(A_i)$  and  $D$  the subgroup of  $\text{Br}_p(F)$  generated by the  $[A_i]$ ,  $i = 1, \dots, s$ . As the classes of splitting fields for  $\mathcal{X}$ ,  $D$  and  $P$  coincide, we have

$$(1) \quad \text{cdim}(\mathcal{X}) = \text{cdim}(P) = \text{cdim}(D) = \text{cdim}_p(D) = \text{cdim}_p(P) = \text{cdim}_p(\mathcal{X})$$

by Theorem 2.1. We shall prove the inequalities  $\text{ed}_p(\mathcal{X}) \geq \text{cdim}(P) + s \geq \text{ed}(\mathcal{X})$ .

Let  $S_i$  be a divisor on  $P_i$  of degree  $p$ . Let  $L/F$  be a field extension and  $f_i \in L(P_i)^\times$  with  $\text{div}(f_i) = pH_i - (S_i)_L$ , where  $H_i$  is a hyperplane in  $(P_i)_L$  for  $i = 1, \dots, s$ . We write  $\langle f_i \rangle_{i=1}^s$  for the corresponding element in  $\widehat{\mathcal{X}}(L)$  (cf. §1.2).

By Example 1.4.2, there is a closed subvariety  $Z \subset P$  and a rational dominant morphism  $P \dashrightarrow Z$  with  $\dim(Z) = \text{cdim}(P) = \text{cdim}_p(P)$ . We view  $F(Z)$  as a subfield of  $F(P)$ . As  $P(L) \neq \emptyset$  and  $P$  is regular, there is an  $F$ -place  $\gamma : F(P) \rightsquigarrow L$  (cf. [11, §4.1]). Since  $Z$  is complete, the valuation ring of the restriction  $\gamma|_{F(Z)} : F(Z) \rightsquigarrow L$  dominates a point in  $Z$ . It follows that  $Z(L) \neq \emptyset$ . Choose a point  $y \in Z$  such that  $F' := F(y) \subset L$ .

Since  $P(F') \neq \emptyset$ , the  $P_i$  are split over  $F'$ , hence  $\text{Pic}(P_i)_{F'} = \mathbb{Z}$  and there are functions  $g_i \in F'(P_i)^\times$  with  $\text{div}(g_i) = pH'_i - (S_i)_{F'}$ , where  $H'_i$  is a hyperplane in  $P_i$  for  $i = 1, \dots, s$ . As  $\text{Pic}(P_i)_L = \mathbb{Z}$ , there are functions  $h_i \in L(P_i)^\times$  with  $\text{div}(h_i) = (H'_i)_L - H_i$ . We have

$$\text{div}(g_i)_L = \text{div}(f_i) + \text{div}(h_i^p),$$

hence

$$a_i g_i = f_i h_i^p$$

for some  $a_i \in L^\times$ . It follows that  $\langle f_i \rangle_{i=1}^s = \langle a_i g_i \rangle_{i=1}^s$  in  $\mathcal{X}(L)$ , therefore  $\langle f_i \rangle_{i=1}^s$  is defined over the field  $F'(a_1, a_2, \dots, a_s)$ . Hence

$$\text{ed}\langle f_i \rangle_{i=1}^s \leq \text{tr. deg}_F(F') + s \leq \dim(Z) + s = \text{cdim}(P) + s,$$

and therefore  $\text{ed}(\mathcal{X}) \leq \text{cdim}(P) + s$ .

We shall prove the inequality  $\text{ed}_p(\mathcal{X}) \geq \text{cdim}(P) + s$ . As  $P(F(Z)) \neq \emptyset$ , there are functions  $f_i \in F(Z)(P_i)^\times$  with  $\text{div}(f_i) = pH_i - (S_i)_{F(Z)}$ , where  $H_i$  is a hyperplane in  $(P_i)_{F(Z)}$ . Let  $L := F(Z)(t_1, t_2, \dots, t_s)$ , where the  $t_i$  are variables, and consider the point  $\langle t_i f_i \rangle_{i=1}^s \in \widehat{\mathcal{X}}(L)$ .

We claim that  $\text{ed}_p\langle t_i f_i \rangle_{i=1}^s \geq \text{cdim}(P) + s$ . Let  $L'$  be a finite extension of  $L$  of degree prime to  $p$  and  $L'' \subset L'$  a subfield such that the image of  $\langle t_i f_i \rangle_{i=1}^s$  in  $\widehat{\mathcal{X}}(L')$  is defined over  $L''$ , i.e., there are functions  $g_i \in L''(P_i)^\times$  and  $h_i \in L'(P_i)^\times$  with  $t_i f_i = g_i h_i^p$ . We shall show that  $\text{tr. deg}_F(L'') \geq \text{cdim}(P) + s$ .

Let  $L_i := F(Z)(t_i, \dots, t_s)$  and  $v_i$  be the discrete valuation of  $L_i$  corresponding to the variable  $t_i$  for  $i = 1, \dots, s$ . We construct a sequence of field extensions  $L'_i/L_i$  of degree prime to  $p$  and discrete valuations  $v'_i$  of  $L'_i$  for  $i = 1, \dots, s$  by induction on  $i$  as follows. Set  $L'_1 = L$ . Suppose the fields  $L'_1, \dots, L'_i$  and the valuations  $v'_1, \dots, v'_{i-1}$  are constructed. By Lemma 1.1, there is a valuation  $v'_i$  of  $L'_i$  with residue field  $L'_{i+1}$  extending the discrete valuation  $v_i$  of  $L_i$  with the ramification index  $e_i$  and the degree  $[L'_{i+1} : L_{i+1}]$  prime to  $p$ .

The composition  $v'$  of the discrete valuations  $v'_i$  is a valuation of  $L'$  with residue field of degree over  $F(Z)$  prime to  $p$ . A choice of prime elements in all the  $L'_i$  identifies the group of values of  $v'$  with  $\mathbb{Z}^s$ . Moreover, for every  $i = 1, \dots, s$ , we have

$$v'(t_i) = e_i \varepsilon_i + \sum_{j>i} a_{ij} \varepsilon_j$$

where the  $\varepsilon_i$ 's denote the standard basis elements of  $\mathbb{Z}^s$  and  $a_{ij} \in \mathbb{Z}$ .

Write  $v''$  for the restriction of  $v'$  on  $L''$ . Let  $K = F(P)$ . We extend canonically the valuations  $v'$  and  $v''$  to valuations  $\tilde{v}'$  and  $\tilde{v}''$  of  $KL'$  and  $KL''$  respectively (cf. §1.5). Note that  $f_i \in K(Z)^\times$ ,  $g_i \in (KL'')^\times$  and  $h_i \in (KL')^\times$ . We

have

$$e_i \varepsilon_i + \sum_{j>i} a_{ij} \varepsilon_j = v'(t_i) = \tilde{v}'(t_i f_i) \equiv \tilde{v}''(g_i) \pmod{p}.$$

Since  $e_i$  are prime to  $p$ , the elements  $\tilde{v}''(g_i)$  generate a subgroup of  $\mathbb{Z}^s$  of finite index. It follows that the value group of  $\tilde{v}''$  is of rank  $s$ , hence  $\text{rank}(v'') = \text{rank}(\tilde{v}'') = s$ .

Let  $R''$  and  $R'$  be residue fields of  $v''$  and  $v'$  respectively. We have the inclusions  $R'' \subset R' \supset F(Z)$  and  $[R' : F(Z)]$  is prime to  $p$ . By [20, Ch. VI, Th. 3, Cor. 1],

$$(2) \quad \text{tr. deg}_F(L'') \geq \text{tr. deg}_F(R'') + \text{rank}(v'') = \text{tr. deg}_F(R'') + s.$$

As  $P(L'') \neq \emptyset$ , there is an  $F$ -place  $F(P) \rightsquigarrow L''$ . Composing it with the place  $L'' \rightsquigarrow R''$  given by  $v''$ , we get an  $F$ -place  $F(P) \rightsquigarrow R''$ . As  $P$  is complete, we have  $P(R'') \neq \emptyset$ , i.e.,  $R''$  is a splitting field of  $P$ .

We prove that  $R''$  is a  $p$ -generic splitting field of  $P$ . Let  $M$  be a splitting field of  $P$ . A regular system of parameters at the image of a morphism  $\alpha : \text{Spec } M \rightarrow P$  yields an  $F$ -place  $F(P) \rightsquigarrow M$  that is a composition of places associated with discrete valuations (cf. [11, §1.4]). By [11, Lemma 3.2] applied to the restriction of  $\alpha$  to  $F(Z)$ , there is a finite field extension  $M'$  of  $M$  and an  $F$ -place  $R' \rightsquigarrow M'$ . Restricting to  $R''$  we get an  $F$ -place  $R'' \rightsquigarrow M'$ , i.e.,  $R''$  is a  $p$ -generic splitting field of  $P$ .

By the definition of the canonical  $p$ -dimension,

$$\text{cdim}(P) = \text{tr. deg}_F F(Z) = \text{tr. deg}_F R' \geq \text{tr. deg}_F(R'') \geq \text{cdim}_p(P).$$

It follows that  $\text{tr. deg}_F(R'') = \text{cdim}(P)$  by (1) and therefore,  $\text{tr. deg}_F(L'') \geq \text{cdim}(P) + s$  by (2). The claim is proved.

It follows from the claim that  $\text{ed}_p(\mathcal{X}) \geq \text{cdim}(P) + s$ .  $\square$

#### 4. MAIN THEOREM

The main result of the paper is the following

**Theorem 4.1.** *Let  $G$  be a  $p$ -group and  $F$  a field of characteristic different from  $p$  containing a primitive  $p$ -th root of unity. Then  $\text{ed}_p(G)$  over  $F$  is equal to  $\text{ed}(G)$  over  $F$  and coincides with the least dimension of a faithful representation of  $G$  over  $F$ .*

The rest of the section is devoted to the proof of the theorem. As was mentioned in the introduction, we have  $\text{ed}_p(G) \leq \text{ed}(G) \leq \dim(V)$  for any faithful representation  $V$  of  $G$  over  $F$ . We shall construct a faithful representation  $V$  of  $G$  over  $F$  with  $\text{ed}_p(G) \geq \dim(V)$ .

Denote by  $C$  the subgroup of all central elements of  $G$  of exponent  $p$  and set  $H = G/C$ , so we have an exact sequence

$$(3) \quad 1 \rightarrow C \rightarrow G \rightarrow H \rightarrow 1.$$

Let  $E \rightarrow \text{Spec } F$  be an  $H$ -torsor and  $\text{Spec } F \rightarrow BH$  be the corresponding morphism. Set  $\mathcal{X}^E := BG \times_{BH} \text{Spec } F$ . Then  $\mathcal{X}^E$  is a gerbe over  $F$  banded by

$C$  and its class in  $H^2(F, C)$  coincides with the image of the class of  $E$  under the connecting map  $H^1(F, H) \rightarrow H^2(F, C)$  (cf. [13, Ch. 4, §2]). An object of  $\mathcal{X}^E$  over a field extension  $L/F$  is a pair  $(E', \alpha)$ , where  $E'$  is a  $G$ -torsor over  $L$  and  $\alpha : E'/C \xrightarrow{\sim} E_L$  is an isomorphism of  $H$ -torsors over  $L$ .

Alternatively,  $\mathcal{X}^E = [E/G]$  with objects (over  $L$ )  $G$ -equivariant morphisms  $E' \rightarrow E_L$ , where  $E'$  is a  $G$ -torsor over  $L$  (cf. [19]).

A lower bound for  $\text{ed}(G)$  was established in [4, Prop. 2.20]. We give a similar bound for  $\text{ed}_p(G)$ .

**Theorem 4.2.** *For any  $H$ -torsor  $E$  over  $F$ , we have  $\text{ed}_p(G) \geq \text{ed}_p(\mathcal{X}^E)$ .*

*Proof.* Let  $L/F$  be a field extension and  $x = (E', \alpha)$  an object of  $\mathcal{X}^E(L)$ . Choose a field extension  $L'/L$  of degree prime to  $p$  and a subfield  $L'' \subset L'$  over  $F$  such that  $\text{tr. deg}(L'') = \text{ed}_p(E')$  and there is a  $G$ -torsor  $E''$  over  $L''$  with  $E''_{L'} \simeq E'_{L'}$ .

Let  $Z$  be the (zero-dimensional) scheme of isomorphisms  $\text{Iso}_{L''}(E''/C, E_{L''})$  of  $H$ -torsors over  $L''$ . The image of the morphism  $\text{Spec } L' \rightarrow Z$  over  $L''$  representing the isomorphism  $\alpha_{L'}$  is a one point set  $\{z\}$  of  $Z$ . The field extension  $L''(z)/L''$  is algebraic since  $\dim Z = 0$ .

The isomorphism  $\alpha_{L'}$  descends to an isomorphism of the  $H$ -torsors  $E''/C$  and  $E$  over  $L''(z)$ . Hence the isomorphism class of  $x_{L'}$  belongs to the image of the map  $\widehat{\mathcal{X}}^E(L''(z)) \rightarrow \widehat{\mathcal{X}}^E(L')$ . Therefore,

$$\text{ed}_p(G) \geq \text{ed}_p(E') = \text{tr. deg}(L'') = \text{tr. deg}(L''(z)) \geq \text{ed}_p(x).$$

It follows that  $\text{ed}_p(G) \geq \text{ed}_p(\mathcal{X}^E)$ .  $\square$

Let  $C^* := \text{Hom}(C, \mathbf{G}_m)$  denote the character group of  $C$ . An  $H$ -torsor  $E$  over  $F$  yields a homomorphism

$$\beta^E : C^* \rightarrow \text{Br}(F)$$

taking a character  $\chi : C \rightarrow \mathbf{G}_m$  to the image of the class of  $E$  under the composition

$$H^1(F, H) \xrightarrow{\partial} H^2(F, C) \xrightarrow{\chi^*} H^2(F, \mathbf{G}_m) = \text{Br}(F),$$

where  $\partial$  is the connecting map for the exact sequence (3). Note that as  $\mu_p \subset F^\times$ , the intersection of  $\text{Ker}(\chi^*)$  over all characters  $\chi \in C^*$  is trivial. It follows that the classes of splitting fields of the gerbe  $\mathcal{X}^E$  and the subgroup  $\text{Im}(\beta^E)$  coincide. It follows that

$$(4) \quad \text{cdim}_p(\mathcal{X}^E) = \text{cdim}_p(\text{Im}(\beta^E)).$$

Let  $\chi_1, \chi_2, \dots, \chi_s$  be a basis of  $C^*$  over  $\mathbb{Z}/p\mathbb{Z}$  such that  $\{\beta^E(\chi_1), \dots, \beta^E(\chi_r)\}$  is a minimal basis of  $\text{Im}(\beta^E)$  for some  $r$  and  $\beta^E(\chi_i) = 1$  for  $i > r$ . By Theorem 2.1, we have

$$(5) \quad \text{cdim}_p(\text{Im}(\beta^E)) = \left( \sum_{i=1}^r \text{ind } \beta^E(\chi_i) \right) - r = \left( \sum_{i=1}^s \text{ind } \beta^E(\chi_i) \right) - s.$$

In view of (4) and Theorems 3.1 and 4.2, we shall find an  $H$ -torsor  $E$  (over a field extension of  $F$ ) so that the integer in (5) is as large as possible. Let  $U$  be a faithful representation of  $H$  and  $X$  an open subset of the affine space  $\mathbb{A}(U)$  of  $U$  where  $H$  acts freely. Set  $Y := X/H$ . Let  $E$  be the generic fiber of the  $H$ -torsor  $\pi : X \rightarrow Y$ . It is a “generic”  $H$ -torsor over the function field  $L := F(Y)$ .

Let  $\chi : C \rightarrow \mathbf{G}_m$  be a character and  $\text{Rep}^{(\chi)}(G)$  the category of all finite dimensional representations  $\rho$  of  $G$  such that  $\rho(c)$  is multiplication by  $\chi(c)$  for any  $c \in C$ . Fix a representations  $\rho : G \rightarrow \mathbf{GL}(W)$  in  $\text{Rep}^{(\chi)}(G)$ . The conjugation action of  $G$  on  $B := \text{End}(W)$  factors through an  $H$ -action. By descent (cf. [13, Ch. 1, §2]), there is (a unique up to canonical isomorphism) Azumaya algebra  $\mathcal{A}$  over  $Y$  and an  $H$ -equivariant algebra isomorphism  $\pi^*(\mathcal{A}) \simeq B_X := B \times X$ . Let  $A$  be the generic fiber of  $\mathcal{A}$ ; it is a central simple algebra over  $L = F(Y)$ .

Consider the homomorphism  $\beta^E : C^* \rightarrow \text{Br}(L)$ .

**Lemma 4.3.** *The class of  $A$  in  $\text{Br}(L)$  coincides with  $\beta^E(\chi)$ .*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \\ & & \chi \downarrow & & \rho \downarrow & & \alpha \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{GL}(W) & \longrightarrow & \mathbf{PGL}(W) & \longrightarrow & 1 \end{array}$$

The image of the  $H$ -torsor  $\pi : X \rightarrow Y$  under  $\alpha$  is the  $\mathbf{PGL}(W)$ -torsor

$$E' := \mathbf{PGL}(W)_X/H \rightarrow Y$$

where  $\mathbf{PGL}(W)_X := \mathbf{PGL}(W) \times X$  and  $H$  acts on  $\mathbf{PGL}(W)_X$  by  $h(a, x) = (ah^{-1}, hx)$ . The conjugation action of  $\mathbf{PGL}(W)$  on  $B$  gives rise to an isomorphism between  $\mathbf{PGL}(W)_X$  and the  $H$ -torsor  $\text{Iso}_X(B_X, \text{End}(W)_X)$  of isomorphisms between the (split) Azumaya  $\mathcal{O}_X$ -algebras  $B_X$  and  $\text{End}(W)_X$ . Note that this isomorphism is  $H$ -equivariant if  $H$  acts by conjugation on  $B_X$  and trivially on  $\text{End}(W)_X$ . By descent,

$$E' \simeq \text{Iso}_Y(\mathcal{A}, \text{End}(W)_Y).$$

Therefore, the image of the class of the torsor  $E' \rightarrow Y$  under the connecting map for the bottom row of the diagram coincides with the class of the Azumaya algebra  $\mathcal{A}$ . Restricting to the generic fiber yields  $[A] = \beta^E(\chi)$ .  $\square$

**Theorem 4.4.** *For any character  $\chi \in C^*$ , we have  $\text{ind } \beta^E(\chi) = \min \dim(V)$  over all representations  $V$  in  $\text{Rep}^{(\chi)}(G)$ .*

*Proof.* We follow the approach given in [12]. Let  $H$  act on a scheme  $Z$  over  $F$ . We also view  $Z$  as a  $G$ -scheme. Denote by  $\mathcal{M}(G, Z)$  the (abelian) category of left  $G$ -modules on  $Z$  that are coherent  $\mathcal{O}_Z$ -modules (cf. [18, §1.2]). In particular,  $\mathcal{M}(G, \text{Spec } F) = \text{Rep}(G)$ , the category of all finite dimensional representations of  $G$ .

Note that  $C$  acts trivially on  $Z$ . For a character  $\chi : C \rightarrow \mathbf{G}_m$ , let  $\mathcal{M}^{(\chi)}(G, Z)$  be the full subcategory of  $\mathcal{M}(G, Z)$  consisting of  $G$ -modules on which  $C$  acts via  $\chi$ . For example,  $\mathcal{M}^{(\chi)}(G, \text{Spec } F) = \text{Rep}^{(\chi)}(G)$ .

We write  $K_0(G, Z)$  and  $K_0^{(\chi)}(G, Z)$  for the Grothendieck groups of  $\mathcal{M}(G, Z)$  and  $\mathcal{M}^{(\chi)}(G, Z)$  respectively.

Every  $M$  in  $\mathcal{M}(G, Z)$  is a direct sum of unique submodules  $M^{(\chi)}$  of  $M$  in  $\mathcal{M}^{(\chi)}(G, Z)$  over all characters  $\chi$  of  $C$ . It follows that

$$K_0(G, Z) = \coprod K_0^{(\chi)}(G, Z).$$

Let  $q$  be the order of  $G$ . By [17, Th. 24], every irreducible representation of  $G$  is defined over the field  $F(\mu_q)$ . Since  $F$  contains  $p$ -th roots of unity, the degree  $[F(\mu_q) : F]$  is a power of  $p$ . Hence the dimension of any irreducible representation of  $G$  over  $F$  is a power of  $p$ . It follows by Lemma 4.3 that it suffices to show  $\text{ind}(A) = \text{gcd dim}(V)$  over all representations  $V$  in  $\text{Rep}^{(\chi)}(G)$ .

The image of the map  $\text{dim} : K_0(A) \rightarrow \mathbb{Z}$  given by the dimension over  $L$  is equal to  $\text{ind}(A) \cdot \text{dim}(W) \cdot \mathbb{Z}$ . To finish the proof of the theorem it suffices to construct a surjective homomorphism

$$(6) \quad K_0(\text{Rep}^{(\chi)}(G)) \rightarrow K_0(A)$$

such that the composition  $K_0(\text{Rep}^{(\chi)}(G)) \rightarrow K_0(A) \xrightarrow{\text{dim}} \mathbb{Z}$  is given by the dimension times  $\text{dim}(W)$ .

First of all we have

$$(7) \quad K_0(\text{Rep}^{(\chi)}(G)) \simeq K_0^{(\chi)}(G, \text{Spec } F).$$

Recall that  $X$  an open subset of  $\mathbb{A}(U)$  where  $H$  acts freely. By homotopy invariance in the equivariant  $K$ -theory [18, Cor. 4.2],

$$K_0(G, \text{Spec } F) \simeq K_0(G, \mathbb{A}(U)).$$

It follows that

$$(8) \quad K_0^{(\chi)}(G, \text{Spec } F) \simeq K_0^{(\chi)}(G, \mathbb{A}(U)).$$

By localization [18, Th. 2.7], the restriction homomorphism

$$(9) \quad K_0^{(\chi)}(G, \mathbb{A}(U)) \rightarrow K_0^{(\chi)}(G, X).$$

is surjective.

Denote by  $\mathcal{M}^{(1)}(G, X, B_X)$  the category of left  $G$ -modules  $M$  on  $X$  that are coherent  $\mathcal{O}_X$ -modules and right  $B_X$ -modules such that  $C$  acts trivially on  $M$  and the  $G$ -action on  $M$  and the conjugation  $G$ -action on  $B_X$  agree. The corresponding Grothendieck group is denoted by  $K_0^{(1)}(G, X, B_X)$ . For any object  $L$  in  $\mathcal{M}^{(\chi)}(G, X)$ , the group  $C$  acts trivially on  $L \otimes_F W^*$  and  $B$  acts on the right on  $L \otimes_F W^*$ . We have Morita equivalence

$$\mathcal{M}^{(\chi)}(G, X) \xrightarrow{\sim} \mathcal{M}^{(1)}(G, X, B_X)$$

given by  $L \mapsto L \otimes_F W^*$  (with the inverse functor  $M \mapsto M \otimes_B W$ ). Hence

$$(10) \quad K_0^{(\chi)}(G, X) \simeq K_0^{(1)}(G, X, B_X).$$

Now, as  $C$  acts trivially on  $X$  and  $B_X$ , the category  $\mathcal{M}^{(1)}(G, X, B_X)$  is equivalent to the category  $\mathcal{M}(H, X, B_X)$  of left  $H$ -modules  $M$  on  $X$  that are coherent  $\mathcal{O}_X$ -modules and right  $B_X$ -modules such that the  $G$ -action on  $M$  and the conjugation  $G$ -action on  $B_X$  agree. Hence

$$(11) \quad K_0^{(1)}(G, X, B_X) \simeq K_0(H, X, B_X).$$

Recall that  $Y = X/H$ . By descent, the category  $\mathcal{M}(H, X, B_X)$  is equivalent to the category  $\mathcal{M}(Y, \mathcal{A})$  of coherent  $\mathcal{O}_Y$ -modules that are right  $\mathcal{A}$ -modules. Hence

$$(12) \quad K_0(H, X, B_X) \simeq K_0(Y, \mathcal{A}).$$

The restriction to the generic point of  $Y$  gives a surjective homomorphism

$$(13) \quad K_0(Y, \mathcal{A}) \rightarrow K_0(A).$$

The homomorphism (6) is the composition of (7), (8), (9), (10), (11), (12) and (13). It takes the class of a representation  $V$  to the class in  $K_0(A)$  of the generic fiber of the vector bundle  $((V \otimes W^*) \times X)/H$  over  $Y$  of rank  $\dim(V) \cdot \dim(W)$ .  $\square$

**Remark 4.5.** The theorem holds with  $\min$  replaced by the  $\gcd$  (with the same proof) in a more general context when the sequence (3) is an arbitrary exact sequence of algebraic groups with  $C$  a central diagonalizable subgroup of  $G$ .

**Example 4.6** (cf. [6], [4, §14], [16, Th. 7.3.8]). Let  $p$  be a prime integer,  $F$  be a field of characteristic different from  $p$  and  $C_m$  the cyclic group  $\mathbb{Z}/p^m\mathbb{Z}$ . Let  $K = F(t_1, \dots, t_{p^m})$  and  $C_m$  act on the variables  $t_1, \dots, t_{p^m}$  by cyclic permutations. Then  $K$  is a Galois  $C_m$ -algebra over  $K^{C_m}$ . Assume that  $F$  contains a primitive root of unity  $\xi_{p^k}$  for some  $k$ . The image of the class of  $K$  under the connecting map  $H^1(F, C_m) \rightarrow H^2(F, C_k) \simeq \text{Br}_{p^k}(F)$  for the exact sequence

$$1 \rightarrow C_k \rightarrow C_n \rightarrow C_m \rightarrow 1,$$

where  $n = k + m$ , is the class of the cyclic algebra  $A = (K/K^{C_m}, \xi_{p^k})$ . The group  $C_n$  acts  $F$ -linearly on  $F(\xi_{p^n})$  by multiplication by roots of unity making the  $F$ -space  $F(\xi_{p^n})$  a faithful representation of  $C_n$  of the smallest dimension. By Theorem 4.4 and Remark 4.5, we have

$$\text{ind}(A) = [F(\xi_{p^n}) : F].$$

We can now complete the proof of Theorem 4.1. By Theorem 4.4, there are representations  $V_i$  in  $\text{Rep}^{(\chi_i)}(G)$  such that  $\text{ind} \beta^E(\chi_i) = \dim(V_i)$ ,  $i = 1, \dots, s$ . Let  $V$  be the direct sum of all the  $V_i$ . By Theorem 4.2 (applied to the group  $G$  over  $L$  and the generic torsor  $E$ ), Theorem 3.1, (4) and (5), we have

$$\begin{aligned} \text{ed}_p(G) &\geq \text{ed}_p(G_L) \geq \text{ed}_p(\mathcal{X}^E) = \text{cdim}_p(\mathcal{X}^E) + s = \text{cdim}_p(\text{Im}(\beta^E)) + s \\ &= \sum_{i=1}^s \text{ind} \beta^E(\chi_i) = \sum_{i=1}^s \dim(V_i) = \dim(V). \end{aligned}$$

Since  $\chi_1, \chi_2, \dots, \chi_s$  generate  $C^*$ , the restriction of  $V$  on  $C$  is faithful. As every nontrivial normal subgroup of  $G$  intersects  $C$  nontrivially, the  $G$ -representation  $V$  is faithful. We have constructed a faithful representation  $V$  of  $G$  over  $F$  with  $\text{ed}_p(G) \geq \dim(V)$ . The theorem is proved.

**Remark 4.7.** The proof of Theorem 4.1 shows how to compute the essential dimension of  $G$  over  $F$ . For every character  $\chi \in C^*$  choose a representation  $V_\chi \in \text{Rep}^{(x)}(G)$  of the smallest dimension. It appears as an irreducible component of the smallest dimension of the induced representation  $\text{Ind}_C^G(\chi)$ . We construct a basis  $\chi_1, \dots, \chi_s$  of  $C^*$  by induction as follows. Let  $\chi_1$  be a nonzero character with the smallest  $\dim(V_{\chi_1})$ . If the characters  $\chi_1, \dots, \chi_{i-1}$  are already constructed for some  $i \leq s$ , then we take for  $\chi_i$  a character with minimal  $\dim(V_{\chi_i})$  among all the characters outside of the subgroup generated by  $\chi_1, \dots, \chi_{i-1}$ . Then  $V$  is a faithful representation of the least dimension and  $\text{ed}(G) = \sum_{i=1}^s \dim(V_{\chi_i})$ .

**Remark 4.8.** We can compute the essential  $p$ -dimension of an arbitrary finite group  $G$  over a field  $F$  of characteristic different from  $p$ . (We don't assume that  $F$  contains  $p$ -th roots of unity.) Let  $G'$  a Sylow  $p$ -subgroup of  $G$ . One can prove that  $\text{ed}_p(G) = \text{ed}_p(G')$  and  $\text{ed}_p(G')$  does not change under field extensions of degree prime to  $p$ . In particular  $\text{ed}_p(G') = \text{ed}_p(G'_{F'})$  where  $F' = F(\mu_p)$ . It follows from Theorem 4.1 that  $\text{ed}_p(G)$  coincides with the least dimension of a faithful representation of  $G'$  over  $F'$ .

## 5. AN APPLICATION

**Theorem 5.1.** *Let  $G_1$  and  $G_2$  be two  $p$ -groups and  $F$  a field of characteristic different from  $p$  containing a primitive  $p$ -th root of unity. Then*

$$\text{ed}(G_1 \times G_2) = \text{ed}(G_1) + \text{ed}(G_2).$$

*Proof.* The index  $j$  in the proof takes the values 1 and 2. If  $V_j$  is a faithful representation of  $G_j$  then  $V_1 \oplus V_2$  is a faithful representation of  $G_1 \times G_2$ . Hence  $\text{ed}(G_1 \times G_2) \leq \text{ed}(G_1) + \text{ed}(G_2)$  (cf. [5, Lemma 4.1(b)]).

Denote by  $C_j$  the subgroup of all central elements of  $G_j$  of exponent  $p$ . Set  $C = C_1 \times C_2$ . We identify  $C^*$  with  $C_1^* \oplus C_2^*$ .

For every character  $\chi \in C^*$  choose a representation  $\rho_\chi : G_1 \times G_2 \rightarrow \mathbf{GL}(V_\chi)$  in  $\text{Rep}^{(x)}(G_1 \times G_2)$  of the smallest dimension. We construct a basis  $\{\chi_1, \chi_2, \dots, \chi_s\}$  of  $C^*$  following Remark 4.7. We claim that all the  $\chi_i$  can be chosen in one of the  $C_j^*$ . Indeed, suppose the characters  $\chi_1, \dots, \chi_{i-1}$  are already constructed, and let  $\chi_i$  be a character with minimal  $\dim(V_{\chi_i})$  among the characters outside of the subgroup generated by  $\chi_1, \dots, \chi_{i-1}$ . Let  $\chi_i = \chi_i^{(1)} + \chi_i^{(2)}$  with  $\chi_i^{(j)} \in C_j^*$ . Denote by  $\varepsilon_1$  and  $\varepsilon_2$  the endomorphisms of  $G_1 \times G_2$  taking  $(g_1, g_2)$  to  $(g_1, 1)$  and  $(1, g_2)$  respectively. The restriction of the representation  $\rho_{\chi_i} \circ \varepsilon_j$  on  $C$  is given by the character  $\chi_i^{(j)}$ . We replace  $\chi_i$  by  $\chi_i^{(j)}$  with  $j$  such that  $\chi_i^{(j)}$  does not belong to the subgroup generated by  $\chi_1, \dots, \chi_{i-1}$ . The claim is proved.

Let  $W_j$  be the direct sum of all the  $V_{\chi_i}$  with  $\chi_i \in C_j^*$ . Then the restriction of  $W_j$  on  $C_j$  is faithful, hence so is the restriction of  $W_j$  on  $G_j$ . It follows that  $\text{ed}(G_j) \leq \dim(W_j)$ . As  $W_1 \oplus W_2 = V$ , we have

$$\text{ed}(G_1) + \text{ed}(G_2) \leq \dim(W_1) + \dim(W_2) = \dim(V) = \text{ed}(G_1 \times G_2). \quad \square$$

**Corollary 5.2.** *Let  $F$  be a field as in Theorem 5.1. Then*

$$\text{ed}(\mathbb{Z}/p^{n_1}\mathbb{Z} \times \mathbb{Z}/p^{n_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_s}\mathbb{Z}) = \sum_{i=1}^s [F(\xi_{p^{n_i}}) : F].$$

*Proof.* By Theorem 5.1, it suffices to consider the case  $s = 1$ . This case has been done in [6]. It is also covered by Theorem 4.1 as the natural representation of the group  $\mathbb{Z}/p^n\mathbb{Z}$  in the  $F$ -space  $F(\xi_{p^n})$  is faithful irreducible of the smallest dimension (cf. Remark 4.6).  $\square$

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