LOCAL-GLOBAL PRINCIPLE FOR REDUCED NORMS OVER FUNCTION FIELDS OF p-ADIC CURVES

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Abstract. Let F be the function field of a p-adic curve. Let D be a central simple algebra over F of period n and $\lambda \in F^*$. We show that if n is coprime to p and $D \cdot (\lambda) = 0$ in $H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm. This leads to a Hasse principle for the $SL_1(D)$, namely an element $\lambda \in F^*$ is a reduced norm from D if and only if it is a reduced norm locally at all discrete valuations of F.

1. Introduction

Let K be a p-adic field and F a function field in one-variable over K. Let Ω_F be the set of all discrete valuations of F. Let G be a semi-simple simply connected linear algebraic group defined over F. It was conjectured in ([5]) that the Hasse principle holds for principal homogeneous spaces under G over F with respect to Ω_F; i.e. if X is a principal homogeneous space under G over F with $X(F_\nu) \neq \emptyset$ for all $\nu \in \Omega_F$, then $X(F) \neq \emptyset$. If G is $SL_1(D)$, where D is a central simple algebra over F of square free index, it follows from the injectivity of the Rost invariant ([19]) and a Hasse principle for $H^3(F, \mu_n)$ due to Kato ([16]), that this conjecture holds. This conjecture has been subsequently settled for classical groups of type B_n, C_n and D_n ([14], [23]). It is also settled for groups of type ${}^2A_n$ with the assumption that $n+1$ is square free ([14], [23]).

The main aim of this paper is to prove that the conjecture holds for $SL_1(D)$ for any central simple algebra D over F with index coprime to p. In fact we prove the following (11.1)

Theorem 1.1. *Let K be a p-adic field and F a function field in one-variable over K. Let D be a central simple algebra over F of index coprime to p and $\lambda \in F^*$. If $D \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.*

This together with Kato's result on the Hasse principle for $H^3(F, \mu_n)$ gives the following (11.2)

Theorem 1.2. *Let K be p-adic field and F the function field of a curve over K. Let Ω_F be the set of discrete valuations of F. Let D be a central simple algebra over F of index coprime to p and $\lambda \in F^*$. If λ is a reduced norm from $D \otimes F_\nu$ for all $\nu \in \Omega_F$, then λ is a reduced norm from D.*

In fact we may restrict the set of discrete valuations to the set of divisorial discrete valuations of F; namely those discrete valuations of F centered on a regular proper model of F over the ring of integers in K.

Here are the main steps in the proof. We reduce to the case where D is a division algebra of period ℓ^d with ℓ a prime not equal to p. Given a central division algebra D over F of period $n = \ell^d$ with $\ell \neq p$ and $\lambda \in F^*$ with $D \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, we construct a degree ℓ extension L of F and $\mu \in L^*$ such that $N_{L/F}(\mu) = \lambda$, $(D \otimes L) \cdot (\mu) = 0$ and the index of $D \otimes L$ is strictly smaller than the index of D.

Then, by induction on the index of D, μ is a reduced norm from $D \otimes L$ and hence $N_{L/F}(\mu) = \lambda$ is a reduced norm from D.

Let $\mathscr{X}$ be a regular proper 2-dimensional scheme over the ring of integers in K with function field F and X_0 the reduced special fibre of $\mathscr{X}$. By the patching techniques of Harbater-Hartman-Krashen ([9], [10]), construction of such a pair (L, μ) is reduced to the construction of compatible pairs (L_x, μ_x) over F_x for all $x \in X_0$ (7.5), where for any $x \in X_0$, F_x is the field of fractions of the completion of the regular local ring at x on $\mathscr{X}$. We use local and global class field theory to construct such local pairs (L_x, μ_x). Thus this method cannot be extended to the more general situation where F is a function field in one variable over a complete discretely valued field with arbitrary residue field.

Here is the brief description of the organization of the paper. In §3, we prove a few technical results concerning central simple algebras and reduced norms over global fields. These results are key to the later patching construction of the fields L_x and $\mu_x \in L_x$ with required properties.

In §4 we prove the following local variant of (1.1)

Theorem 1.3. *Let F be a complete discrete valued field with residue field κ. Suppose that κ is a local field or a global field. Let D be a central simple algebra over F of period n. Suppose that n is coprime to char(κ). Let $\alpha \in H^2(F, \mu_n)$ be the class of D and $\lambda \in F^*$. If $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.*

Let A be a complete regular local ring of dimension 2 with residue field κ finite, field of fractions F and maximal ideal $m = (\pi, \delta)$. Let ℓ be a prime not equal to char(κ). Let D be a central simple algebra over F of index ℓ^n with $n \geq 1$ and α the class of D in $H^2(F, \mu_{\ell^n})$. Suppose that D is unramified on A except possibly at π and δ. In §5, we analyze the structure of D. We prove that index of D is equal to the period of D. A similar analysis is done by Saltman ([25]) with the additional assumption that F contains all the primitive ℓ^n-roots of unity, where ℓ^n is the index of D. Let $\lambda \in F^*$. Suppose that $\lambda = u\pi^r\delta^t$ for some unit $u \in A$ and $r, s \in \mathbb{Z}$ and $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_{\ell^n}^{\otimes 2})$. In §6, we construct possible pairs (L, μ) with L/F of degree ℓ, $\mu \in L$ such that $N_{L/F}(\mu) = \lambda$, ind$(D \otimes L) <$ ind(D) and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_{\ell^n}^{\otimes 2})$.

Let K be a p-adic field and F a function field of a curve over K. Let ℓ be a prime not equal to p, D a central division algebra over F of index ℓ^n and α the class of D in $H^2(F, \mu_{\ell^n})$. Let $\lambda \in F^*$ with $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_{\ell^n}^{\otimes 2})$. Let $\mathscr{X}$ be a normal proper model of F over the ring of integers in K and X_0 its reduced special fibre. In §7, we reduce the construction of (L, μ) to the construction of local (L_x, μ_x) for all $x \in X_0$ with some compatible conditions along the "branches".

Further assume that $\mathscr{X}$ is regular and ram$_{\mathscr{X}}(\alpha) \cup$ supp$_{\mathscr{X}}(\lambda) \cup X_0$ is a union of regular curves with normal crossings. We group the components of X_0 into 8 types depending on the valuation of λ, index of D and the ramification type of D along those components. We call some nodal points of X_0 as special points depending on the type of components passing through the point. We also say that two components of X_0 are type 2 connected if there is a sequence of type 2 curves connecting these two components. We prove that there is a regular proper model of F with no special points and no type 2 connection between certain types of component (8.6).

Starting with a model constructed in (8.6), in §9, we construct (L_P, μ_P) for all nodal points of X_0 (9.8) with the required properties. In §10, using the class field results of §3, we construct (L_η, μ_η) for each of the components η of X_0 which are compatible with (L_P, μ_P) when P is in the component η.

Finally in §11, we prove the main results by piecing together all the constructions of §7, §9 and §10.

2. Preliminaries

In this section we recall a few definitions and facts about Brauer groups, Galois cohomology groups, residue homomorphisms and unramified Galois cohomology groups. We refer the reader to ([4]) and ([8]).

Let K be a field and $n \geq 1$. Let ${}_n\mathrm{Br}(K)$ be the n-torsion subgroup of the Brauer group $\mathrm{Br}(K)$. Assume that n is coprime to the characteristic of K. Let μ_n be the group of n^{th} roots of unity. For $d \geq 1$ and $m \geq 0$, let $H^d(K, \mu_n^{\otimes m})$ denote the d^{th} Galois cohomology group of K with values in $\mu_n^{\otimes m}$. We have $H^1(K, \mu_n) \simeq K^*/K^{*n}$ and $H^2(K, \mu_n) \simeq {}_n\mathrm{Br}(K)$. For $a \in K^*$, let $(a)_n \in H^1(K, \mu_n)$ denote the image of the class of a in K^*/K^{*n}. When there is no ambiguity of n, we drop n and denote $(a)_n$ by (a). If K is a product of finitely many fields K_i, we denote $\prod H^d(K_i, \mu_n^{\otimes m})$ by $H^d(K, \mu_n^{\otimes m})$.

Every element of $H^1(K, \mathbb{Z}/n\mathbb{Z})$ is represented by a pair (E, σ), where E/F is a cyclic extension of degree dividing n and σ a generator of $\mathrm{Gal}(E/F)$. Let $r \geq 1$. Then $(E, \sigma)^r \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ is represented by the pair (E', σ') where E' is the fixed field of the subgroup of $\mathrm{Gal}(E/F)$ generated by $\sigma^{n/d}$, where $d = \gcd(n, r)$ and $\sigma' = \sigma^r$. In particular if r is coprime to n, then $(E, \sigma)^r = (E, \sigma^r)$. Let $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ and $\lambda \in K^*$. Let $(E, \sigma, \lambda) = (E/F, \sigma, \lambda)$ denote the cyclic algebra over K

$$(E, \sigma, \lambda) = E \oplus Ly \oplus \cdots \oplus Ey^{n-1}$$

with $y^n = \lambda$ and $ya = \sigma(a)y$. The cyclic algebra (E, σ, λ) is a central simple algebra and its index is the order of λ in $K^*/N_{E/K}(E^*)$ ([1, Theorem 18, p. 98]). The pair (E, σ) represents an element in $H^1(K, \mathbb{Z}/n\mathbb{Z})$ and the element $(E, \sigma)\cdot(\lambda) \in H^2(K, \mu_n)$ is represented by the central simple algebra (E, σ, λ). In particular $(E, \sigma, \lambda) \otimes E$ is a matrix algebra and hence $\mathrm{ind}(E, \sigma, \lambda) \leq [E : F]$.

For $\lambda, \mu \in K^*$ we have ([1, p. 97])

$$(E, \sigma, \lambda) + (E, \sigma, \mu) = (E, \sigma, \lambda\mu) \in H^2(K, \mu_n).$$

In particular $(E, \sigma, \lambda^{-1}) = -(E, \sigma, \lambda)$.

Let (E, σ, λ) be a cyclic algebra over a field K and L/K be a field extension. Since E/K is separable, $E \otimes_K L$ is a product of field extensions E_i, $1 \leq i \leq t$, of L with E_i and E_j isomorphic over L and E_i/L is cyclic with Galois group a subgroup of the Galois group of E/K. Then $(E, \sigma, \lambda) \otimes L$ is Brauer equivalent to (E_i, σ_i, λ) for any i, with a suitable σ_i. In particular if L is a finite extension of K and EL is the composite of E and L in an algebraic closure of K, then EL/L is cyclic with Galois group isomorphic to a subgroup of the Galois group of E/K and $(E, \sigma, \lambda) \otimes L$ is Brauer equivalent to (EL, σ', λ) for a suitable σ'.

Lemma 2.1. *Let E/F be a cyclic extension of degree n, σ a generator of Gal(E/F) and $\lambda \in F^*$. Let m be a factor of n and $d = n/m$. Let M/F be the subextension of E/F with $[M : F] = m$. Then* $(E/F, \sigma, \lambda) \otimes F(\sqrt[d]{\lambda}) = (M(\sqrt[d]{\lambda})/F(\sqrt[d]{\lambda}), \sigma \otimes 1, \sqrt[d]{\lambda})$.

Proof. We have $(E,\sigma)^d = (M,\sigma) \in H^1(F,\mathbb{Z}/n\mathbb{Z})$ and hence

$$\begin{aligned}(E,\sigma,\lambda)\otimes F(\sqrt[d]{\lambda}) &= (E(\sqrt[d]{\lambda})/F(\sqrt[d]{\lambda}),\sigma\otimes 1,\lambda)\\ &= (E(\sqrt[d]{\lambda})/F(\sqrt[d]{\lambda}),\sigma\otimes 1,(\sqrt[d]{\lambda})^d)\\ &= (E(\sqrt[d]{\lambda})/F(\sqrt[d]{\lambda}),\sigma\otimes 1)^d\cdot(\sqrt[d]{\lambda})\\ &= (M(\sqrt[d]{\lambda})/F(\sqrt[d]{\lambda}),\sigma\otimes 1,\sqrt[d]{\lambda}).\end{aligned}$$

□

Let K be a field with a discrete valuation ν, residue field κ and valuation ring R. Suppose that n is coprime to the characteristic of κ. For any $d \geq 1$, we have the residue map $\partial_K : H^d(K,\mu_n^{\otimes i}) \to H^{d-1}(\kappa,\mu_n^{\otimes i-1})$. We also denote ∂_K by ∂. An element α in $H^d(K,\mu_n^{\otimes i})$ is called *unramified* at ν or R if $\partial(\alpha)=0$. The subgroup of all unramified elements is denoted by $H^d_{nr}(K/R,\mu_n^{\otimes i})$ or simply $H^d_{nr}(K,\mu_n^{\otimes i})$. Suppose that K is complete with respect to ν. Then we have an isomorphism $H^d(\kappa,\mu_n^{\otimes i}) \stackrel{\sim}{\to} H^d_{nr}(K,\mu_n^{\otimes i})$ and the composition $H^d(\kappa,\mu_n^{\otimes i}) \stackrel{\sim}{\to} H^d_{nr}(K,\mu_n^{\otimes i}) \hookrightarrow H^d(K,\mu_n^{\otimes i})$ is denoted by ι_κ or simply ι.

Let K be a complete discretely valued field with residue field κ, ν the discrete valuation on K and $\pi \in K^*$ a parameter. Suppose that n is coprime to the characteristic of κ. Let $\partial : H^2(K,\mu_n) \to H^1(\kappa,\mathbb{Z}/n\mathbb{Z})$ be the residue homomorphism. Let E/K be a cyclic unramified extension of degree n with residue field E_0 and σ a generator of $\mathrm{Gal}(E/K)$ with $\sigma_0 \in \mathrm{Gal}(E_0/\kappa)$ induced by σ. Then (E,σ,π) is a division algebra over K of degree n. For any $\lambda \in K^*$, we have

$$\partial(E,\sigma,\lambda) = (E_0,\sigma_0)^{\nu(\lambda)}.$$

For $\lambda,\mu \in K^*$, we have

$$\partial((E,\sigma,\lambda)\cdot(\mu)) = (E_0,\sigma_0)\cdot((-1)^{\nu(\lambda)\nu(\mu)}\theta),$$

where θ is the image of $\frac{\lambda^{\nu(\mu)}}{\mu^{\nu(\lambda)}}$ in the residue field.

Suppose E_0 is a cyclic extension of κ of degree n. Then there is a unique unramified cyclic extension E of K of degree n with residue field E_0. Let σ_0 be a generator of $\mathrm{Gal}(E_0/\kappa)$ and $\sigma \in \mathrm{Gal}(E/K)$ be the lift of σ_0. Then σ is a generator of $\mathrm{Gal}(E/K)$. We call the pair (E,σ) the *lift of* (E_0,σ_0).

Let X be an integral regular scheme with function field F. For every point x of X, let $\mathscr{O}_{X,x}$ be the regular local ring at x and $\kappa(x)$ the residue field at x. Let $\hat{\mathscr{O}}_{X,x}$ be the completion of $\mathscr{O}_{X,x}$ at its maximal ideal m_x and F_x the field of fractions of $\hat{\mathscr{O}}_{X,x}$. Then every codimension one point x of X gives a discrete valuation ν_x on F. Let $n \geq 1$ be an integer which is a unit on X. For any $d \geq 1$, the residue homomorphism $H^d(F,\mu_n^{\otimes j}) \to H^{d-1}(\kappa(x),\mu_n^{\otimes(j-1)})$ at the discrete valuation ν_x is denoted by ∂_x. An element $\alpha \in H^d(F,\mu_n^{\otimes m})$ is said to be *ramified* at x if $\partial_x(\alpha)\neq 0$ and *unramified* at x if $\partial_x(\alpha)=0$. If $X=\mathrm{Spec}(A)$ and x a point of X given by (π), π s prime element, we also denote F_x by F_π and $\kappa(x)$ by $\kappa(\pi)$.

Lemma 2.2. *Let K be a complete discretely valued field and ℓ a prime not equal to the characteristic of the residue field of K. Suppose that K contains a primitive ℓ^{th} root of unity. Let L/K be a cyclic field extension or the split extension of degree ℓ. Let $\mu \in L$ and $\lambda = N_{L/K}(\mu) \in K$. Then there exists $\theta \in L$ with $N_{L/K}(\theta)=1$ such that $L=K(\mu\theta)$ and θ is sufficiently close to 1.*

Proof. Since $[L:K]$ is a prime, if $\mu \notin K$, then $L=K(\mu)$. In this case $\theta=1$ has the required properties. Suppose that $\mu\in K$. If $L=\prod K$, let $\theta_0 \in K^*\setminus\{\pm 1\}$

sufficiently close to 1 and $\theta = (\theta_0, \theta_0^{-1}, 1, \cdots, 1)$. Suppose L is a field. Let σ be a generator of $\mathrm{Gal}(L/K)$. Since L/K is cyclic, we have $L = K(\sqrt[\ell]{a})$ for some $a \in K^*$. For any sufficiently large n, $\theta = (1 + \pi^n \sqrt[\ell]{a})^{-1}\sigma(1 + \pi^n\sqrt[\ell]{a}) \in L$ has the required properties. □

Lemma 2.3. *Let K be a field and E/K be a finite extension of degree coprime to char(K). Let L/K be a sub-extension of E/K and $e = [E : L]$. Suppose L/K is Galois and $E = L(\sqrt[e]{\pi})$ for some $\pi \in L^*$. Then E/K is Galois if and only if E contains a primitive e^{th} root of unity and for every $\tau \in Gal(L/K)$, $\tau(\pi) \in E^{*e}$.*

Proof. Suppose that E/K is Galois. Let $f(X) = X^e - \pi \in L[X]$. Since $[E : L] = e$ and $E = L(\sqrt[e]{\pi})$, $f(X)$ is irreducible in $L[X]$. Since $f(X)$ has one root in E and E/L is Galois, $f(X)$ has all the roots in E. Hence E contains a primitive e^{th} root of unity. Let $\tau \in \mathrm{Gal}(L/K)$. Then τ can be extended to an automorphism $\tilde{\tau}$ of E. We have $\tau(\pi) = \tilde{\tau}(\pi) = (\tilde{\tau}(\sqrt[e]{\pi}))^e \in E^{*e}$.

Conversely, suppose that E contains a primitive e^{th} root of unity and $\tau(\pi) \in E^{*e}$ for every $\tau \in \mathrm{Gal}(L/K)$. Let

$$g(X) = \prod_{\tau \,\in\, Gal(L/K)} (X^e - \tau(\pi)).$$

Then $g(X) \in K[X]$ and $g(X)$ splits completely in E. Since e is coprime to char(K), the splitting field E_0 of $g(X)$ over K is Galois. Since L/K is Galois and E is the composite of L and E_0, E/K is Galois. □

Lemma 2.4. *Let F be a complete discretely valued field with residue field κ and $\pi \in F$ a parameter. Let e be a natural number coprime to the characteristic of κ. If L/F is a totally ramified extension of degree e, then $L = F(\sqrt[e]{v\pi})$ for some $v \in F$ which is a unit in the valuation ring of F. Further if e is a power of a prime ℓ and $\theta \in F^* \setminus F^{*\ell}$ is a norm form L, then $L = F(\sqrt[e]{\theta})$.*

Proof. Since F is a complete discretely valued field, there is a unique extension of the valuation ν on F to a valuation ν_L on L. Since L/F is totally ramified extension of degree e and e is coprime to char(κ), the residue field of L is κ and $\nu_L(\pi) = e$. Let $\pi_L \in L$ with $\nu_L(\pi_L) = 1$. Then $\pi = w\pi_L^e$ for some $w \in L$ with $\nu_L(w) = 0$. Since the residue field of L is same as the residue field of F, there exists $w_1 \in F$ with $\nu(w_1) = 0$ and the image of w_1 is same as the image of w in the residue field κ. Since L is complete and e is coprime to char(κ), by Hensel's Lemma, there exists $u \in L$ such that $w = w_1u^e$. Thus $\pi = w\pi_L^e = w_1u^e\pi_L^e = w_1(u\pi_L)^e$. In particular $w_1^{-1}\pi \in L^{*e}$ and hence $L = F(\sqrt[e]{v\pi})$ with $v = w_1^{-1}$.

Let $\theta \in F^* \setminus F^{*e}$. Suppose that θ is a norm from L. Let $\mu \in L$ with $N_{L/F}(\mu) = \theta$. Since $L = F(\sqrt[e]{v\pi})$ with $v \in F$ a unit in the valuation ring of F and $\pi \in F$ a parameter, $\sqrt[e]{v\pi} \in L$ is a parameter at the valuation of L. Write $\mu = w_0(\sqrt[e]{v\pi})^s$ for some $w_0 \in L$ a unit at the valuation of L and $s \in \mathbb{Z}$. As above, we have $w_0 = v_1u_1^e$ for some $v_1 \in K$ and $u_1 \in L$. Since $v_1 \in F$, we have

$$\theta = N_{L/F}(\mu) = N_{L/F}(w_0(\sqrt[e]{v\pi})^s) = N_{L/F}(v_1u_1^e(\sqrt[e]{v\pi})^s) = v_1^eN_{L/F}(u_1)^e(v\pi)^s.$$

Since e is a power of a prime ℓ and $\theta \notin F^{*\ell}$, s is coprime to ℓ and hence $L = F(\sqrt[e]{\theta})$. □

Lemma 2.5. *Let k be a local field and ℓ a prime not equal to the characteristic of the residue field of k. Let L_0/k be a an extension of degree ℓ and $\theta_0 \in k^*$. If $\theta_0 \notin k^{*\ell}$ and θ_0 is a norm from L_0, then $L_0 = k(\sqrt[\ell]{\theta_0})$.*

Proof. Suppose that L_0/k is ramified. Since $\theta_0 \notin k^{*\ell}$, by (2.4), $L_0 = k(\sqrt[\ell]{\theta_0})$.

Suppose that L_0/k is unramified. Let π be a parameter in k and write $\theta_0 = u\pi^r$ with u a unit in the valuation ring of k. Since θ_0 is a norm from L_0, ℓ divides r. Since θ_0 not an ℓ^{th} power in k, u is not an ℓ^{th} power in k and $k(\sqrt[\ell]{\theta_0}) = k(\sqrt[\ell]{u})$ is an unramified extension of k of degree ℓ. Since k is a local field, there is only one unramified field extension of k of degree ℓ and hence $L_0 = k(\sqrt[\ell]{u}) = k(\sqrt[\ell]{\theta_0})$. □

Lemma 2.6. *Suppose F is a complete discretely valued field with residue feld κ a local field. Let ℓ be prime not equal to char(κ). Let L/F be a degree ℓ field extension with θ a norm from L. If $\theta \notin F^{*\ell}$, then $L \simeq F(\sqrt[\ell]{\theta})$.*

Proof. If L/F is a ramified extension, then by (2.4), $L \simeq F(\sqrt[\ell]{\theta})$. Suppose that L/F is an unramified extension. Let L_0 be the residue field of L. Then L_0/κ is a field extension of degree ℓ and the image $\overline{\theta}$ of θ in κ is a norm from L_0. Since $\theta \notin F^{*\ell}$, $\overline{\theta}$ is not an ℓ^{th} power in κ. Since κ is a local field, $L_0 \simeq \kappa(\sqrt[\ell]{\overline{\theta}})$ (2.5) and hence $L \simeq F(\sqrt[\ell]{\theta})$. □

Lemma 2.7. *Let F be a complete discretely valued field with residue field k a global field. Let L/F be an unramified cyclic extension of degree coprime to char(k) and L_0 the residue field of L. Let $\theta \in F$ be a unit in the valuation ring of F and $\overline{\theta}$ be the image of θ in k. Suppose that θ is a norm from L. If $\mu_0 \in L_0$ with $N_{L_0/k}(\mu_0) = \overline{\theta}$, then there exists $\mu \in L$ such that $N_{L/F}(\mu) = \theta$ and the image of μ in L_0 is μ_0.*

Proof. Let σ be a generator of the Galois group of L/F and σ_0 be the induced automorphism of L_0/k. Since $\theta \in F$ is a norm from L, there exists $\mu' \in L$ with $N_{L/F}(\mu') = \theta$. Since θ is a unit at the discrete valuation of F, $\mu' \in L$ is a unit at the discrete valuation of L. Let $\overline{\mu}'$ be the image of μ' in L_0. Then $N_{L_0/k}(\overline{\mu'}) = \overline{\theta}$ and hence $\overline{\mu}'\mu_0^{-1} \in L_0$ is a norm one element. Thus there exist $a \in L_0$ such that $\overline{\mu}'\mu_0^{-1} = a^{-1}\sigma_0(a)$. Let $b \in L$ be a lift of a and $\mu = \mu' b\sigma(b)^{-1}$. Then $N_{L/F}(\mu) = N_{L/F}(\mu') = \theta$ and the image of μ in L_0 is μ_0. □

For $L = \prod_1^\ell F$, let σ be the automorphism of L given by $\sigma(a_1, \cdots, a_\ell) = (a_2, \cdots, a_\ell, a_1)$. Then any σ^i, $1 \leq i \leq \ell - 1$ is called a *generator* of Gal(L/F).

Lemma 2.8. *Let F be a field and ℓ a prime not equal to the characteristic of F. Let L be a cyclic extension of F or the split extension of degree ℓ and σ a generator of the Galois group of L/F. Suppose that there exists an integer $t \geq 1$ such that F does not contain a primitive $\ell^{t^{\text{th}}}$ root of unity. Let $\mu \in L$ with $N_{L/F}(\mu) = 1$ and $m \geq t$. If $\mu \in L^{*\ell^{2m}}$, then there exists $b \in L^*$ such that $\mu = b^{-\ell^m}\sigma(b^{\ell^m})$.*

Proof. Suppose $L = \prod F$ and $\mu \in L^{*\ell^s}$ for some $s \geq 1$ with $N_{L/F}(\mu) = 1$. Then $\mu = (\theta_1^s, \cdots, \theta_\ell^s) \in L$ with $\theta_1^s \cdots \theta_\ell^s = 1$. Let $b = (1, \theta_1, \cdots, \theta_{\ell-1}) \in L^*$. Then $\mu = b^{-s}\sigma(b^s)$.

Suppose L/F is a cyclic field extension. Write $\mu = \mu_0^{\ell^{2m}}$ for some $\mu_0 \in L$. Let $\mu_1 = \mu_0^{\ell^m}$. Then $\mu = \mu_1^{\ell^m}$. Let $\theta_0 = N_{L/F}(\mu_0)$ and $\theta_1 = N_{L/F}(\mu_1)$. Then $\theta_1 = \theta_0^{\ell^m}$. Since $N_{L/F}(\mu) = 1$, we have $\theta_1^{\ell^m} = N_{L/F}(\mu_1^{\ell^m}) = 1$. If $\theta_1 \neq 1$, then F contains a primitive $\ell^{m^{\text{th}}}$ root of unity. Since $m \geq t$ and F has no primitive $\ell^{t^{\text{th}}}$ root of unity, $\theta_1 = 1$. Hence $N_{L/F}(\mu_1) = 1$ and by Hilbert 90, $\mu_1 = b^{-1}\sigma(b)$ for some $b \in L$. Thus $\mu = \mu_1^{\ell^m} = b^{-\ell^m}\sigma(b^{\ell^m})$. □

3. Global fields

In this a section we prove a few technical results concerning Brauer group of global fields and reduced norms. We begin with the following.

Lemma 3.1. *Let k be a global field, ℓ a prime not equal char(k), $n, d \geq 2$ and $r \geq 1$ be integers. Let E_0 be a cyclic extension of k, σ_0 a generator of the Galois group of E_0/k and $\theta_0 \in k^*$. Let $\beta \in H^2(k, \mu_{\ell^n})$ be such that $r\ell\beta = (E_0, \sigma_0, \theta_0) \in H^2(k, \mu_{\ell^n})$. Let S be a finite set of places of k containing all the places of κ with $\beta \otimes k_\nu \neq 0$. Suppose for each $\nu \in S$, there is a field extension L_ν of k_ν of degree ℓ or L_ν is the split extension of k_ν of degree ℓ and $\mu_\nu \in L_\nu^*$ such that*
1) $N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0$
2) $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$
3) ind$(\beta \otimes E_0 \otimes L_\nu) < d$.
Then there exists a field extension L_0/k of degree ℓ and $\mu_0 \in L_0$ such that
1) $N_{L_0/k}(\mu_0) = \theta_0$
2) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$
3) ind$(\beta \otimes E_0 \otimes L_0) < d$
4) $L_0 \otimes k_\nu \simeq L_\nu$ for all $\nu \in S$.
5) μ_0 is close to μ_ν for all $\nu \in S$.

Proof. Let Ω_k be the set of all places of k and

$$S' = S \cup \{\nu \in \Omega_k \mid \theta_0 \text{ is not a unit at } \nu \text{ or } E_0/k \text{ is ramified at } \nu\}$$

Let $\nu \in S' \setminus S$. Then $\beta \otimes k_\nu = 0$. Let L_ν be a field extension of k_ν of degree ℓ such that $\theta_0 \in N(L_\nu^*)$. Let $\mu_\nu \in L_\nu$ with $N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0$. Since $\beta \otimes k_\nu = 0$, ind$(\beta \otimes E_0 \otimes L_\nu) = 1 < d$. Since the corestriction map cor : $H^2(L_\nu, \mu_{\ell^n}) \to H^2(k_\nu, \mu_{\ell^n})$ is injective (cf. [17, Theorem 10, p. 237]) and cor$(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = (E_0 \otimes k_\nu, \sigma_0 \otimes 1, \theta_0) = r\ell\beta \otimes k_\nu = 0$, $(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = 0 = r\beta \otimes L_\nu$. Thus, if necessary, by enlarging S, we assume that S contains all those places ν of k with either θ_0 is not a unit at ν or E_0/k is ramified at ν and that there is at least one $\nu \in S$ such that L_ν is a field extension of k_ν of degree ℓ.

Let $\nu \in S$. By (2.2), there exists $\theta_\nu \in L_\nu$ such that $N_{L_\nu/k_\nu}(\theta_\nu) = 1$, $L_\nu = k_\nu(\theta_\nu\mu_\nu)$ and θ_ν is sufficiently close to 1. In particular $\theta_\nu \in L_\nu^{\ell^n}$ and hence $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu\theta_\nu)$. Thus, replacing μ_ν by $\mu_\nu\theta_\nu$, we assume that $L_\nu = k_\nu(\mu_\nu)$. Let $f_\nu(X) = X^\ell + b_{\ell-1,\nu}X^{\ell-1} + \cdots + b_{1,\nu}X + (-1)^\ell\theta_0 \in k_\nu[X]$ be the minimal polynomial of μ_ν over k_ν.

By Chebotarev density theorem ([7, Theorem 6.3.1]), there exists $\nu_0 \in \Omega_k \setminus S$ such that $E_0 \otimes k_{\nu_0}$ is the split extension of k_{ν_0}. By the strong approximation theorem ([3, p. 67]), choose $b_j \in k$, $0 \leq j \leq \ell - 1$ such that each b_j is sufficiently close enough to $b_{j,\nu}$ for all $\nu \in S$ and each b_j is an integer at all $\nu \notin S \cup \{\nu_0\}$. Let $L_0 = k[X]/(X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + b_1X + (-1)^\ell\theta_0)$ and $\mu_0 \in L_0$ be the image of X. We now show that L_0 and μ_0 have the required properties.

Since each b_j is sufficiently close enough to $b_{j,\nu}$ at each $\nu \in S$, it follows from Krasner's lemma that $L_0 \otimes k_\nu \simeq L_\nu$ and the image of $\mu_0 \otimes 1$ in L_ν is close to μ_ν for all $\nu \in S$ (cf. [26, Ch. II, §2]). Since L_ν is a field extension of k_ν of degree ℓ for at least one $\nu \in S$, L_0 is a field extension of degree ℓ over k. Since $X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + (-1)^\ell\theta_0$ is the minimal polynomial of μ_0, we have $N(\mu_0) = \theta_0$.

To show that ind$(\beta \otimes E_0 \otimes L_0) < d$ and $r\beta = (E_0, \sigma_0, \mu_0) \in H^2(L_0, \mu_{\ell^n})$, by Hasse-Brauer-Noether theorem (cf. [3, p. 187]), it is enough to show that for every place w of L_0, ind$(\beta \otimes E_0 \otimes L_w) < d$ and $r\beta \otimes L_w = (E_0, \sigma_0, \mu_0) \otimes L_w \in H^2(L_w, \mu_{\ell^n})$.

Let w be a place of L_0 and ν a place of k lying below w. Suppose that $\nu \in S$. Then $L_0 \otimes k_\nu \simeq L_\nu$. Suppose $L_\nu = \prod k_\nu$ is the split extension. Then $L_w \simeq k_\nu$. By the assumption on L_ν, we have ind$(\beta \otimes E_0 \otimes k_\nu) < d$. Since μ_ν is close to μ_0, we have $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0, \mu_\nu) = (E_0 \otimes L \otimes k_\nu, \sigma_0, \mu_0)$.

Suppose that L_ν is a field extension of k_ν of degree ℓ. Then $L_w \simeq L_0 \otimes k_\nu \simeq L_\nu$ and by the assumption on L_ν, we have $r\beta \otimes L_\nu = (E_0, \sigma_0, \mu_\nu) \otimes L_\nu$ and ind$(\beta \otimes E_0 \otimes L_\nu) < d$. Since μ_0 is close to μ_ν, we have $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0, \mu_\nu) = (E_0 \otimes L \otimes k_\nu, \sigma_0, \mu_0)$.

Suppose that $\nu \notin S$ and $\nu \neq \nu_0$. Then θ_0 is a unit at ν, E_0/k is unramified at ν and $\beta \otimes k_\nu = 0$. Since each b_j is an integer at ν and μ_0 is a root of the polynomial $X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + b_1 X + (-1)^\ell \theta_0$, μ_0 is an integer at w. Since θ_0 is a unit at ν, μ_0 is a unit at w. In particular $(E_0 \otimes L_w, \sigma_0, \mu_0) = 0 = r\beta \otimes L_w$. If $\nu = \nu_0$, then by the choice of ν_0, $\beta \otimes k_\nu = 0$, $E_0 \otimes k_\nu$ is the split extension of k_ν and hence $(E_0, \sigma_0, \mu_0) \otimes L_w = 0 = r\beta \otimes L_w$. □

Corollary 3.2. *Let k be a global field, ℓ a prime not equal char(k), n and $r \geq 1$ be integers. Let $\theta_0 \in k^*$, $r \geq 1$ and $\beta \in H^2(k, \mu_{\ell^n})$. Suppose that $r\ell\beta = 0 \in H^2(k, \mu_{\ell^n})$ and $\beta \neq 0$. Then there exists a field extension L_0/k of degree ℓ and $\mu_0 \in L_0$ such that $N_{L_0/k}(\mu_0) = \theta_0$, $r\beta \otimes L_0 = 0$ and ind$(\beta \otimes L_0) <$ ind(β).*

Proof. Let S be a finite set of places of k containing all the places of k with $\beta \neq 0$. Let $\nu \in S$. If $\theta_0 \notin k_\nu^{*\ell}$, then, let $L_\nu = k_\nu(\sqrt[\ell]{\theta_0})$ and $\mu_v = \sqrt[\ell]{\theta_0} \in L_\nu$. If $\theta_0 \in k_\nu^{*\ell}$, then, let L_ν/k_ν be any field extension of degree ℓ and $\mu_\nu = \sqrt[\ell]{\theta_0} \in k_\nu \subset L_\nu$. In both the cases, we have $N_{L_\nu/k_\nu}(\mu_v) = \theta_0$. Since L_ν/k_ν is a degree ℓ field extension, ℓ divides ind(β) and k_ν is a local field, ind$(\beta \otimes L_\nu) <$ ind(β) ([3, p. 131]). Since $r\ell\beta = 0$ and L_ν/k_ν is a field extension of degree ℓ, $r\beta \otimes L_\nu = 0$. Let $E_0 = k$. Then, by (3.1), there exist a field extension L_0/k of degree ℓ and $\mu \in L_0$ with required properties. □

Lemma 3.3. *Let k be a global field and ℓ a prime not equal to char(k). Let E_0/k be a cyclic extension of degree a power of ℓ and σ_0 a generator of Gal(E_0/k). Let $n \geq 1$, $\theta_0 \in k^*$ and $\beta \in H^2(k, \mu_{\ell^n})$ be such that $r\ell\beta = (E_0, \sigma_0, \theta_0)$ for some $r \geq 1$. Suppose that $r\beta \otimes E_0 \neq 0$. If ν is a place of k such that $\sqrt[\ell]{\theta_0} \notin k_\nu$, then ind$(\beta \otimes E_0 \otimes k_\nu(\sqrt[\ell]{\theta_0})) <$ ind$(\beta \otimes E_0)$.*

Proof. Write $r\ell = m\ell^d$ with m coprime to ℓ. Then $d \geq 1$. Since $m\ell^d\beta = r\ell\beta = (E_0, \sigma_0, \theta_0)$, we have $m\ell^d\beta \otimes E_0 = 0$. Since m is coprime to ℓ and the period of β is a power of ℓ, it follows that $\ell^d\beta \otimes E_0 = 0$. Since $r\beta \otimes E_0 \neq 0$, $\ell^{d-1}\beta \otimes E_0 \neq 0$ and per$(\beta \otimes E_0) = \ell^d$.

Let ν be a place of k. Suppose that $\sqrt[\ell]{\theta_0} \notin E_0 \otimes k_\nu$. Then $[E_0 \otimes k_\nu(\sqrt[\ell]{\theta_0}) : E_0 \otimes k_\nu] = \ell$ and hence ind$(\beta \otimes E_0 \otimes k_\nu(\sqrt[\ell]{\theta_0})) <$ ind$(\beta \otimes E_0)$ ([3, p. 131]). Suppose that $\sqrt[\ell]{\theta_0} \in E_0 \otimes k_\nu$. Then $E_0 \otimes k_\nu(\sqrt[\ell]{\theta_0}) = E_0 \otimes k_\nu$. Write $E_0 \otimes k_\nu = \prod E_i$ with each E_i a cyclic field extension of k_ν. Since E_0/k is a Galois extension, $E_i \simeq E_j$ for all i and j and $m\ell^d\beta \otimes k_\nu = (E_0, \sigma_0, \theta_0) \otimes k_\nu = (E_i, \sigma_i, \theta_0)$ for all i, for suitable generators σ_i of Gal(E_i/k_ν). Since $\sqrt[\ell]{\theta_0} \in E_0 \otimes k_\nu$, $\sqrt[\ell]{\theta_0} \in E_i$ for all i and hence $\theta_0^{[E_i:\kappa_\nu]/\ell} \in N_{E_i/k_\nu}(E_i^*)$. Since the period of $(E_i, \sigma_i, \theta_0)$ is equal to the order of the class of θ_0 in the group $k_\nu^*/N_{E_i/k_\nu}(E_i^*)$ ([1, p. 75]), per$(E_i, \sigma_i, \theta_0) \leq [E_i : k_\nu]/\ell < [E_i : k_\nu]$.

Suppose that per$(\beta \otimes k_\nu) \leq [E_i : k_\nu]$. Since k_ν is a local field, per$(\beta \otimes E_i) = 1$. Thus per$(\beta \otimes E_0 \otimes k_\nu) =$ per$(\beta \otimes E_i) = 1 < \ell^d =$ per$(\beta \otimes E_0)$.

Suppose that $\mathrm{per}(\beta \otimes k_\nu) > [E_i : k_\nu]$. Since $m\ell^d\beta \otimes k_\nu = (E_i, \sigma_i, \theta_0)$ and m is coprime to ℓ, we have $\mathrm{per}(\beta \otimes k_\nu) \leq \ell^d \mathrm{per}(E_i, \sigma_i, \theta_0)$. Since k_ν is a local-field,

$$\mathrm{per}(\beta \otimes E_0 \otimes k_\nu) = \mathrm{per}(\beta \otimes E_i) = \frac{\mathrm{per}(\beta \otimes k_\nu)}{[E_i : k_\nu]} \leq \frac{\ell^d \mathrm{per}(E_i, \sigma_i, \theta_0)}{[E_i : k_\nu]} < \ell^d = \mathrm{per}(\beta \otimes E_0).$$

Since k_ν is a local field, period equals index and hence the lemma follows. □

Proposition 3.4. *Let k be a global field and ℓ a prime not equal to char(k). Let E_0/k be a cyclic extension of degree a power of ℓ and σ_0 a generator of Gal(E_0/k). Let $\theta_0 \in k^*$ and $\beta \in H^2(k, \mu_{\ell^n})$ be such that $r\ell\beta = (E_0, \sigma_0, \theta_0)$ for some $r \geq 1$. Suppose that $r\beta \otimes E_0 \neq 0$. Then there exist a field extension L_0/κ of degree ℓ and $\mu_0 \in L_0$ such that*
1) $N_{L_0/k}(\mu_0) = \theta_0$
2) ind$(\beta \otimes E_0 \otimes L_0) <$ ind$(\beta \otimes E_0)$
2) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$.

Proof. Let S be the finite set of places of k consisting of all those places ν with $\beta \otimes k_\nu \neq 0$. Let $\nu \in S$. Suppose that $\theta_0 \notin k_\nu^{\ell}$. Let $L_\nu = k_\nu(\sqrt[\ell]{\theta_0})$ and $\mu_\nu = \sqrt[\ell]{\theta_0} \in L_\nu$. Then $N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0$. By (3.3), $\mathrm{ind}(\beta \otimes E_0 \otimes k_\nu) < \mathrm{ind}(\beta \otimes E_0)$. In particular $\mathrm{ind}(\beta \otimes E_0 \otimes L_\nu) \leq \mathrm{ind}(\beta \otimes E_0 \otimes k_\nu) < \mathrm{ind}(\beta \otimes E_0)$. Since $\mathrm{cor}_{L_\nu/k_\nu}(r\beta \otimes L_\nu) = r\ell\beta = (E_0 \otimes k_\nu, \sigma_0, \theta_0) = \mathrm{cor}_{L_\nu/k_\nu}(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$ and corestriction is injective (cf. [17, Theorem 10, p. 237]), we have $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$.

Suppose that $\theta_0 = \mu_\nu^\ell$ for some $\mu_\nu \in k_\nu$. Since k_ν is local field containing a primitive ℓ^{th} root of unity and $E_0 \otimes k_\nu$ is a cyclic extension, there exists a cyclic field extension L_ν/k_ν of degree ℓ which is not contained in $E_0 \otimes k_\nu$. Then $N_{L_\nu/k_\nu}(\mu_\nu) = \mu_\nu^\ell = \theta_0$. Since L_ν is not a subfield of $E_0 \otimes k_\nu$, $\mathrm{ind}(\beta \otimes E_0 \otimes L_\nu) < \mathrm{ind}(\beta \otimes E_0 \otimes k_\nu) \leq \mathrm{ind}(\beta \otimes E_0)$ ([3, p. 131]). Since $\mathrm{cor}_{L_\nu/k_\nu}(r\beta \otimes L_\nu) = r\ell\beta \otimes k_\nu = (E_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_\nu^\ell) = \mathrm{cor}_{L_\nu/k_\nu}(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$, by (cf. [17, Theorem 10, p. 237]), we have $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$.

By (3.1), we have the required L_0 and μ_0. □

Proposition 3.5. *Let k be a global field and ℓ a prime not equal to char(k). Let E_0/k be a cyclic extension of degree a positive power of ℓ and σ_0 a generator of Gal(E_0/k). Let $\theta_0 \in k^*$ and $\beta \in H^2(k, \mu_{\ell^n})$ be such that $r\ell\beta = (E_0, \sigma_0, \theta_0)$ for some $r \geq 1$. Suppose that $r\beta \otimes E_0 = 0$. Let L_0 be the unique subfield of E_0 of degree ℓ over k. Then there exists $\mu_0 \in L_0$ such that*
1) $N_{L_0/k}(\mu_0) = \theta_0$
2) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$.

Proof. Since $r\beta \otimes E_0 = 0$ and E_0/k is a cyclic extension, we have $r\beta = (E_0, \sigma_0, \mu')$ for some $\mu' \in k$. We have $(E_0, \sigma_0, \mu'^\ell) = \ell r\beta = (E_0, \sigma_0, \theta_0)$. Thus $\theta_0 = N_{E_0/k}(y)\mu'^\ell$. Let $\mu_0 = N_{E_0/L_0}(y)\mu' \in L_0$. Since $L_0 \subset E_0$, we have $r\beta \otimes L_0 = (E_0/L_0, \sigma_0^\ell, \mu') = (E_0/L_0, \sigma_0^\ell, N_{E_0/L_0}(y)\mu') = (E_0/L_0, \sigma_0^\ell, \mu_0)$ (cf. §2) and

$$N_{L_0/k}(\mu_0) = N_{E_0/k}(N_{E_0/L_0}(y))\mu'^\ell = \theta_0.$$

□

Corollary 3.6. *Let k be a global field and ℓ a prime not equal to char(k). Let E_0/k be a cyclic extension of degree a power of ℓ and σ_0 a generator of Gal(E_0/k). Let*

$\theta_0 \in k^*$ and $\beta \in H^2(k, \mu_{\ell^n})$ be such that $r\ell\beta = (E_0, \sigma_0, \theta_0)$ for some $r \geq 1$. Suppose that $r\beta \otimes E_0 = 0$. Let L_0 be the unique subfield of E_0 of degree ℓ over k. Let S be a finite set of places of k. Suppose for each $\nu \in S$ there exists $\mu_\nu \in L_0 \otimes k_\nu$ such that
- $N_{L_0 \otimes k_\nu / k_\nu}(\mu_\nu) = \theta_0$
- $r\beta \otimes L_0 \otimes k_\nu = (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_\nu)$.

Then there exists $\mu \in L_0$ such that
1) $N_{L_0/k}(\mu) = \theta_0$
2) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu)$
3) μ is close to μ_ν for all $\nu \in S$.

Proof. By (3.5), there exists $\mu_0 \in L_0$ such that
- $N_{L_0/k}(\mu_0) = \theta_0$
- $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$.

Let $\nu \in S$. Then we have
- $N_{L_0/k}(\mu_0) = \theta_0 = N_{L_0 \otimes k_\nu / k_\nu}(\mu_\nu)$
- $(E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_0) = (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_\nu)$.

Let $b_\nu = \mu_0 \mu_\nu^{-1} \in L_0 \otimes k_\nu$. Then $N_{L_0 \otimes k_\nu / k_\nu}(b_\nu) = 1$ and $(E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, b_\nu) = 1$. Thus, there exists $a_\nu \in E_0 \otimes L_0 \otimes k_\nu$ with $N_{E_0 \otimes L_0 \otimes k_\nu / L_0 \otimes k_\nu}(a_\nu) = b_\nu$. We have $N_{E_0 \otimes L_0 \otimes k_\nu / k_\nu}(a_\nu) = N_{L_0 \otimes k_\nu / \otimes k_\nu}(b_\nu) = 1$. Since E_0/k is a cyclic extension with σ_0 a generator of $\mathrm{Gal}(E_0/k)$, for each $\nu \in S$, there exists $c_\nu \in E_0 \otimes L_0 \otimes k_\nu$ such that $a_\nu = c_\nu^{-1}(\sigma_0 \otimes 1)(c_\nu)$. By the weak approximation, there exists $c \in E_0 \otimes L_0$ such that c is close to c_ν for all $\nu \in S$. Let $a = c^{-1}(\sigma \otimes 1)(c) \in E_0 \otimes L_0$ and $\mu = \mu_0 N_{E_0 \otimes L_0 / L_0}(c) \in L_0$. Then μ has all the required properties. □

4. COMPLETE DISCRETELY VALUED FIELDS

Let F be a complete discretely valued field with residue field κ. Let D be a central simple algebra over F of period n coprime to $\mathrm{char}(\kappa)$. Let $\lambda \in F^*$ and $\alpha \in H^2(F, \mu_n)$ be the class of D. In this section we analyze the condition $\alpha \cdot (\lambda) = 0$ and we use this analysis in the proof of our main result (§10). As a consequence, we also deduce that if κ is either a local field or a global field and $\alpha \cdot (\lambda) = 0$ in $H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.

We use the following notation throughout this section:
- F a complete discretely valued field.
- κ the residue field of F.
- ν the discrete valuation on F.
- $\pi \in F^*$ a parameter at ν.
- $n \geq 2$ an integer coprime to $\mathrm{char}(\kappa)$
- D a central simple algebra over F of period n.
- $\alpha \in H^2(F, \mu_n)$ the class representing D.

Let E_0 be the cyclic extension of κ and $\sigma_0 \in Gal(E_0/\kappa)$ be such that $\partial(\alpha) = (E_0, \sigma_0)$. Let (E, σ) be the lift of (E_0, σ_0) (cf. §2). The pair (E, σ) or E is called the *lift of the residue* of α. The following is well known.

Lemma 4.1. *Let $\alpha \in H^2(F, \mu_n)$, (E, σ) the lift of the residue of α. Then $\alpha = \alpha' + (E, \sigma, \pi)$ for some $\alpha' \in H^2_{nr}(F, \mu_n)$. Further $\alpha' \otimes E = \alpha \otimes E$ is independent of the choice of π.*

Proof. Since $\partial(E, \sigma, \pi) = \partial(\alpha)$, $\alpha' = \alpha - (E, \sigma, \pi) \in H^2_{nr}(F, \mu_n)$ and $\alpha = \alpha' + (E, \sigma, \pi)$. □

Lemma 4.2. *Let $n \geq 2$ be coprime to char(κ) and $\alpha \in H^2(F, \mu_n)$. If $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1), then ind(α) = ind$(\alpha' \otimes E)[E : F]$ = ind$(\alpha \otimes E)[E : F]$.*

Proof. Cf. ([6, Proposition 1(3)] and [15, 5.15]). □

Lemma 4.3. *Let E be the lift of the residue of α. Suppose there exists a totally ramified extension M/F which splits α, then $\alpha \otimes E = 0$.*

Proof. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1). Since $\alpha' \otimes E = \alpha \otimes E$, we have $\alpha' \otimes E \otimes M = 0$. Since $E \otimes M/E$ is totally ramified, the residue field of $E \otimes M$ is same as the residue field of E. Since $\alpha' \otimes E \otimes M = 0$ and $\alpha' \otimes E$ is unramified, it follows from ([28, 7.9 and 8.4]) that $\alpha \otimes E = \alpha' \otimes E = 0$. □

Lemma 4.4. *Let $n \geq 2$ be coprime to char(κ). Let $\alpha \in H^2(F, \mu_n)$ and (E, σ) be the lift of the residue of α. If $\alpha \otimes E = 0$, then $\alpha = (E, \sigma, u\pi)$ for some $u \in F^*$ which is a unit at the discrete valuation and per(α) = ind(α).*

Proof. We have $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1). Since $\alpha' \otimes E = \alpha \otimes E = 0$, we have $\alpha' = (E, \sigma, u)$ for $u \in F^*$. Since E/F and α' are unramified at the discrete valuation of F, u is a unit at the discrete valuation of F. We have $\alpha = (E, \sigma, u) + (E, \sigma, \pi) = (E, \sigma, u\pi)$. Since E/F is an unramified extension and $u\pi$ is a parameter, $(E, \sigma, u\pi)$ is a division algebra and its period is $[E : F]$. In particular ind(α) = per(α). □

Theorem 4.5. *Let F be a complete discretely valued field with residue field κ. Suppose that κ is a local field. Let ℓ be a prime not equal to the characteristic of κ, $n = \ell^d$ and $\alpha \in H^2(F, \mu_n)$. Then per(α) = ind(α).*

Proof. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1). Then E is an unramified cyclic extension of F with $\partial(\alpha) = (E_0, \sigma_0)$ and α' is unramified at the discrete valuation of F. Let $\overline{\alpha}'$ be the image of α' in $H^2(\kappa, \mu_n)$

Suppose that per$(\partial(\alpha))$ = per(α). Then per$(\partial(\alpha)) = [E_0 : \kappa]$. Since F is complete discretely valued field and E/F unramified extension, we have $[E_0 : \kappa] = [E : F]$. Thus,

$$\begin{aligned} 0 &= per(\alpha)\alpha \\ &= per(\alpha)(\alpha' + (E, \sigma, \pi)) \\ &= per(\alpha)\alpha' + per(\alpha)(E, \sigma, \pi) \\ &= per(\alpha)\alpha' + [E : F](E, \sigma, \pi) \\ &= per(\alpha)\alpha'. \end{aligned}$$

In particular, per(α') divides per$(\alpha) = [E_0, \kappa] = [E : F]$. Since κ is a local field, $\overline{\alpha}' \otimes E_0$ is zero ([3, p.131]) and hence $\alpha' \otimes E$ is zero. By (4.4), we have $\alpha = (E, \sigma, \theta\pi)$ for some $\theta \in F$ which is a unit in the valuation ring. In particular, α is cyclic and ind(α) = per$(\alpha) = [E : F]$.

Suppose that per$(\partial(\alpha)) \neq$ per(α). Then per$(\partial(\alpha))$ < per(α). Since per$(\partial(\alpha))$ = per(E, σ, π), we have per(α) = per(α'). Since κ is a local field, per$(\overline{\alpha}')$ = ind$(\overline{\alpha}')$. Let E_0 be the residue field of E. Since per$(\overline{\alpha}')$ = per(α') and per$(\partial(\alpha)) = [E_0 : \kappa]$, we have $[E_0 : \kappa]$ < per$(\overline{\alpha}')$. Since κ is a local field,

$$\text{ind}(\overline{\alpha}' \otimes E_0) = \frac{\text{per}(\overline{\alpha}')}{[E_0 : \kappa]}.$$

Since E is a complete discrete valued field with residue field E_0 and α' is unramified at the discrete valuation of E, we have $\mathrm{ind}(\alpha' \otimes E) = \mathrm{ind}(\overline{\alpha}' \otimes E_0)$. Thus, we have

$$\begin{array}{rcll} \mathrm{ind}(\alpha) & = & ind(\alpha' \otimes E)[E : F] & (\text{by}(4.2)) \\ & = & ind(\overline{\alpha}' \otimes E_0)[E_0 : \kappa] & \\ & = & \frac{\mathrm{per}(\overline{\alpha}')}{[E_0:\kappa]}[E_0 : \kappa] & \\ & = & \mathrm{per}(\overline{\alpha}') = \mathrm{per}(\alpha). & \end{array}$$

□

Proposition 4.6. *Suppose that κ is a local field. Let $n \geq 2$ be coprime to char(κ). If L/F is a finite field extension, then the corestriction homomorphism $H^3(L, \mu_n^{\otimes 2}) \to H^3(F, \mu_n^{\otimes 2})$ is bijective.*

Proof. Let k' be the residue field of L. Since k and k' are local fields, $H^3(k, \mu_n^{\otimes 2}) = H^3(k', \mu_n^{\otimes 2}) = 0$ ([27, p. 86]). Since F and L are complete discrete valued fields, the residue homomorphisms $H^3(F, \mu_n^{\otimes 2}) \overset{\partial_F}{\to} H^2(k, \mu_n)$ and $H^3(L, \mu_n^{\otimes 2}) \overset{\partial_L}{\to} H^2(k', \mu_n)$ are isomorphisms (cf. [28, 7.9]). The proposition follows from the commutative diagram

$$\begin{array}{ccc} H^3(L, \mu_n^2) & \overset{\partial_L}{\to} & H^2(k', \mu_n) \\ \downarrow & & \downarrow \\ H^3(F, \mu_n^{\otimes 2}) & \overset{\partial_F}{\to} & H^2(k, \mu_n), \end{array}$$

where the vertical arrows are the corestriction maps ([28, 8.6]). □

Lemma 4.7. *Let ℓ be a prime not equal to char(κ) and $n = \ell^d$ for some $d \geq 1$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. Write $\lambda = \theta\pi^r$ for some $\theta, \pi \in F$ with $\nu(\theta) = 0$ and $\nu(\pi) = 1$. Let (E, σ) be the lift of the residue of α and $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1). Then*

$$\partial(\alpha \cdot (\lambda)) = 0 \iff r\alpha' = (E, \sigma, \theta) \iff r\alpha = (E, \sigma, \lambda).$$

In particular, if $\partial(\alpha \cdot (\lambda)) = 0$ and $r = \nu(\lambda)$ is coprime to ℓ, then ind$(\alpha \otimes F(\sqrt[\ell]{\lambda})) <$ ind(α) and $\alpha \cdot (\sqrt[\ell]{\lambda}) = 0 \in H^3(F(\sqrt[\ell]{\lambda}), \mu_n^{\otimes 2})$.

Proof. Since $r\alpha = r\alpha' + (E, \sigma, \pi^r)$ and $\lambda = \theta\pi^r$, $r\alpha = (E, \sigma, \lambda)$ if and only if $r\alpha' = (E, \sigma, \theta)$.

We have

$$\partial(\alpha \cdot (\lambda)) = \partial((\alpha' + (E, \sigma, \pi)) \cdot (\theta\pi^r)) = r\overline{\alpha}' + (E_0, \sigma_0, \overline{\theta}^{-1}),$$

where $\partial(\alpha) = (E_0, \sigma_0)$.

Thus $\partial(\alpha \cdot (\lambda)) = 0$ if and only if $r\overline{\alpha}' + (E_0, \sigma_0, \overline{\theta}^{-1}) = 0$ if and only if $r\overline{\alpha}' = (E_0, \sigma_0, \overline{\theta})$ if and only if $r\alpha = (E, \sigma, \theta)$ (F being complete).

□

Lemma 4.8. *Let $n \geq 2$ be coprime to char(κ) and ℓ a prime which divides n. Let $\alpha \in H^2(F, \mu_n)$, $\lambda = \theta\pi^{\ell r} \in F^*$ with θ a unit in the valuation ring of F, π a parameter and $\alpha = \alpha' + (E, \sigma, \pi)$ be as in (4.1). Suppose that $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$ and there exist an extension L_0 of κ of degree ℓ and $\mu_0 \in L_0$ such that*
- *$N_{L_0/\kappa}(\mu_0) = \overline{\theta}$,*
- *$r\overline{\alpha}' \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$.*

Then, there exist an unramified extension L of F of degree ℓ and $\mu \in L$ such that
- *residue field of L is L_0,*
- *μ a unit in the valuation ring of L,*

- $\overline{\mu} = \mu_0$,
- $N_{L/F}(\mu) = \theta$,
- $\alpha \cdot (\mu\pi^r) \in H^3(L, \mu_n^{\otimes 2})$ *is unramified.*

Proof. Since ℓ is a prime and $[L_0 : \kappa] = \ell$, $L_0 = \kappa(\mu'_0)$ for any $\mu'_0 \in L_0 \setminus \kappa$. Let $g(X) = X^\ell + b_{\ell-1}X^{\ell-1} + \cdot + b_1X + b_0 \in \kappa[X]$ be the minimal polynomial of μ'_0 over κ. Let a_i be in the valuation ring of F mapping to b_i and $f(X) = X^\ell + a_{\ell-1}X^{\ell-1} + \cdots + a_1X + a_0 \in F[X]$. If $\mu_0 \notin \kappa$, then we take $\mu'_0 = \mu_0$. Since $N_{L_0/\kappa}(\mu_0) = \overline{\theta}$, we have $b_0 = (-1)^\ell\overline{\theta}$. In this case we take $a_0 = (-1)^\ell\theta$. Since $g(X)$ is irreducible in $\kappa[X]$, $f(X) \in F[X]$ is irreducible. Let $L = F[X]/(f)$. Then L/F is the unramified extension with residue field L_0. If $\mu_0 \in \kappa$, then $\overline{\theta} = N_{L_0/\kappa}(\mu_0) = \mu_0^\ell$. Since F is a complete discretely valued field and ℓ is coprime to char(κ), there exists $\mu \in F$ which is a unit in the valuation ring of F which maps to μ_0 and $\mu^\ell = \theta$. If $\mu_0 \notin \kappa$, then let $\mu \in L$ be the image of X. Then the image of μ is μ_0 and $N_{L/F}(\mu) = \theta$.

Since L/F, E/F and α' are unramified at the discrete valuation of F, we have $\partial_L(\alpha' \cdot (\mu\pi^r)) = r\overline{\alpha}' \otimes L_0$ and $\partial_L((E, \sigma, \pi) \cdot (\mu\pi^r)) = \partial_L((E \otimes L, \sigma \otimes 1, \mu^{-1}) \cdot (\pi)) = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0^{-1})$. Since $\alpha = \alpha' + (E, \sigma, \pi)$, we have

$$\begin{aligned} \partial_L(\alpha \cdot (\mu\pi^r)) &= \partial_L((\alpha' \otimes L) \cdot (\mu\pi^r)) + \partial_L((E, \sigma, \pi) \cdot (\mu\pi^r)) \\ &= r\overline{\alpha}' \otimes L_0 + (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0^{-1}) \\ &= 0 \end{aligned}$$

□

Lemma 4.9. *Suppose that κ is a local field. Let ℓ be prime not equal to char(k) and $n = \ell^d$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. Suppose $\lambda \notin F^{*\ell}$, $\alpha \neq 0$ and $\alpha \cdot (\lambda) = 0$. Then ind$(\alpha \otimes F(\sqrt[\ell]{\lambda})) <$ ind(α) and $\alpha \cdot (\sqrt[\ell]{\lambda}) = 0 \in H^3(F(\sqrt[\ell]{\lambda}), \mu_n^{\otimes 2})$.*

Proof. Suppose $\nu(\lambda)$ is coprime to ℓ. Then, by (4.7), we have ind$(\alpha \otimes F(\sqrt[\ell]{\lambda})) <$ ind(α) and $\alpha \cdot (\sqrt[\ell]{\lambda}) = 0 \in H^3(F(\sqrt[\ell]{\lambda}), \mu_n^{\otimes 2})$.

Suppose that $\nu(\lambda)$ is divisible by ℓ. Write $\lambda = \theta\pi^{\ell d}$ with $\theta \in F$ a unit in the valuation ring of F.

Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1). Then ind$(\alpha) =$ ind$(\alpha' \otimes E)[E : F]$ and ind$(\alpha \otimes F(\sqrt[\ell]{\theta})) \leq$ ind$(\alpha' \otimes E(\sqrt[\ell]{\theta}))[E(\sqrt[\ell]{\theta}) : F(\sqrt[\ell]{\theta})]$ (cf. 4.2). If $\sqrt[\ell]{\theta} \in E$, then $F(\sqrt[\ell]{\theta}) \subset E = E(\sqrt[\ell]{\theta})$. In particular $[E(\sqrt[\ell]{\theta}) : F(\sqrt[\ell]{\theta})] = [E : F(\sqrt[\ell]{\theta})] < [E : F]$. Since ind$(\alpha' \otimes E(\sqrt[\ell]{\theta})) \leq$ ind$(\alpha' \otimes E)$, it follows that ind$(\alpha \otimes F(\sqrt[\ell]{\theta})) <$ ind(α).

Suppose $\sqrt[\ell]{\theta} \notin E$. Since E is unramified extension of F and θ is a unit in the valuation ring of E, $E(\sqrt[\ell]{\theta})$ is an unramified extension of F with residue field $E_0(\sqrt[\ell]{\overline{\theta}})$, where E_0 is the residue field of E and $\overline{\theta}$ is the image of θ in the residue field. Since F is a complete discretely valued field and θ is not an ℓ^{th} power in E, $\overline{\theta}$ is not an ℓ^{th} power in E_0 and $[E_0(\sqrt[\ell]{\overline{\theta}}) : E_0] = \ell$.

Suppose $\alpha' \otimes E \neq 0$. Then $\overline{\alpha}' \otimes E_0 \neq 0$. Since E_0 is a local field and ind$(\overline{\alpha}')$ is a power of ℓ, ind$(\overline{\alpha}' \otimes E_0(\sqrt[\ell]{\overline{\theta}})) <$ ind$(\overline{\alpha}' \otimes E_0)$ ([3, p. 131]). Hence ind$(\alpha' \otimes E(\sqrt[\ell]{\theta})) <$ ind$(\alpha' \otimes E)$ and ind$(\alpha \otimes F(\sqrt[\ell]{\theta})) <$ ind(α).

Suppose that $\alpha' \otimes E = 0$. Then, by (4.4), $\alpha = (E, \sigma, u\pi)$ for some unit u in the valuation ring of F. Since $\alpha \cdot (\lambda) = 0$, $(E, \sigma, u\pi) \cdot (\lambda) = 0$. Since E/F is unramified with residue field E_0, u, θ are units in the valuation ring of F and π a parameter, by taking the residue of $\alpha \cdot (\lambda) = 0$, we see that $(E_0, \sigma_0, \overline{\theta}^{-1}\overline{u}^{\ell d}) = 0 \in H^2(\kappa, \mu_n)$ (cf. 4.7). In particular, $\overline{\theta}\overline{u}^{-\ell d}$ is a norm from E_0. Since $[E_0 : k]$ is a power of ℓ and E_0/κ is cyclic, there exists a sub extension L of E_0 such that $[L : \kappa] = \ell$. Then

$\overline{\theta}\overline{u}^{-\ell d}$ is a norm from L and hence $\overline{\theta}$ is a norm from L. Since $\overline{\theta}$ is not in $\kappa^{*\ell}$, by (2.5), $L = \kappa(\sqrt[\ell]{\overline{\theta}})$. In particular $\sqrt[\ell]{\overline{\theta}} \in E_0$ and hence $\sqrt[\ell]{\theta} \in E$. Thus $\text{ind}(\alpha \otimes F(\sqrt[\ell]{\theta})) = \text{ind}((E, \sigma, u\pi) \otimes F(\sqrt[\ell]{\theta})) < \text{ind}(E, \sigma, u\pi) = \text{ind}(\alpha)$. □

Lemma 4.10. *Suppose κ is a local field. Let ℓ be a prime not equal to char(κ) and $n = \ell^d$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. Suppose that κ contains a primitive ℓ^{th} root of unity. If $\alpha \neq 0$ and $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then there exist a cyclic field extension L/F of degree ℓ and $\mu \in L^*$ such that $N_{L/F}(\mu) = \lambda$, ind$(\alpha \otimes L) <$ ind(α) and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$. Further, if $\lambda \in F^{*\ell}$, then L/F is unramified and $\mu \in F$.*

Proof. Suppose λ is not an ℓ^{th} power in F. Let $L = F(\sqrt[\ell]{\lambda})$ and $\mu = \sqrt[\ell]{\lambda}$. Then, by (4.9), $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$.

Suppose $\lambda = \mu^\ell$ for some $\mu \in F^*$. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1).

Suppose that $\alpha' \otimes E = 0$. Then, by (4.4), $\alpha = (E, \sigma, u\pi)$ for some $u \in F^*$ which is a unit in the valuation ring of F. Let L be the unique subfield of E with L/F of degree ℓ. Then $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$. Since $\text{cor}_{L/F}(\alpha \cdot (\mu)) = \alpha \cdot (\mu^\ell) = \alpha \cdot (\lambda) = 0$, by (4.6), $\alpha \cdot (\mu) = 0$ in $H^3(L, \mu_n)$. We also have $\lambda = \mu^\ell = N_{L/F}(\mu)$.

Suppose that $\alpha' \otimes E \neq 0$. Let E_0 be the residue field of E. Then E_0/κ is a cyclic field extension of κ of degree equal to the degree of E/F. Since κ is a local field and contains a primitive ℓ^{th} root of unity, there are at least three distinct cyclic field extensions of κ of degree ℓ. Since E_0/κ is a cyclic extension, there is at most one sub extension of E_0 of degree ℓ over κ. Thus there exists a cyclic field extension L_0/κ of degree ℓ such that $E_0 \otimes L_0$ is a field. Let L/F be the unramified extension with residue field L_0. Then $E \otimes L$ is a field. Let $\overline{\alpha}'$ be the image of α' in $H^2(\kappa, \mu_n)$. Since E is a complete discretely valued field, $\overline{\alpha}' \otimes E_0 \neq 0$. Since $E_0 \otimes L_0/E_0$ is a field extension of degree ℓ and κ is a local field, $\text{ind}(\overline{\alpha}' \otimes E_0 \otimes L_0) < \text{ind}(\overline{\alpha}' \otimes E_0)$ ([3, p. 131]). Since E is a complete discretely valued field, $\text{ind}(\alpha' \otimes E \otimes L) < \text{ind}(\alpha' \otimes E)$. Since L/F is unramified, $\partial(\alpha \otimes L) = \partial(\alpha) \otimes L_0$ (cf. [4, Proposition 3.3.1]) and hence the decomposition $\alpha \otimes L = \alpha' \otimes L + (E \otimes L, \sigma \otimes 1, \pi)$ is as in (4.1). Thus, by (4.2), $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$. As above, we also have $\lambda = N_{L/F}(\mu)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$. □

Lemma 4.11. *Suppose κ is a global field. Let ℓ be a prime not equal to char(κ) and $n = \ell^d$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. If $\alpha \neq 0$ and $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then there exist a field extension L/F of degree ℓ and $\mu \in L^*$ such that $N_{L/F}(\mu) = \lambda$, ind$(\alpha \otimes L) <$ ind(α) and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$.*

Proof. Suppose that $\nu(\lambda)$ is coprime to ℓ. Then, by (4.7), $L = F(\sqrt[\ell]{\lambda})$ and $\mu = \sqrt[\ell]{\lambda}$ has the required properties.

Suppose that $\nu(\lambda)$ is divisible by ℓ. Let π be a parameter in F. Then $\lambda = \theta\pi^{r\ell}$ with $\nu(\theta) = 0$. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in (4.1). Let $\overline{\alpha}'$ be the image of α' in $H^2(\kappa, \mu_n)$ and θ_0 the image of θ in κ. Since $\alpha \cdot (\lambda) = 0$, by (4.7), we have $r\ell\overline{\alpha}' = (E_0, \sigma_0, \theta_0)$, where E_0 is the residue field of E and σ_0 induced by σ.

Suppose that $r\overline{\alpha}' \otimes E_0 \neq 0$. Then, by (3.4) , there exist a extension L_0/κ of degree ℓ and $\mu_0 \in L_0$ such that $N_{L_0/\kappa}(\mu_0) = \theta_0$, $\text{ind}(\overline{\alpha}' \otimes E_0 \otimes L_0) < \text{ind}(\overline{\alpha}' \otimes E_0)$ and $r\overline{\alpha}' \otimes L_0 = (E_0 \otimes L_0, \sigma_0, \mu_0)$.

Suppose that $r\overline{\alpha}' \otimes E_0 = 0$. Suppose that $E_0 \neq \kappa$. Let L_0 be the unique subfield field of E_0 of degree ℓ over κ. Then, by (3.5), there exists $\mu_0 \in L_0$ such that $N_{L_0/\kappa}(\mu_0) = \theta_0$ and $r\overline{\alpha}' \otimes L_0 = (E_0, \sigma_0, \mu_0)$.

Suppose that $E_0 = \kappa$. Then, by (3.2), there exist a field extension L_0/κ of degree ℓ and $\mu_0 \in L_0$ such that $N_{L_0/k}(\mu_0) = \theta_0$ and $\mathrm{ind}(\overline{\alpha}' \otimes L_0) < \mathrm{ind}(\overline{\alpha}')$.

By (4.8), we have the required L and μ. □

Theorem 4.12. *Let F be a complete discrete valued field with residue field κ. Suppose that κ is a local field or a global field. Let D be a central simple algebra over F of period n. Suppose that n is coprime to char(κ). Let $\alpha \in H^2(F, \mu_n)$ be the class of D and $\lambda \in F^*$. If $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.*

Proof. Write $n = \ell_1^{d_1} \cdots \ell_r^{d_r}$, ℓ_i distinct primes, $d_i > 0$, $D = D_1 \otimes \cdots \otimes D_r$ with each D_i a central simple algebra over F of period power of ℓ_i ([1, Ch. V, Theorem 18]). Let α_i be the corresponding cohomology class of D_i. Since ℓ_i's are distinct primes, $\alpha \cdot (\lambda) = 0$ if and only if $\alpha_i \cdot (\lambda) = 0$ and λ is a reduced norm from D if and only if λ is a reduced norm from each D_i. Thus without loss of generality we assume that $\mathrm{per}(D) = \ell^d$ for some prime ℓ.

We prove the theorem by the induction on the index of D. Suppose that $\deg(D) = 1$. Then every element of F^* is a reduced norm from D. We assume that $\deg(D) = n = \ell^d \geq 2$.

Let $\lambda \in F^*$ with $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$. Let ρ be a primitive ℓ^{th} root of unity. Since $[F(\rho) : F]$ is coprime to n, λ is a reduced norm from F is and only if λ is a reduced from $D \otimes F(\rho)$. Thus, replacing F by $F(\rho)$, we assume that $\rho \in F$.

Since κ is either a local field or a global field, by (4.10, 4.11), there exist an extension L/F of degree ℓ and $\mu \in L^*$ such that $N_{L/F}(\mu) = \lambda$, $\alpha \cdot (\mu) = 0$ and $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. Thus, by induction, μ is a reduced norm from $D \otimes L$. Since $N_{L/F}(\mu) = \lambda$, λ is a reduced norm from D. □

The following technical lemma is used in §6.

Lemma 4.13. *Let κ be a finite field and K a function field of a curve over κ. Let $u, v, w \in \kappa^*$ and $\theta \in K^*$. Let ℓ a prime not equal to char(κ) and $\theta = wu\lambda$. If κ contains a primitive ℓ^{th} root of unity and $w \notin \kappa^{*\ell}$, then for $r \geq 1$, the element $(v, \sqrt[\ell^r]{\theta})_\ell$ is $H^2(K(\sqrt[\ell^e]{\theta})$ is trivial over $K(\sqrt[\ell^r]{\theta}, \sqrt[\ell]{v + u\lambda})$.*

Proof. Let $L = K(\sqrt[\ell^r]{\theta}, \sqrt[\ell]{v + u\lambda})$ and $\beta = (v, \sqrt[\ell^r]{\theta})_\ell$. Since L is a global field, to show that $\beta \otimes L$ is trivial, it is enough to show that $\beta \otimes L_\nu$ is trivial for every discrete valuation ν of L. Let ν be a discrete valuation of L. Since $v \in \kappa^*$, v is a unit at ν. If θ is a unit at ν, then $\beta \otimes L$ is unramified at ν and hence $\beta \otimes L_\nu$ is trivial. Suppose that θ is not a unit at ν. Since u and v are units at ν, λ is not a unit. Suppose that $\nu(\lambda) > 0$. Then $v \in L_\nu^{*\ell}$ and hence $\beta \otimes L_\nu$ is trivial. Suppose that $\nu(\lambda) < 0$. Then $\sqrt[\ell]{u\lambda} \in L_\nu$. Since $r \geq 1$, $\theta = uw\lambda$ and $\sqrt[\ell^r]{\theta} \in L_\nu$, we have $\sqrt[\ell]{\theta} = \sqrt[\ell]{wu\lambda} \in L_\nu$. Hence $\sqrt[\ell]{w} \in L_\nu$. Since $w \in \kappa^* \setminus \kappa^{*\ell}$, $v \in \kappa^*$ and κ is a finite field, $\sqrt[\ell]{v} \in \kappa(\sqrt[\ell]{w})$. Since $\kappa(\sqrt[\ell]{w}) \subset L_\nu$, $\beta \otimes L_\nu$ is trivial. □

We end this section with the following well known facts.

Lemma 4.14. *Let F be a complete discrete valued field with the residue field κ. Let $\alpha \in \mathrm{Br}(F)$ and L/F an unramified extension with residue field L_0. Suppose that per(α) is coprime to char(κ). Let $\partial(\alpha) = (E_0, \sigma_0)$. If $\partial(\alpha \otimes L)$ is trivial, then E_0 is isomorphic to a subfield of L_0.*

Proof. Let L_0 be the residue field of L. Since L/F is unramified, $(E_0, \sigma_0) \otimes L_0 = \partial(\alpha) \otimes L_0 = \partial(\alpha \otimes L)$ (cf. [4, Proposition 3.3.1]). Since $\alpha \otimes L = 0$, $\partial(\alpha \otimes L) = 0$ and hence E_0 is isomorphic to a subfield of L_0. □

Corollary 4.15. *Let L/F be a cyclic extension of degree n, τ a generator of Gal(L/F) and $\theta \in F^*$. If $\nu(\theta)$ is coprime to n and ind$(L/F, \tau, \theta) = [L : F]$, then $[L : F] =$ per$(\partial(L/F, \tau, \theta))$.*

Proof. Let $\beta = (L/F, \tau, \theta)$ and $m = \mathrm{per}(\partial(\beta))$. Since $n = [L : F] = \mathrm{ind}(\beta)$, m divides n. Since $\nu(\theta)$ is coprime to n, $F(\sqrt[m]{\theta})/F$ is a totally ramified extension of degree m with residue field equal to the residue field κ of F. Since $\partial(\beta \otimes F(\sqrt[m]{\theta})) = m\partial(\beta)$, $\beta \otimes F(\sqrt[m]{\theta})$ is unramified. Since $F(\sqrt[n]{\theta})/F(\sqrt[m]{\theta})$ is totally ramified and $\beta \otimes F(\sqrt[n]{\theta})$ is trivial, $\beta \otimes F(\sqrt[m]{\theta})$ is trivial (cf. 4.3). Hence $n = m$. □

5. Brauer group - complete two dimensional regular local rings

Through out this section A denotes a complete regular local ring of dimension 2 with residue field κ and F its field of fractions. Let ℓ be a prime not equal to the characteristic of κ and $n = \ell^d$ for some $d \geq 1$. Let $m = (\pi, \delta)$ be the maximal ideal of A. For any prime $p \in A$, let F_p be the completion of the field of fractions of the completion of the local ring $A_{(p)}$ at p and $\kappa(p)$ the residue field at p.

Lemma 5.1. *Let E_π be a Galois extension of F_π of degree coprime to char(κ). Then there exists a Galois extension E of F of degree $[E_\pi : F_\pi]$ which is unramified on A except possibly at δ and Gal$(E/F) \simeq$Gal(E_π/F_π).*

Proof. Since A is complete and $m = (\pi, \delta)$, $\kappa(\pi)$ is a complete discretely valued field with residue field κ and the image $\overline{\delta}$ of δ as a parameter. Let E_0 be the residue field of E_π. Then $E_0/\kappa(\pi)$ is a Galois extension with $\mathrm{Gal}(E_0/\kappa(\pi)) \simeq \mathrm{Gal}(E_\pi/F_\pi)$. Let L_0 be the maximal unramified extension of $\kappa(\pi)$ contained in E_0. Then L_0 is also a complete discretely valued field with $\overline{\delta}$ as a parameter. Since E_0/L_0 is a totally ramified extension of degree coprime to char(κ), we have $E_0 = L_0(\sqrt[e]{v\overline{\delta}})$ for some $v \in L_0$ which is a unit at the discrete valuation of L_0 (cf. 2.4).

Since $E_0/\kappa(\pi)$ is a Galois extension, E_0/L_0 and $L_0/\kappa(\pi)$ are Galois extensions. Let κ_0 be the residue field of E_0. Then the residue field of L_0 is also κ_0. Since κ_0 is a Galois extension of κ and A is complete, there exists a Galois extension L of F which is unramified on A with residue field κ_0. Let B be the integral closure of A in L. Then B is a regular local ring with residue field κ_0 (cf. [21, Lemma 3.1]). Let $u \in B$ be a lift of v.

Let $E = L(\sqrt[e]{u\delta})$. Since L/F is unramified on A, E/F is unramified on A except possibly at δ. In particular E/F is unramified at π with residue field E_0. By the construction $[E : F] = [E_0 : \kappa(\pi)]$. Hence $E \otimes F_\pi \simeq E_\pi$.

Since L/F is a Galois extension which is unramified at π, we have $\mathrm{Gal}(L/F) \simeq \mathrm{Gal}(L_0/\kappa(\pi))$. Let $\tau \in \mathrm{Gal}(L/F)$ and $\overline{\tau} \in$Gal$(L_0/\kappa(\pi))$ be the image of τ. Since $E_0/\kappa(\pi)$ is Galois and $E_0 = L_0(\sqrt[e]{v\overline{\delta}})$, by (2.3), E_0 contains a primitive e^{th} root of unity ρ and $\overline{\tau}(v\overline{\delta})) \in E_0^e$. In particular $\rho \in \kappa_0$. Since B is complete with residue field κ_0, $\rho \in B$ and hence $\rho \in L \subseteq E$. Since $\overline{\tau}(v\overline{\delta}) = \overline{\tau}(v)\overline{\delta}$ and $v\overline{\delta}$, $\overline{\tau}(v\overline{\delta}) \in E_0^e$, $\overline{\tau}(v)/v \in E_0^e$. Since $\overline{\tau}(v)$ and v are units at the discrete valuation of L_0 and E_0/L_0 is totally ramified, $\overline{\tau}(v)/v \in L_0^e$. Since B is complete and the image of $\tau(u)/u$ in L_0 is $\overline{\tau}(v)/v$, $\tau(u)/u \in L^e$. Since $E = L(\sqrt{u\delta})$, $\tau(u\delta) \in E^e$. Thus, by (2.3), E/F is Galois. Since $E \otimes F_\pi \simeq E_\pi$, Gal$(E/F) \simeq$Gal$(E_\pi/F_\pi)$. □

Since A is complete and (π, δ) is the maximal ideal of A, $A/(\pi)$ is a complete discrete valuation ring with $\overline{\delta}$ is a parameter and $A/(\delta)$ is a complete discrete valuation ring with $\overline{\pi}$. The following follows from ([16, Proposition 1.7]).

Lemma 5.2. *([16, Proposition 1.7]) Let $m \geq 1$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$. Suppose that α is unramified on A except possibly at π and δ. Then*

$$\partial_{\overline{\delta}}(\partial_\pi(\alpha)) = -\partial_{\overline{\pi}}(\partial_\delta(\alpha)).$$

Let $H^m_{nr}(F, \mu_n^{\otimes(m-1)})$ be the intersections of the kernels of the residue homomorphisms $\partial_\theta : H^m(F, \mu_n^{\otimes(m-1)}) \to H^{m-1}(\kappa(\theta), \mu_n^{\otimes(m-2)})$ for all primes $\theta \in A$. The following lemma follows from the purity theorem of Gabber.

Lemma 5.3. *For $m \geq 1$, $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) \simeq H^m(\kappa, \mu_n^{\otimes(m-1)})$.*

Proof. By the purity theorem of Gabber (cf. [24, CH. XVI]), we have $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) \simeq H^m_{ét}(A, \mu_n^{\otimes(m-1)})$. Since A is complete, we have $H^m_{ét}(A, \mu_n^{\otimes(m-1)}) \simeq H^m(\kappa, \mu_n^{\otimes(m-1)})$ (cf. [20, Corollary 2.7, p.224]). □

Lemma 5.4. *Let $m \geq 1$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$. Suppose that α is unramified except possibly at π. Then there exist $\alpha_0 \in H^m(F, \mu_n^{\otimes(m-1)})$ and $\beta \in H^{m-1}(F, \mu_n^{\otimes(m-2)})$ which are unramified on A such that*

$$\alpha = \alpha_0 + \beta \cdot (\pi).$$

Proof. Let $\beta_0 = \partial_\pi(\alpha)$. By (5.2), $\beta_0 \in H^{m-1}(\kappa(\pi), \mu_n^{\otimes(m-2)})$ is unramified on $A/(\pi)$. Since $A/(\pi)$ is a complete discrete valuation ring with residue field κ, we have $H^{m-1}_{nr}(\kappa(\pi), \mu_n^{\otimes(m-2)}) \simeq H^{m-1}(\kappa, \mu_n^{\otimes(m-2)})$. Since A is a complete regular local ring of dimension 2, $H^{m-1}_{nr}(F, \mu_n^{\otimes(m-2)}) \simeq H^{m-1}(\kappa, \mu_n^{\otimes(m-2)})$ (5.3). Thus, there exists $\beta \in H^{m-1}_{nr}(F, \mu^{\otimes(m-1)})$ which is a lift of β_0. Then $\alpha_0 = \alpha - \beta \cdot (\pi)$ is unramified on A. Hence $\alpha = \alpha_0 + \beta \cdot (\pi)$. □

Corollary 5.5. *Let $m \geq 1$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$ is unramified on A except possibly at π and δ. If $\alpha \otimes F_\delta = 0$, then $\alpha = 0$. In particular if $\alpha_1, \alpha_2 \in H^m(F, \mu_n^{\otimes(m-1)})$ unramified on A except possibly at π and δ and $\alpha_1 \otimes F_\delta = \alpha_2 \otimes F_\delta$, then $\alpha_1 = \alpha_2$.*

Proof. Since $\alpha \otimes F_\delta = 0$, α is unramified at δ. Thus α is unramified on A except possibly at π. By (5.4), we have $\alpha = \alpha_0 + \beta \cdot (\pi)$ for some $\alpha_0 \in H^m(F, \mu_n^{\otimes(m-1)})$ and $\beta \in H^{m-1}(F, \mu_n^{\otimes(m-2)})$ which are unramified on A. Since $\alpha \otimes F_\delta = 0$, we have $(\beta \cdot (\pi)) \otimes F_\delta = -\alpha_0 \otimes F_\delta$. Since $\beta \cdot (\pi)$ and α_0 are unramified at δ, we have $\overline{\beta} \cdot (\overline{\pi}) = -\overline{\alpha}_0$, where $\overline{}$ denotes the image over $\kappa(\delta)$. Since $\kappa(\delta)$ is a complete discrete valued field with $\overline{\pi}$ as a parameter, by taking the residues, we see that the image of β is 0 in $H^{m-1}(\kappa, \mu_n^{\otimes(m-2)})$. Since A is a complete regular local ring, $\beta = 0$ (5.3). Hence $\alpha = \alpha_0$ is unramified on A. Since $\alpha \otimes F_\delta = 0$, $\overline{\alpha} = 0 \in H^m(\kappa(\delta), \mu_n^{\otimes(m-1)})$. In particular the image of α in $H^m(\kappa, \mu_n^{\otimes(m-1)})$ is zero. Since A is a complete regular local ring, $\alpha = 0$ (5.3). □

Corollary 5.6. *Let $m \geq 1$ and $\alpha \in H^m(F, \mu_n^{m-1})$. If α is unramified on A except possibly at π and δ, then per(α) = per$(\alpha \otimes F_\pi)$ = per$(\alpha \otimes F_\delta)$.*

Proof. Suppose $t = \text{per}(\alpha \otimes F_\delta)$. Then $t\alpha \otimes F_\delta = 0$ and hence, by (5.5), $t\alpha = 0$. Since $\text{per}(\alpha \otimes F_\delta) \leq \text{per}(\alpha)$, it follows that $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\delta)$. Similarly, $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\pi)$. □

Corollary 5.7. *Suppose that κ is a finite field. Let $\alpha \in H^2(F, \mu_n)$. If α is unramified except at π and δ, then there exist a cyclic extension E/F and $\sigma \in$ Gal(E/F) a* *generator, $u \in A$ a unit, and $0 \leq i, j < n$ such that $\alpha = (E, \sigma, u\pi^i\delta^j)$ with E/F is unramified on A except at δ and $i = 1$ or E/F is unramified on A except at π and $j = 1$.*

Proof. Since n is a power of the prime ℓ and $n\alpha = 0$, $\mathrm{per}(\partial_\pi(\alpha))$ and $\mathrm{per}(\partial_\delta(\alpha))$ are powers of ℓ. Let d' be the maximum of $\mathrm{per}(\partial_\pi(\alpha))$ and $\mathrm{per}(\partial_\delta(\alpha))$. Then $\partial_\pi(d'\alpha) = d'\partial_\pi(\alpha) = 0$ and $\partial_\delta(d'\alpha) = d'\partial_\delta(\alpha) = 0$. In particular $d'\alpha$ is unramified on A. Since κ is a finite field, $d'\alpha = 0$. Hence $\mathrm{per}(\alpha)$ divides d' and $d' = \mathrm{per}(\alpha)$. Thus $\mathrm{per}(\alpha) = \mathrm{per}(\partial_\pi(\alpha))$ or $\mathrm{per}(\partial_\delta(\alpha))$.

Suppose that $\mathrm{per}(\alpha) = \mathrm{per}(\partial_\pi(\alpha))$. Since $\partial_\pi(\alpha \otimes F_\pi) = \partial_\pi(\alpha)$, we have $\mathrm{per}(\partial_\pi(\alpha)) \leq \mathrm{per}(\alpha \otimes F_\pi) \leq \mathrm{per}(\alpha)$. Thus $\mathrm{per}(\alpha \otimes F_\pi) = \mathrm{per}(\partial_\pi(\alpha \otimes F_\pi))$. Thus, by (4.4), we have $\alpha \otimes F_\pi = (E_\pi/F_\pi, \sigma, \theta\pi)$ for some cyclic unramified extension E_π/F_π and $\theta \in F_\pi$ a unit in the valuation ring of F_π.

By (5.1), there exists a Galois extension E/F which is unramified on A except possibly at (δ) such that $E \otimes F_\pi \simeq E_\pi$. Since E_π/F_π is cyclic, E/F is cyclic. Since $\theta \in F_\pi$ is a unit in the valuation ring of F_π and the residue field of F_π is a complete discrete valued field with $\overline{\delta}$ as parameter, we can write $\theta = u\delta^j\theta_1^n$ for some unit $u \in A$, $\theta_1 \in F_\pi$ and $0 \leq j \leq n-1$. Then $\alpha \otimes F_\pi \simeq (E, \sigma, u\delta^j\pi) \otimes F_\pi$. Thus, by (5.5), we have $\alpha = (E, \sigma, u\delta^i\pi)$.

If $\mathrm{per}(\alpha) = \mathrm{per}(\partial_\delta(\alpha))$, then, as above, we get $\alpha = (E, \sigma, u\pi^j\delta)$ for some cyclic extension E/F which is unramified on A except possibly at π. □

The following is proved in ([29, 2.4]) under the assumption that F contains a primitive n^{th} root of unity.

Proposition 5.8. *Suppose that κ is a finite field. Let $\alpha \in H^2(F, \mu_n)$. If α is unramified on A except possibly at (π) and (δ). Then $ind(\alpha) = ind(\alpha \otimes F_\pi) = ind(\alpha \otimes F_\delta)$.*

Proof. Suppose that α is unramified on A except possibly at (π) and (δ). Then, by (5.7), we assume without loss of generality that $\alpha = (E/F, \sigma, \pi\delta^j)$ with E/F unramified on A except possibly at δ. Then $\mathrm{ind}(\alpha) \leq [E : F]$. Since E/F is unramified on A expect possibly at δ, we have $[E : F] = [E_\pi : F_\pi]$ and $\mathrm{ind}(\alpha \otimes F_\pi) = [E_\pi : F_\pi]$. Thus $[E : F] = [E_\pi : F_\pi] = \mathrm{ind}(\alpha \otimes F_\pi) \leq \mathrm{ind}(\alpha) \leq [E : F]$ and hence $[E : F] = \mathrm{ind}(\alpha \otimes F_\pi) = \mathrm{ind}(\alpha)$.

□

Corollary 5.9. *Suppose that κ is a finite field. Let $\alpha \in H^2(F, \mu_n)$. If α is unramified on A except possibly at (π) and (δ). Then $ind(\alpha) = per(\alpha)$.*

Proof. By (5.6), $\mathrm{per}(\alpha) = \mathrm{per}(\alpha \otimes F_\pi)$ and by (4.5), $\mathrm{ind}(\alpha \otimes F_\pi) = \mathrm{per}(\alpha \otimes F_\pi)$. Thus $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha \otimes F_\pi)$. By (5.8), we have $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$. □

The following follows from ([11] and [13]).

Proposition 5.10. *Let $\alpha \in H^2(F, \mu_n)$. Let $\phi : \mathscr{X} \to \mathrm{Spec}(A)$ be a sequence of blow-ups and $V = \phi^{-1}(m)$. Then $ind(\alpha) = l.c.m\{\mathrm{ind}(\alpha \otimes F_x) \mid x \in V\}$.*

Proof. Follows from similar arguments as in the proof of ([11, Theorem 9.11]) and using ([13, Theorem 4.2.1]). □

We end this section with the following well known result

Lemma 5.11. *Let E/F be a cyclic extension of degree ℓ^d for some $d \geq 1$. If E/F is unramified on A except possibly at δ, then there exist a subextension E_{nr} of E/F and $w \in E_{nr}$ which is a unit in the integral closure of A in E_{nr} such that E_{nr}/F is unramified on A and $E = E_{nr}(\sqrt[\ell^e]{w\delta})$. Further if κ is a finite field, κ contains a primitive ℓ^{th} root of unity and $0 < e < d$, then $N_{E_{nr}/F}(w) \in A$ is not an ℓ^{th} power in A.*

Proof. Let $E(\pi)$ be the residue field of E at π. Since E/F is unramified at A except possibly at δ, by (5.6), $[E(\pi) : \kappa(\pi)] = [E : F]$. Since E/F is cyclic, $E(\pi)/\kappa(\pi)$ is cyclic. As in the proof of (5.1), there exist a cyclic extension E_0/F unramified on A and a unit w in the integral closure of A in E_0 such that the residue field of $E_0(\sqrt[\ell^e]{w\delta})$ at π is $E(\pi)$. By (5.5), we have $E \simeq E_0(\sqrt[\ell^e]{w\delta})$. Let $E_{nr} = E_0$. Then E_{nr} has the required properties.

Suppose that κ is a finite field and contains a primitive ℓ^{th} root of unity. Let B be the integral closure of A in E_{nr}. Then B is a complete regular local ring with residue field κ' a finite extension of κ.

Let $w_0 = N_{E_{nr}/F}(w) \in A^*$ and $\overline{w}_0 \in \kappa^*$. Suppose that $w_0 \in A^{*\ell}$. Then $\overline{w}_0 \in \kappa^{*\ell}$. Since κ contains a primitive ℓ^{th} root of unity, we have $\mid \kappa'^*/\kappa'^{*\ell} \mid = \mid \kappa^*/\kappa^{*\ell} \mid = \ell$. Since norm map is surjective from κ' to κ, the norm map induces an isomorphism from $\kappa'^*/\kappa'^{*\ell} \to \kappa^*/\kappa^{*\ell}$. Thus the image of w in κ' is an ℓ^{th} power. Since B is a complete regular local ring, $w \in B^{*\ell}$. Suppose $0 < e < f$. Then $\sqrt[\ell]{\delta} \in E$. Since E_{nr}/F is nontrivial unramified extension and $F(\sqrt[\ell]{\delta})/F$ is a nontrivial totally ramified extension of F, we have two distinct degree ℓ subextensions of E/F, which is a contradiction to the fact that E/F is cyclic. Hence $w_0 \notin A^{*\ell}$. □

6. Reduced norms - complete two dimensional regular local rings

Throughout this section we fix the following notation:

- A a complete two dimensional regular local ring
- F the field of fractions of A
- $m = (\pi, \delta)$ the maximal ideal of A
- $\kappa = A/m$ a finite field
- ℓ a prime not equal to char(κ)
- $n = \ell^d$
- $\alpha \in H^2(F, \mu_n)$ is unramified on A except possibly at (π) and (δ)
- $\lambda = w\pi^s\delta^t$, $w \in A$ a unit and $s, t \in \mathbb{Z}$ with $1 \leq s, t < n$.

The aim of this section is to prove that if $\alpha \neq 0$ and $\alpha \cdot (\lambda) = 0$, then there exist an extension L/F of degree ℓ and $\mu \in L$ such that $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$ and $N_{L/F}(\mu) = \lambda$. We assume that

- F contains a primitive ℓ^{th} root of unity.

We begin with the following

Lemma 6.1. *If $\alpha \cdot (\lambda) = 0$, then $s\alpha = (E, \sigma, \lambda)$ for some cyclic extension E of F which is unramified on A except possibly at δ. In particular, if s is coprime to ℓ, then $\alpha = (E', \sigma', \lambda)$ for some cyclic extension E' of F which is unramified on A except possibly at δ.*

Proof. By (4.7), there exists an unramified cyclic extension E_π of F_π such that $s\alpha \otimes F_\pi = (E_\pi, \sigma, \lambda)$. Let $E(\pi)$ be the residue field of E_π. Then $E(\pi)$ is a cyclic extension of $\kappa(\pi)$. By (5.1), there exists a cyclic extension E of F which is unramified on

A except possibly at δ with $E \otimes F_\pi \simeq E_\pi$. Since E/F is unramified on A except possibly at δ and $\lambda = w\pi^s\delta^t$ with w a unit in A, (E, σ, λ) is unramified on A except possibly at (π) and (δ). Since α is unramified on A except possibly at (π) and (δ), $s\alpha - (E, \sigma, \lambda)$ is unramified on A except possibly at (π) and (δ). Since $s\alpha \otimes F_\pi = (E_\pi, \sigma, \lambda) = (E, \sigma, \lambda) \otimes F_\pi$, by (5.5), $s\alpha = (E, \sigma, \lambda)$. □

Lemma 6.2. *Suppose that $\alpha \cdot (\lambda) = 0$ and $\lambda \notin F^{*\ell}$. If $\alpha \neq 0$, then ind$(\alpha \otimes F(\sqrt[\ell]{\lambda})) <$ ind(α) and $\alpha \cdot (\sqrt[\ell]{\lambda}) = 0 \in H^3(F(\sqrt[\ell]{\lambda}), \mu_n^{\otimes 2})$.*

Proof. Suppose that s is coprime to ℓ. Then, by (6.1), $\alpha = (E', \sigma', \lambda)$ for some cyclic extension E' of F which is unramified on A except possibly at δ. Since $\nu_\pi(\lambda) = s$ is coprime to ℓ and E'/F is unramified at π, it follows that $\mathrm{ind}(\alpha) = [E' : F]$. In particular, $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) \leq [E' : F]/\ell < \mathrm{ind}(\alpha)$. Similarly, if t is coprime to ℓ, then $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \mathrm{ind}(\alpha)$. Further $\alpha \cdot (\sqrt[\ell]{\lambda}) = (E', \sigma', \lambda) \cdot (\sqrt[\ell]{\lambda}) = 0$.

Suppose that s and t are divisible by ℓ. Since $\lambda = w\pi^s\delta^t$, we have $F(\sqrt[\ell]{\lambda}) = F(\sqrt[\ell]{w})$. Let $L = F(\sqrt[\ell]{\lambda}) = F(\sqrt[\ell]{w})$ and B be the integral closure of A in L. Since w is a unit in A, by ([21, Lemma 3.1]), B is a complete regular local ring with maximal ideal generated by π and δ. Since w is not an ℓ^{th} power in F and A is a complete regular local ring, the image of w in A/m is not an ℓ^{th} power. Since $A/(\pi)$ is also a complete regular local ring with residue field A/m, the image of w in $A/(\pi)$ is not an ℓ^{th} power. Since F_π is a complete discrete valued field with residue field the field of fractions of $A/(\pi)$, w is not an ℓ^{th} power in F_π. Since $\alpha \cdot (\lambda) = 0$ and the residue field of F_π is a local field, by (4.9), $\mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha)$. Hence, by (5.8), $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.

Since $L_\pi = L \otimes F_\pi$ and $L_\delta = L \otimes F_\delta$ are field extension of degree ℓ over F_π and F_δ respectively and $\mathrm{cores}(\alpha \cdot (\sqrt[\ell]{\lambda})) = \alpha \cdot (\lambda) = 0$, by (4.6), $(\alpha \cdot (\sqrt[\ell]{\lambda})) \otimes L_\pi = 0$ and $(\alpha \cdot (\sqrt[\ell]{\lambda})) \otimes L_\delta = 0$. Hence, by (5.5), $\alpha \cdot (\sqrt[\ell]{\lambda}) = 0$. □

Lemma 6.3. *Suppose $\alpha = (E/F, \sigma, u\pi\delta^{\ell m})$ for some $m \geq 0$, u a unit in A, E/F a cyclic extension of degree ℓ^d which is unramified on A except possibly at δ and σ a generator of Gal(E/F). Let ℓ^e be the ramification index of E/F at δ and $f = d - e$. Let $i \geq 0$ be such that $\ell^f + \ell^{di} > \ell m$. Let $v \in A$ be a unit which is not in $F^{*\ell}$ and $L = F(\sqrt[\ell]{v\delta^{\ell^f + \ell^{di} - \ell m} + u\pi})$. If $f > 0$, then ind$(\alpha \otimes L) <$ ind(α)*

Proof. Let B be the integral closure of A in L and $r = \ell^f + \ell^{di} - \ell m$. Since $\ell^f + \ell^{di} > \ell m$, $L = F(\sqrt[\ell]{v\delta^r + u\pi})$ and $v\delta^r + u\pi$ is a regular prime in A. Thus B is a complete regular local ring (cf. [21, Lemma 3.2]) and π, δ remain primes in B. Note that π and δ may not generate the maximal ideal of B. Let L_π and L_δ be the completions of L at the discrete valuations given by π and δ respectively. Since $v \notin F^{*\ell}$, $F(\sqrt[\ell]{v})$ is the unique degree ℓ extension of F_π which is unramified on A. Since $f > 0$, there is a subextension E of degree ℓ over F which is unramified on A and hence $F(\sqrt[\ell]{v}) \subset E$.

Since E/F is unramified on A except possibly at δ, by (5.8), $[E : F] = [E_\pi : F_\pi]$ and hence $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha) = [E : F]$.

Since r is divisible by ℓ, $L_\pi \simeq F_\pi(\sqrt[\ell]{v})$ and hence $L_\pi \subset E_\pi$. Thus $\mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha)$. Since $r > 0$, $L_\delta \simeq F_\delta(\sqrt[\ell]{u\pi})$. Since $\alpha = (E/F, \sigma, u\pi\delta^{\ell m})$, $\mathrm{ind}(\alpha \otimes L_\delta) < [E \otimes L_\delta : L_\delta] \leq [E : F]$. In particular by (4.5), $\mathrm{per}(\alpha \otimes F_\pi) < \mathrm{ind}(\alpha)$ and $\mathrm{per}(\alpha \otimes F_\delta) < \mathrm{ind}(\alpha)$. Since $\alpha \otimes L$ is unramified on B except possibly at π and δ and $H^2(B, \mu_\ell) = 0$, $\mathrm{per}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. If $d = 1$, then $\mathrm{per}(\alpha \otimes L) < \mathrm{ind}(\alpha) = \ell$ and hence $\mathrm{per}(\alpha \otimes L) = \mathrm{ind}(\alpha \otimes L) = 1 < \mathrm{ind}(\alpha)$. Suppose that $d \geq 2$.

Let $\phi : \mathscr{X} \to \mathrm{Spec}(B)$ be a sequence of blow-ups such that the ramification locus of $\alpha \otimes L$ is a union of regular curves with normal crossings. Let $V = \phi^{-1}(P)$. To show that $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$, by (5.10), it is enough to show that for every point x of V, $\mathrm{ind}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$.

Let $x \in V$ be a closed point. Then, by (5.9), $\mathrm{ind}(\alpha \otimes L_x) = \mathrm{per}(\alpha \otimes L_x)$. Since $\mathrm{per}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$, $\mathrm{ind}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$.

Let $x \in V$ be a codimension zero point. Then $\phi(x)$ is the closed point of $\mathrm{Spec}(B)$. Let $\tilde{\nu}$ be the discrete valuation of L given by x. Then $\kappa(\tilde{\nu}) \simeq \kappa'(t)$ for some finite extension κ' over κ and a variable t over κ. Let ν be the restriction of $\tilde{\nu}$ to F.

Suppose that $\nu(\delta^r) < \nu(\pi)$. Then $L \otimes F_\nu = F_\nu(\sqrt[\ell]{v\delta^r})$. Since ℓ divides r, $L \otimes F_\nu = F_\nu(\sqrt[\ell]{v})$. Since $F(\sqrt[\ell]{v}) \subset E$, $\mathrm{ind}(\alpha \otimes L \otimes F_\nu) < \mathrm{ind}(\alpha)$. Suppose that $\nu(\delta^r) > \nu(\pi)$. Then $L \otimes F_\nu = F_\nu(\sqrt[\ell]{u\pi})$ and as above $\mathrm{ind}(\alpha \otimes L \otimes F_\nu) < \mathrm{ind}(\alpha)$. Suppose that $\nu(\delta^r) = \nu(\pi)$. Let $\lambda = \pi/\delta^r$. Then λ is a unit at ν and $L_{\tilde{\nu}} = F_\nu(\sqrt[\ell]{v + u\lambda})$. We have $u\pi\delta^{\ell m} = u\lambda\delta^{r+\ell m} = u\lambda\delta^{\ell^f + \ell^{d}i}$ and

$$\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\pi\delta^{\ell m}) = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\lambda\delta^{\ell^f + \ell^{d}i}).$$

Since $[E : F] = \ell^d$, $\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\lambda\delta^{\ell^f})$. Suppose that $f = d$. Then E/F is unramified and hence every element of A^* is a norm from E. Thus $(E\otimes F_\nu/F_\nu, \sigma\otimes 1, w_0u\lambda)$ with $w_0 \in A^* \setminus A^{*\ell}$. Suppose that $f < d$. Then $e = d - f > 0$ and hence by (5.11), we have $E = E_{nr}(\sqrt[\ell^e]{w\delta})$, for some unit w in the integral closure of A in E_{nr}, with $N(\sqrt[\ell^d]{w\delta}) = w_1\delta^{\ell^f}$ with $w_1 \in A^* \setminus A^{*\ell}$. Thus

$$\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\lambda\delta^{\ell^f}) = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, w_0u\lambda).$$

with $w_0 = w_1^{-1}$.

If $E \otimes F_\nu$ is not a field, then $\mathrm{ind}(\alpha \otimes F_\nu) < [E : F]$. Suppose $E \otimes F_\nu$ is a field. Let $\theta = w_0u\lambda$. Since $\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, \theta)$, $\mathrm{ind}(\alpha \otimes L \otimes F_\nu) \leq \mathrm{ind}(\alpha \otimes L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta})) \cdot [L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta}) : L \otimes F_\nu]$. Since $[L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta}) : L \otimes F_\nu] \leq \ell^{d-1} < [E : F]$, it is enough to show that $\alpha \otimes L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta})$ is trivial.

Since $F(\sqrt[\ell]{v})/F$ is the unique subextension of E/F degree ℓ and $[E : F] = \ell^d$, we have $\alpha \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta}) = (F_\nu(\sqrt[\ell^{d-1}]{\theta}, \sqrt[\ell]{v})/F(\sqrt[\ell^{d-1}]{\theta}), \sigma, \sqrt[\ell^{d-1}]{\theta})$ (cf. 2.1). Let $M = F_\nu(\sqrt[\ell^{d-1}]{\theta})$. Since κ contains a primitive ℓ^{th} root of unity, we have $\alpha\otimes M = (v, \sqrt[\ell^{d-1}]{\theta})_\ell$. Then M is a complete discrete valuation field. Since λ is a unit at ν, θ is a unit at ν. Hence the residue field of M is $\kappa(\nu)(\sqrt[\ell^{d-1}]{\overline{\theta}})$. Since θ and v are units at ν, $\alpha \otimes M = (v, \sqrt[\ell^{d-1}]{\theta})$ is unramified at the discrete valuation of M. Hence it is enough to show that the specialization β of $\alpha \otimes M$ is trivial over $\kappa(\nu)(\sqrt[\ell^{d-1}]{\overline{\theta}}) \otimes L_0$, where L_0 is the residue field of $L \otimes F_\nu$ at ν.

Suppose that $L_{\tilde{\nu}}/F_\nu$ is ramified. Since $L_{\tilde{\nu}} = F_\nu(\sqrt[\ell]{u + v\lambda})$, $v + u\lambda$ is not a unit at ν. Thus $v = -u\lambda$ modulo $F_\nu^{*\ell^d}$ and $\theta = w_0u\lambda = -w_0v$ modulo $F_\nu^{*\ell^d}$. In particular $\sqrt[\ell^{d-1}]{\theta} = \sqrt[\ell^{d-1}]{-w_0v}$ modulo $M^{*\ell}$. Since $\overline{v}, \overline{w_0} \in \kappa$ and κ a finite field, $\beta = (\sqrt[\ell]{\overline{v}}, \sqrt[\ell^{d-1}]{\theta}) = (\sqrt[\ell]{\overline{v}}, \sqrt[\ell^{d-1}]{-\overline{w_0}\overline{v}})$ is trivial.

Suppose that $L_{\tilde{\nu}}/F_\nu$ is unramified. Then $L_0 = \kappa(\pi)(\sqrt[\ell]{\overline{v} + \overline{u\lambda}})$. Since $\kappa(\pi)$ is a global field and $d - 1 \geq 1$, by (4.13), $\beta \otimes L_0(\sqrt[\ell^{d-1}]{\overline{\theta}}) = 0$. □

Lemma 6.4. *Suppose L_π/F_π and L_δ/F_δ are unramified cyclic field extensions of degree ℓ and $\mu_\pi \in L_\pi$, $\mu_\delta \in L_\delta$ such that*

- *$\mathrm{ind}(\alpha \otimes L_\pi) < d_0$ for some d_0,*

• $\lambda = N_{L_\pi/F_\pi}(\mu_\pi)$ *and* $\lambda = N_{L_\delta/F_\delta}(\mu_\delta)$*,*
• $\alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_n^{\otimes 2})$*,* $\alpha \cdot (\mu_\delta) = 0 \in H^3(L_\delta, \mu_n^{\otimes 2})$*,*
• *if* $\lambda \in F_P^{*\ell}$ *and* $\alpha = (E/F, \sigma, v\pi)$ *for some cyclic extension* E/F *which is unramified on* A *except possibly at* δ*, then* $L_\delta/F_\delta = F_\delta(\sqrt[\ell]{v\pi})$*.*

Then there exists a cyclic extension L/F *of degree* ℓ *and* $\mu \in L$ *such that*
• *ind*$(\alpha \otimes L) < d_0$*,*
• $\lambda = N_{L/F}(\mu)$*,*
• $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$*,*
• $L \otimes F_\pi \simeq L_\pi$ *and* $L \otimes F_\delta \simeq L_\delta$*.*

Proof. Since $\alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_n^{\otimes 2})$ and $\lambda = N_{L_\pi/F_\pi}(\mu_\pi)$, by taking the corestriction, we see that $\alpha \cdot (\lambda) = 0 \in H^3(F_\pi, \mu_n^{\otimes 2})$. Since $\alpha \cdot (\lambda)$ is unramified on A except possibly at π and δ, by (5.5), $\alpha \cdot (\lambda) = 0$.

Suppose that $\lambda \notin F^{*\ell}$. Then, by (2.6) and (6.2), $L = F(\sqrt[\ell]{\lambda})$ and $\mu = \sqrt[\ell]{\lambda}$ have the required properties.

Suppose that $\lambda \in F^{*\ell}$. Let $L(\pi)$ and $L(\delta)$ be the residue fields of L_π and L_δ respectively. Since L_π/F_π and L_δ/F_δ are unramified cyclic extensions of degree ℓ, $L(\pi)/\kappa(\pi)$ and $L(\delta)/\kappa(\delta)$ are cyclic extensions of degree ℓ. Since F contains a primitive ℓ^{th} root of unity, we have $L(\pi) = \kappa(\pi)[X]/(X^\ell - a)$ and $L(\delta) = \kappa(\delta)[X]/(X^\ell - b)$ for some $a \in \kappa(\pi)$ and $b \in \kappa(\delta)$. Since $\kappa(\pi)$ is a complete discretely valued field with $\overline{\delta}$ a parameter, without loss of generality we assume that $a = \overline{u_1}\overline{\delta}^{\epsilon}$ for some unit $u_1 \in A$ and $\epsilon = 0$ or 1. Similarly we have $b = \overline{u_2}\overline{\pi}^{\epsilon'}$ for some unit $u_2 \in A$ and $\epsilon' = 0$ or 1.

By (5.7), we assume that $\alpha = (E/F, \sigma, u\pi\delta^j)$ for some cyclic extension E/F which is unramified on A except possibly at δ, u a unit in A and $j \geq 0$. Then $\text{ind}(\alpha) = [E : F]$. Let E_0 be the residue field of E at π. Then $[E : F] = [E_0 : \kappa(\pi)]$. Since $\partial_\pi(\alpha) = (E_0/\kappa(\pi), \overline{\sigma})$, $\text{per}(\partial_\pi(\alpha)) = [E : F] = \text{ind}(\alpha)$. Since L_π/F_π is an unramified cyclic extension of degree ℓ and $\text{ind}(\alpha \otimes L_\pi) < \text{ind}(\alpha)$, the residue field $L(\pi)$ of L_π is the unique degree ℓ subextension of $E_0/\kappa(\pi)$.

Suppose that $\epsilon = \epsilon' = 0$. Since L_π and L_δ are fields, u_1 and u_2 are not ℓ^{th} powers. Let L/F be the unique cyclic field extension of degree ℓ which is unramified on A. Then $L \otimes F_\pi \simeq L_\pi$ and $L \otimes F_\delta \simeq L_\delta$. Let B be the integral closure of A in L. Then B is a regular local ring with maximal ideal (π, δ) and hence by (5.8) $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$.

Suppose $\epsilon = 1$. Then $L_\pi = F_\pi(\sqrt[\ell]{u_1\delta})$ and $L(\pi) = \kappa(\pi)(\sqrt[\ell]{\overline{u_1}\overline{\delta}})$. Since $E_0/\kappa(\pi)$ is a cyclic extension containing a totally ramified extension, $E_0/\kappa(\pi)$ is a totally ramified cyclic extension. Thus $\kappa(\pi)$ contains a primitive $\ell^{d^{\text{th}}}$ root of unity and $E_0 = \kappa(\pi)(\sqrt[\ell^d]{\overline{u_1}\overline{\delta}})$. In particular F contains a primitive $\ell^{d^{\text{th}}}$ root of unity and $\alpha = (u_1\delta, u\pi\delta^j) = (u_1\delta, u'\pi)$. Since L_δ/F_δ is an unramified extension of degree ℓ with $\text{ind}(\alpha \otimes L_\delta) < \text{ind}(\alpha)$, as above, we have $L_\delta = F_\delta(\sqrt{u'\pi})$ and hence $\alpha = (u_1\delta, u_2\pi)$. Let $L = F(\sqrt[\ell]{u_1\delta + u_2\pi})$. Then $L \otimes F_\pi \simeq L_\pi$ and $L \otimes F_\delta \simeq L_\delta$. Since for any $a, b \in F^*$, $(a, b) = (a+b, -a^{-1}b)$, we have $\alpha = (u_1\pi + u_2\delta, -u_1^{-1}\pi^{-1}u_2\delta)$. In particular $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$.

Suppose that $\epsilon = 0$ and $\epsilon' = 1$. Suppose j is coprime to ℓ. Then, by (4.15), $\text{ind}(\alpha) = \text{per}(\partial_\delta(\alpha))$ and as in the proof of (5.7), we have $\alpha = (E'/F, \sigma', v\delta\pi^{j'})$ for some cyclic extension E'/F which is unramified on A except possibly at π. Thus, we have the required extension as in the case $\epsilon = 1$.

Suppose j is divisible by ℓ. Since $\epsilon = 0$, $L_\pi = F_\pi(\sqrt[\ell]{u_1})$. Since the residue field $L_\pi(\pi)$ of L_π is contained in the residue field E_0 of E at π, $F(\sqrt[\ell]{u_1}) \subset E$ and hence E/F is not totally ramified at δ. Since E/F is unramified on A except possibly at δ, by (5.11), $E = E_{nr}(\sqrt[\ell^e]{w\delta})$ for some unit w in the integral closure of A in E_{nr}. Suppose $e = 0$. Then $E = E_{nr}/F$ is unramified on A. Since κ is a finite field and A is complete, every unit in A is a norm from E/F. Thus multiplying $u\pi\delta^j$ by a norm from E/F we assume that $\alpha = (E/F, \sigma, u_2\pi\delta^j)$. Suppose that $e > 0$. Then, by (5.11), $N_{E/F}(w\delta) = w_1\delta^{\ell f}$ with $w_1 \in A^* \setminus A^{*\ell}$. Since $A^*/A^{*\ell}$ is a cyclic group of order ℓ, we have $\alpha = (E/F, \sigma, u_2\pi\delta^{j+j'\ell^f})$ for some j'. Since j is divisible by ℓ and $f \geq 1$, $j + j'\ell^f$ is divisible by ℓ. Hence, we assume that $\alpha = (E/F, \sigma, u_2\pi\delta^{\ell m})$ for some m. Thus, by (6.3), there exists $i \geq 0$ such that $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$ for $L = F(\sqrt[\ell]{u_1\delta^{\ell f+di} + u_2\pi\delta^{\ell m}})$.

By the choice, we have L/F is the unique unramified extension or $L = F(\sqrt[\ell]{u_1\delta + u_2\pi})$ or $L = F(\sqrt[\ell]{u_1\delta^{\ell f+di} + u_2\pi\delta^{\ell m}})$ with $\ell^{f+di} > \ell m$. Let B be the integral closure of A in L. Then B is a complete regular local ring with π and δ remain prime in B. Since $\lambda = w\pi^s\delta^t$ and $\lambda \in F_P^{*\ell}$, we have $\lambda = w_0^\ell\pi^{\ell s_1}\delta^{\ell t_1}$ for some unit $w_0 \in A$. Let $\mu = w_0\pi^{s_1}\delta^{t_1} \in F$. Then $N_{L/F}(\mu) = \mu^\ell = \lambda$. Since $\alpha \cdot (\lambda) = 0$, by (4.6), $\alpha \cdot (\mu) = 0$ in $H^3(L_\pi, \mu_n^{\otimes 2})$ and $H^3(L_\delta, \mu_n^{\otimes 2})$. Hence $\alpha \cdot (\mu)$ is unramified at all height one prime ideals of B. Since B is a complete regular local ring with residue field κ finite, $\alpha \cdot (\mu) = 0$ (5.3). □

Lemma 6.5. *Suppose that $\nu_\pi(\lambda)$ is divisible by ℓ, α is unramified on A except possibly at π and δ, and $\alpha \cdot (\lambda) = 0$. Let L_π be a cyclic unramified or split extension of F_π of degree ℓ, $\mu_\pi \in L_\pi$ and $d_0 \geq 2$ such that*
- $N_{L_\pi/F_\pi}(\mu_\pi) = \lambda$,
- *ind*$(\alpha \otimes L_\pi) < d_0$,
- $\alpha \cdot (\mu_\pi) = 0$ *in* $H^3(L_\pi, \mu_n)$.

Then there exists an extension L over F of degree ℓ and $\mu \in L$ such that
- $N_{L/F}(\mu) = \lambda$,
- *ind*$(\alpha \otimes L) < d_0$,
- $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$ *and*
- *there is an isomorphism $\phi : L_\pi \to L \otimes F_\pi$ with*

$$\phi(\mu_\pi)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_\pi)^{\ell m},$$

for all $m \geq 1$.

Proof. Since $\nu_\pi(\lambda)$ is divisible by ℓ, $\lambda = w\pi^{r\ell}\delta^s$ for some $w \in A$ a unit.

Suppose that $L_\pi = \prod F_\pi$ is a split extension. Let $L = \prod F$ be the split extension of degree ℓ. Since $\mu_\pi \in L_\pi$, we have $\mu_\pi = (\mu_1, \cdots, \mu_\ell)$ with $\mu_i \in F_\pi$. Write $\mu_i = \theta_i\pi^{r_i}$ with $\theta_i \in F_\pi$ a unit at its discrete valuation. Since $N_{L_\pi/F_\pi}(\mu_\pi) = \lambda = w\pi^{r\ell}$, $\theta_1 \cdots \theta_\ell = w$ and $\pi^{r_1+\cdots+r_\ell} = \pi^{r\ell}$. For $2 \leq i \leq \ell$, let $\bar{\theta}_i$ be the image of θ_i in the residue field $\kappa(\pi)$ of F_π. Since $\kappa(\pi)$ is the field of fractions of $A/(\delta)$ and $A/(\delta)$ is a complete discrete valuation ring with $\bar{\delta}$ as a parameter, we have $\bar{\theta}_i = \bar{u}_i\bar{\delta}^{s_i}$ for some unit $u_i \in A$. For $2 \leq i \leq \ell$, let $\tilde{\theta}_i = u_i\delta^{s_i} \in F$, $\tilde{\theta}_1 = w\tilde{\theta}_2^{-1} \cdots \tilde{\theta}_\ell^{-1}$ and $\mu = (\tilde{\theta}_1\pi^{r_1}, \cdots, \tilde{\theta}_\ell\pi^{r_\ell}) \in L = \prod F$. Then $N_{L/F}(\mu) = \lambda$. Since $\tilde{\theta}_i\pi^{r_i}\mu_i^{-1}$ is a unit at π with image 1 in $\kappa(\pi)$, $\tilde{\theta}_i\pi^{r_i}\mu_i^{-1} \in F_\pi^{\ell^m}$ for any $m \geq 1$. In particular $\alpha \cdot (\tilde{\theta}_i\pi^{r_i}) = \alpha \cdot (\mu_i) = 0 \in H^3(F_\pi, \mu_n^{\otimes 2})$. Since α is unramified on A except possibly at π and δ and $\tilde{\theta}_i = u_i\delta^{t_i}$ with $u_i \in A$ a unit, $\alpha \cdot (\tilde{\theta}_i\pi^{r_i})$ is unramified on A except possibly at π and δ. Thus, by (5.5), $\alpha \cdot (\tilde{\theta}_i\pi^{r_i}) = 0 \in H^3(F, \mu_n^{\otimes 2})$. Since $\mathrm{ind}(\alpha \otimes L_\pi) < d_0$

and L_π is the split extension, $\operatorname{ind}(\alpha \otimes F_\pi) < d_0$. Since α is unramified on A except possibly at π and δ, by (5.8), $\operatorname{ind}(\alpha) < d_0$. Thus L and $\mu = (\tilde{\theta}_1\pi^{r_1}, \cdots, \tilde{\theta}_\ell\pi^{r_\ell}) \in L$ have the required properties.

Suppose that L_π is a field extension of F_π. By (5.1), there exists a cyclic extension L of F of degree ℓ which is unramified on A except possibly at δ with $L \otimes F_\pi \simeq L_\pi$. Let B be the integral closure of A in L. By the construction of L, either L/F is unramified A or $L = F(\sqrt[\ell]{u\delta})$ for some unit $u \in A$. Replacing δ by $u\delta$, we assume that L/F is unramified on A or $L = F(\sqrt[\ell]{\delta})$. In particular, B is a regular local ring with maximal ideal generated by (π, δ') for $\delta' = \delta$ or $\delta' = \sqrt[\ell]{\delta}$ (cf. [21, Lemma 3.2]). Since α is unramified on A except possibly at π and δ, $\alpha \otimes L$ is unramified on B except possibly at π and δ'. Since $\operatorname{ind}(\alpha \otimes L_\pi) < d_0$, by (5.8), $\operatorname{ind}(\alpha \otimes L) = \operatorname{ind}(\alpha \otimes L_\pi) < d_0$.

Since L_π/F_π is unramified and $N_{L_\pi/F_\pi}(\mu_\pi) = \lambda = w\pi^{r\ell}\delta^s$, we have $\mu_\pi = \theta_\pi\pi^r$ for some $\theta_\pi \in L_\pi$ which is a unit at its discrete valuation. Let $\overline{\theta}_\pi$ be the image of θ_π in $L(\pi)$. Since $L(\pi)$ is the field of fractions of the complete discrete valuation ring $B/(\pi)$ and $\overline{\delta'}$ is a parameter in $B/(\pi)$, we have $\overline{\theta}_\pi = \overline{v}\overline{\delta'}^t$ for some unit $v \in B$. Since $N_{L(\pi)/\kappa(\pi)}(\overline{\theta}_\pi) = \overline{w}\overline{\delta}^s$, it follows that $N_{L(\pi)/\kappa(\pi)}(\overline{v}) = \overline{w}$. Since w is a unit in A, there exists a unit $\tilde{v} \in B$ with $N_{L/F}(\tilde{v}) = w$ and $\overline{\tilde{v}} = \overline{v}$. Let $\mu = \tilde{v}\pi^r\delta'^t \in L$. Then $\mu\mu_\pi^{-1} \in L_\pi$ is a unit in the valuation ring at π with the image 1 in the residue field $L(\pi)$ and hence $\mu\mu_\pi^{-1} \in L_\pi^{\ell m}$ for all $m \geq 1$. In particular $\alpha \cdot (\mu) = \alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_n^{\otimes 2})$. Since $\operatorname{ind}(\alpha \otimes L_\pi) < d_0$, by (5.8), $\operatorname{ind}(\alpha \otimes L) < d_0$. Since $\alpha \cdot (\mu) = 0$ in $H^3(L_\pi, \mu_n^{\otimes 2})$, α is unramified on A except possibly at π and the support of μ on A is at most π, by (5.5), $\alpha \cdot (\mu) = 0$ in $H^3(L, \mu_n^{\otimes 2})$. ∎

Lemma 6.6. *Suppose that $\alpha \cdot (\lambda) = 0$ and $\nu_\delta(\lambda) = s\ell$. Suppose that $\alpha = (E/F, \sigma, \pi\delta^m)$ for some cyclic extension E/F which is unramified on A except possibly at δ. Let E_δ be the lift of the residue of α at δ. If $s\alpha \otimes E_\delta = 0$, then there exists an integer $r_1 \geq 0$ such that $w_1\delta^{mr_1 - s}$ is a norm from the extension E/F for some unit $w_1 \in A$.*

Proof. Write $\alpha \otimes F_\delta = \alpha' + (E_\delta/F_\delta, \sigma_\delta, \delta)$ as in (4.1). Since $\alpha \otimes E_\delta = \alpha' \otimes E_\delta$, $s\alpha' \otimes E_\delta = 0$. Hence $s\alpha' = (E_\delta, \sigma, \theta)$ for some $\theta \in F_\delta$. Since α' and E_δ/F_δ are unramified at δ, we assume that $\theta \in F_\delta$ is a unit at δ. Since the residue field $\kappa(\delta)$ of F_δ is a complete discrete valued field with the image of π as a parameter, without loss of generality we assume that $\theta = w_0\pi^{r_1}$ for unit $w_0 \in A$ and $r_1 \geq 0$. Let $\lambda_1 = w_0\pi^{r_1}\delta^s$. Since $s\alpha' = (E_\delta, \sigma_\delta, \theta)$, by (4.7), $\alpha \cdot (\lambda_1) = 0 \in H^3(F_\delta, \mu_n^{\otimes 2})$. Since α is unramified on A except possibly at π,δ and $\lambda_1 = w_0\pi^{r_1}\delta^s$ with $w_0 \in A$ a unit, $\alpha \cdot (\lambda_1)$ is unramified in A except possibly at π and δ. Hence, by (5.5), $\alpha \cdot (\lambda_1) = 0 \in H^3(F, \mu_n^{\otimes 2})$. We have

$$0 = \partial_\pi(\alpha \cdot (\lambda_1)) = \partial_\pi((E/F, \sigma, u\pi\delta^m) \cdot (w_0\pi^{r_1}\delta^s)) = (E(\pi)/\kappa(\pi), \overline{\sigma}, (-1)^{r_1}\overline{u}^{r_1}\overline{w}_1^{-1}\overline{\delta}^{mr_1 - s}).$$

Since $(E/F, \sigma, (-1)^{r_1}u^{r_1}w_0^{-1}\delta^{mr_1 - s})$ is unramified on A except possibly at π and δ, by (5.5), $(E/F, \sigma, (-1)^{r_1}u^{r_1}w_0^{-1}\delta^{mr_1 - s}) = 0$. In particular $(-1)^{r_1}u^{r_1}w_0^{-1}\delta^{mr_1 - s}$ is a norm from the extension E/F. ∎

Lemma 6.7. *Suppose that $\alpha \cdot (\lambda) = 0$ and $\lambda = w\pi^r\delta^{s\ell}$ for some unit $w \in A$ and r coprime to ℓ. Let E_δ be the lift of the residue of α at δ. If $s\alpha \otimes E_\delta = 0$, then there exists $\theta \in A$ such that*
- *$\alpha \cdot (\theta) = 0$,*
- *$\nu_\pi(\theta) = 0$,*
- *$\nu_\delta(\theta) = s$.*

Proof. Since r is coprime to ℓ, by (6.1), $\alpha = (E/F, \sigma, \lambda)$ for some cyclic extension E/F which is unramified on A except possible at δ. Let $t = [E : F]$. Since t is a power of ℓ and r is coprime to ℓ, there exists an integer $r' \geq 1$ such that $rr' \equiv 1$ modulo t. We have

$$\alpha = \alpha^{rr'} = (E/F, \sigma, w\pi^r\delta^{s\ell})^{rr'} = (E/F, \sigma)^r \cdot (w\pi^r\delta^{s\ell})^{r'} = (E/F, \sigma)^r \cdot (w^{r'}\pi\delta^{r's\ell}).$$

Since r is coprime to ℓ, we also have $(E/F, \sigma)^r = (E/F, \sigma^{r'})$ (cf. §2) and hence $\alpha = (E/F, \sigma^r, \pi\delta^{r's\ell})$. Thus, by (6.6), there exist a unit $w_1 \in A$ and $r_1 \geq 0$ such that $w_1\delta^{r's\ell r_1 - s}$ is a norm from E/F. Since $r'\ell r_1 - 1$ is coprime to ℓ, $r'\ell r_1 - 1$ is coprime to t and hence there exists an integer $r_1 \geq 0$ such that $(r'\ell r_1 - 1)r_2 \equiv 1$ modulo t. In particular $w_1^{r_2}\delta^s \equiv (w_1\delta^{r's\ell r_1 - s})^{r_2}$ modulo F^{*t} and hence $w_1^{r_2}\delta^s$ is a norm from E/F. Thus $\theta = w_1^{r_2}\delta^s$ has the required properites. □

Lemma 6.8. *Let E_π and E_δ be the lift of the residues of α at π and δ respectively. Suppose that $\alpha \cdot (\lambda) = 0$ and $\lambda = w\pi^{r\ell}\delta^{s\ell}$ for some unit $w \in A$. If $\alpha \cdot (\lambda) = 0$, $r\alpha \otimes E_\pi = 0$ and $s\alpha \otimes E_\delta = 0$, then there exists $\theta \in A$ such that*
- $\alpha \cdot (\theta) = 0$,
- $\nu_\pi(\theta) = r$,
- $\nu_\delta(\theta) = s$.

Proof. By (5.7), we assume that $\alpha = (E/F, \sigma, u\pi\delta^m)$ for some extension E/F which is unramified on A except possible at δ and $m \geq 0$. Without loss of generality, we assume that $0 \leq m < [E : F]$. By (6.6), there exists an integer $r_1 \geq 0$ such that $w_1\delta^{mr_1 - s}$ is a norm from E/F. Let $t = [E - F]$ and $\theta = (-u\pi + \delta^{t-m})^{r_1 - r}w_1^{-1}(-u)^r\pi^r\delta^s$. Since $t - m > o$, we have $\nu_\pi(\theta) = r$ and $\nu_\delta(\theta) = s$.

Now we show that $\alpha \cdot (\theta) = 0$. Since $t - m > 0$, we have $(-u\pi + \delta^{t-m})^{r_1 - r} = (-u\pi)^{r_1 - r}$ modulo δ and hence $\theta \equiv (-u)^{r_1 - r}\pi^{r_1 - r}w_1^{-1}(-u)^r\pi^r\delta^s = w_1^{-1}(-u)^{r_1}\pi^{r_1}\delta^s$ modulo F_δ^{*t}. Since $w_1\delta^{mr_1 - s}$ is a norm from E/F, we have

$$\begin{aligned}
(\alpha \cdot (\theta)) \otimes F_\delta &= (E/F, \sigma, u\pi\delta^m) \cdot (w_1^{-1}(-u)^{r_1}\pi^{r_1}\delta^s) \otimes F_\delta \\
&= (E/F, \sigma, u\pi\delta^m) \cdot (w_1^{-1}(-u)^{r_1}\pi^{r_1}\delta^s w_1\delta^{mr_1 - s}) \otimes F_\delta \\
&= (E/F, \sigma, u\pi\delta^m) \cdot ((-u)^{r_1}\pi^{r_1}\delta^{mr_1}) \otimes F_\delta \\
&= (E/F, \sigma, u\pi\delta^m) \cdot ((-u\pi\delta^m)^{r_1}) \otimes F_\delta = 0.
\end{aligned}$$

Thus $\alpha \cdot (\theta)$ is unramified at δ.

We have $(-u\pi + \delta^{t-m})^{r_1 - r} \equiv \delta^{t(r_1 - r) + m(r - r_1)}$ modulo π and hence

$$\theta \equiv \delta^{t(r_1 - r) + m(r - r_1)}w_1^{-1}(-u)^r\pi^r\delta^s \equiv (-u\pi\delta^m)^r(w_1\delta^{mr_1 - s})^{-1} \text{ modulo } F_\pi^{*t}.$$

Since $w_1\delta^{mr_1 - s}$ is a norm from E/F and $t = [E : F]$, we have

$$\begin{aligned}
(\alpha \cdot (\theta)) \otimes F_\pi &= (E/F, \sigma, u\pi\delta^m) \cdot ((-u\pi\delta^m)^r(w_1\delta^{mr_1 - s})^{-1}) \otimes F_\pi \\
&= (E/F, \sigma, u\pi\delta^m) \cdot ((-u\pi\delta^m)^r) \otimes F_\pi = 0.
\end{aligned}$$

In particular $\alpha \cdot (\theta)$ is unramified at δ.

Let γ be a prime in A with $(\gamma) \neq (\pi)$ and $(\gamma) \neq (\delta)$. Since α is unramified on A except possibly at π and δ, if γ does not divide θ, then $\alpha \cdot (\theta)$ is unramified at γ. Suppose γ divides θ. Then $\gamma = -u\pi + \delta^{t-m}$. Thus $u\pi\delta^m \equiv \delta^t$ modulo γ. Since $\partial_\gamma(\alpha \cdot (\theta)) = (E(\theta), \overline{\sigma}, \overline{u\pi}\overline{\delta}^m)$, where $E(\theta)$ is the residue field of E at θ and $^-$ denotes the image modulo γ, we have $\partial_\gamma(\alpha \cdot (\theta)) = (E(\theta), \overline{\sigma}, \overline{u\pi}\overline{\delta}^m) = (E(\theta), \overline{\sigma}, \overline{\delta}^t) = 0$.

Hence $\alpha \cdot (\theta)$ is unramified on A. Since $\alpha \cdot (\theta) \otimes F_\pi = 0$, by (5.5), we have $\alpha \cdot (\theta) = 0$. □

7. Patching

We fix the following data:

- R a complete discrete valuation ring,
- K the field of fractions of R,
- κ the residue field of R,
- ℓ a prime not equal to $\text{char}(\kappa)$ and $n = \ell^d$ for some $d \geq 1$.
- X a smooth projective geometrically integral variety over K,
- F the function field of X,
- $\alpha \in H^2(F, \mu_n)$, $\alpha \neq 0$,
- $\lambda \in F^*$ with $\alpha \cdot (\lambda) = 0$,
- $\mathscr{X}$ a normal proper model of X over R and X_0 the reduced special fibre of $\mathscr{X}$.
- $\mathscr{P}_0$ a finite set of closed points of X_0 containing all the points of intersection of irreducible components of X_0.

For $x \in \mathscr{X}$, let $\hat{A}_x$ be the completion of the regular local ring at x on $\mathscr{X}$, F_x the field of fractions of $\hat{A}_x$ and $\kappa(x)$ the residue field at x. Let $\eta \in X_0$ be a codimension zero point and $P \in X_0$ be a closed point such that P is in the closure of η. For abuse of the notation we denote the closure of η by η and say that P is a point of η. A pair (P, η) of a closed point P and a codimension zero point of X_0 is called a **branch** if P is in η. Let (P, η) be a branch. Let $F_{P,\eta}$ be the completion of F_P at the discrete valuation on F_P associated to η. Then F_x and F_P are subfields of $F_{P,x}$. Since $\kappa(\eta)$ is the function field of the curve η, any closed point of η gives a discrete valuation on $\kappa(\eta)$. The residue field $\kappa(\eta)_P$ of $F_{P,\eta}$ is the completion of $\kappa(\eta)$ at the discrete valuation on $\kappa(\eta)$ given by P. Let η be a codimension zero point of X_0 and $U \subset \eta$ be a non-empty open subset. Let A_U be the ring of all those functions in F which are regular at every closed point of U. Let t be parameter in R. Then $t \in R_{U_\eta}$. Let $\hat{A}_U$ be the (t)-adic completion of A_U and F_U be the field of fractions of $\hat{A}_U$. Then $F \subseteq F_U \subseteq F_\eta$.

We begin with the following result, which follows from ([11, Theorem 9.11]) (cf. proof of [22, Theorem 2.4]).

Proposition 7.1. *For each irreducible component X_η of X_0, let U_η be a non-empty proper open subset of X_η and $\mathscr{P} = X_0 \setminus \cup_\eta U_\eta$, where η runs over the codimension zero points of X_0. Suppose that $\mathscr{P}_0 \subseteq \mathscr{P}$. Let L be a finite extension of F. Suppose that there exists $N \geq 1$ such that for each codimension zero point η of X_0, ind$(\alpha \otimes L \otimes F_{U_\eta}) \leq N$ and for every closed point $P \in \mathscr{P}$, ind$(\alpha \otimes L \otimes F_P) \leq N$. Then ind$(\alpha \otimes L) \leq N$.*

Proof. Let $\mathscr{Y}$ be the integral closure of $\mathscr{X}$ in L and $\phi : \mathscr{Y} \to \mathscr{X}$ be the induced map. Let $\mathscr{P}'$ be a finite set if closed points of $\mathscr{Y}$ containing points of the intersection of distinct irreducible curves on the special fibre Y_0 of $\mathscr{Y}$ and inverse image of $\mathscr{P}$ under ϕ. Let U be an irreducible component of $Y_0 \setminus \mathscr{P}'_0$. Then $\phi(U) \subset U_\eta$ for some U_η and there is a homomorphism of algebras from $L \otimes F_{U_\eta}$ to L_U. (Note that $L \otimes F_{U_\eta}$ may be a product of fields). Since $\text{ind}(\alpha \otimes L \otimes F_{U_\eta}) \leq d$, we have $\text{ind}(\alpha \otimes L_U) \leq N$. Let $Q \in \mathscr{P}'$. Suppose $\phi(Q) = P \in \mathscr{P}$. Then there is a homomorphism of algebras from $L \otimes F_P$ to L_Q. (Once again note that $L \otimes F_P$ may be a product of fields). Since $\text{ind}(\alpha \otimes L \otimes F_P) \leq N$, $\text{ind}(\alpha \otimes L_Q) \leq N$. Suppose that $\phi(Q) \in U_\eta$ for some U_η. Then there is a homomorphism of algebras from $L \otimes F_{U_\eta}$ to L_Q. Thus $\text{ind}(\alpha \otimes L_Q) \leq N$. Therefore, by ([11, Theorem 9.11]), $\text{ind}(\alpha \otimes L) \leq N$. □

Lemma 7.2. *Let η be a codimension zero point of X_0. Suppose there exists a field extension or split extension L_η/F_η of degree ℓ and $\mu_\eta \in L_\eta$ such that*
1) $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $ind(\alpha \otimes L_\eta) < ind(\alpha)$
3) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$.
Then there exists a non-empty open subset U_η of η, a split or field extension L_{U_η}/F_{U_η} of degree ℓ and $\mu_{U_\eta} \in L_{U_\eta}$ such that
1) $N_{L_{U_\eta}/F_{U_\eta}}(\mu_{U_\eta}) = \lambda$
2) $ind(\alpha \otimes L_\eta) < ind(\alpha)$
3) $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L_{U_\eta}, \mu_n^{\otimes 2})$
4) there is an isomorphism $\phi_{U_\eta} : L_{U_\eta} \otimes F_\eta \to L_\eta$ with $\phi_{U_\eta}(\mu_{U_\eta} \otimes 1)\mu_\eta^{-1} \equiv 1$ modulo the radical of the integral closure of $\hat{R}_\eta$ in L_η.
Further if L_η/F_η is cyclic, then L_{U_η}/F_{U_η} is cyclic.

Proof. Suppose $L_\eta = \prod F_\eta$ is the split extension of degree ℓ. Write $\mu_\eta = (\mu_1, \cdots, \mu_\ell)$ with $\mu_i \in F_\eta$. Then $\lambda = N_{L_\eta/F_\eta}(\mu_\eta) = \mu_1 \cdots \mu_\ell$. Since $\mathrm{ind}(\alpha \otimes L_\eta) = \mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$, by ([11, Proposition 5.8], [18, Proposition 1.17]), there exists a non-empty open subset U_η of η such that $\mathrm{ind}(\alpha) \otimes F_{U_\eta} < \mathrm{ind}(\alpha)$. Since F_η is the completion of F at the discrete valuation given by η, there exist $\theta_i \in F^*$, $1 \leq i \leq \ell$, such that $\theta_i\mu_i^{-1} \equiv 1$ modulo the maximal ideal of $\hat{R}_\eta$. Let $L_{U_\eta} = \prod F_{U_\eta}$ and $\mu_{U_\eta} = (\lambda(\theta_2 \cdots \theta_\ell)^{-1}, \theta_2, \cdots, \theta_\ell) \in L_{U_\eta}$. Then $N_{L_{U_\eta/F_{U_\eta}}}(\mu_{U_\eta}) = \lambda$. Since $\alpha \cdot (\theta_i) \in H^3(F_{U_\eta}, \mu_n^{\otimes 2})$ and $\alpha \cdot (\theta_i) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$, by ([12, Proposition 3.2.2]), there exists a non-empty open subset $V_\eta \subseteq U_\eta$ such that $\alpha \cdot (\theta_i) = 0 \in H^3(F_{V_\eta}, \mu_n^{\otimes 2})$. By replacing U_η by V_η, we have the required L_{U_η} and $\mu_{U_\eta} \in L_{U_\eta}$.

Suppose that L_η/F_η is a field extension of degree ℓ. Let F_η^h be the henselization of F at the discrete valuation η. Then there exists a field extension L_η^h/F_η^h of degree ℓ with an isomorphism $\phi_\eta^h : L_\eta^h \otimes_{F_\eta^h} F_\eta \to L_\eta$. We identify L^h with a subfield of L_η through ϕ^h. Further if L_η/F_η is cyclic extension, then L^h/F^h is also a cyclic extension. Let $\tilde{\pi}_\eta \in L^h$ be a parameter. Then $\tilde{\pi}_\eta$ is also a parameter in L_η.Write $\mu_\eta = u_\eta\tilde{\pi}_\eta^r$ for some $u_\eta \in L_\eta$ a unit at η. Since $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$, we have $\lambda = N_{L_\eta/F_\eta}(u_\eta)N_{L_\eta/F_\eta}(\tilde{\pi}_\eta)$. Since $u_\eta \in L_\eta$ is a unit at η, $N_{L_\eta/F_\eta}(u_\eta) \in F_\eta$ is a unit at η. By ([2, Theorem 1.10]), there exists $u^h \in L_\eta^h$ such that $N_{L_\eta^h/F_\eta^h}(u_\eta^h) = N_{L_\eta/F_\eta}(u_\eta)$. Let $\mu_\eta^h = u_\eta^h\tilde{\pi}_\eta \in L_\eta^h$. Since F_η^h is the filtered direct limit of the fields F_V, where V ranges over the non-empty open subset of η ([12, Lemma 2.2.1]), there exists a non-empty open subset U_η of η, a field extension L_{U_η}/F_{U_η} of degree ℓ and $\mu_{U_\eta} \in L_{U_\eta}$ such that $N_{L_{U_\eta}/F_{U_\eta}}(\mu_{U_\eta}) = \lambda$ and there is an isomorphism $\phi_{U_\eta}^h : L_{U_\eta} \otimes F_\eta \simeq L_\eta^h$ with $\phi_{U_\eta}^h(\mu_{U_\eta}) = \mu_\eta^h$. By shrinking U_η, we assume that $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L_{U_\eta}, \mu_n^{\otimes 2})$ ([12, Proposition 3.2.2]). □

Lemma 7.3. *Suppose that for each codimension zero point η of X_0 there exist a field (not necessarily cyclic) or split extension L_η/F_η of degree ℓ, $\mu_\eta \in F_\eta$ and for every closed point P of X_0 there exist a cyclic or split extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that for every point x of X_0*
1) $N_{L_x/F_x}(\mu_x) = \lambda$
2) $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_x^{\otimes 2})$
3) $ind(\alpha \otimes L_x) < ind(\alpha)$
4) for any branch (P, η) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ such that for a generator σ of $Gal(L_P \otimes F_{P,\eta}/F_{P,\eta})$ there exists $\theta_{P,\eta} \in L_P \otimes F_{P,\eta}$ such that

$\phi_{P,\eta}(\mu_\eta)\mu_P^{-1} = \theta_{P,\eta}^{-\ell^d}\sigma(\theta_{P,\eta})^{\ell^d}$.
Then there exist
- *a field extension L/F of degree ℓ*
- *a non-empty open subset U_η of η for every codimension zero point η of X_0 with $\theta_{U_\eta} \in L \otimes F_{U_\eta}$*
- *for every $P \in \mathscr{P} = X_0 \setminus \cup U_\eta$, $\theta_P \in L \otimes F_P$*

such that
1) $ind(\alpha \otimes L) < ind(\alpha)$
2) $N_{L\otimes F_{U_\eta}/F_{U_\eta}}(\theta_{U_\eta}) = \lambda$ and $\alpha \cdot (\theta_{U_\eta}) = 0 \in H^3(L \otimes F_{U_\eta}, \mu_n^{\otimes 2})$ for all codimension zero points η of X_0
3) $N_{L\otimes F_P/F_P}(\theta_P) = \lambda$ and $\alpha \cdot (\theta_P) = 0 \in H^3(L \otimes F_P, \mu_n^{\otimes 2})$ for all $P \in \mathscr{P}$
4) for any branch (P, η), $L \otimes F_{P,\eta}/F_{P,\eta}$ is cyclic or split and for a generator σ of $Gal(L \otimes F_{P,\eta}/F_{P,\eta})$ there exists $\gamma_{P,\eta} \in L \otimes F_P$ such that $\theta_{U_\eta}\theta_P^{-1} = \gamma_{P,\eta}^{-\ell^d}\sigma(\gamma_{P,\eta})^{\ell^d}$.
Further if for each $x \in X_0$, L_x/F_x is cyclic or split, then L/F is cyclic.

Proof. Let η be a codimension zero point of X_0. By the assumption, there exist a cyclic or split extension L_η/F_η and $\mu_\eta \in L_\eta$ such that $N_{L_x/F_x}(\mu_x) = \lambda$, $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_x^{\otimes 2})$ and $\text{ind}(\alpha \otimes L_\eta) < \text{ind}(\alpha)$. By (7.2), there exist a non-empty open set U_η of η, a cyclic or split extension L_{U_η}/F_{U_η} of degree ℓ and $\mu_{U_\eta} \in L_{U_\eta}$ such that $N_{L_{U_\eta}/F_{U_\eta}}(\mu_\eta) = \lambda$, $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_x^{\otimes 2})$, $\text{ind}(\alpha \otimes L_{U_\eta}) < \text{ind}(\alpha)$, $\phi_\eta : L_{U_\eta} \otimes F_\eta \to L_\eta$ an isomorphism and $\phi_\eta(\mu_{U_\eta}) = \mu_\eta$. By shrinking U_η, if necessary, we assume that $\mathscr{P}_0 \cap U_\eta = \emptyset$.

Let $\mathscr{P} = X_0 \setminus \cup_\eta U_\eta$ and $P \in \mathscr{P}$. Then, by the assumption we have a cyclic or split extension L_P/F_P of degree ℓ and for every branch (P, η) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$. Thus $\phi_{P,U_\eta} = \phi_{P,\eta}(\phi_\eta \otimes 1) : L_{U_\eta} \otimes F_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ is an isomorphism. Thus, by ([9, Theorem 7.1]), there exists an extension L/F of degree ℓ with isomorphisms $\phi_{U_\eta} : L \otimes F_{U_\eta} \to L_{U_\eta}$ for all codimension zero points η of X_0 and $\phi_P : L \otimes F_P \to L_P$ for all $P \in \mathscr{P}$ such that the following commutative diagram

$$\begin{array}{ccc} L \otimes F_{U_\eta} \otimes F_{P,\eta} & \stackrel{\phi_{U_\eta}\otimes 1}{\to} & L_{U_\eta} \otimes F_\eta \otimes F_{P,\eta} \\ \downarrow & & \downarrow \phi_{P,U_\eta} \\ L \otimes F_P \otimes F_{P,\eta} & \stackrel{\phi_P \otimes 1}{\to} & L_P \otimes F_{P,\eta} \end{array}$$

where the vertical arrow on the left side is the natural map. Further if each L_x/F_x is cyclic for all $x \in X_0$, then L/F is cyclic ([9, Theorem 7.1]).

Since $\text{ind}(\alpha \otimes L \otimes F_{U_\eta}) < \text{ind}(\alpha)$ for all codimension zero points of X_0 and $\text{ind}(\alpha \otimes L \otimes F_P) < \text{ind}(\alpha)$ for all $P \in \mathscr{P}$, by (7.1), $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$. In particular L is a field.

For every codimension zero point η of X_0, let $\theta_{U_\eta} = (\phi_{U_\eta})^{-1}(\mu_{U_\eta}) \in L \otimes F_{U_\eta}$ and for every $P \in \mathscr{P}$, let $\theta_P = (\phi_P)^{-1}(\mu_P) \in L \otimes F_P$. Since ϕ_{U_η} and ϕ_P are isomorphisms, we have the required properties. □

Proposition 7.4. *Suppose that for each point x of X_0 there exist a cyclic or split extension L_x/F_x of degree ℓ and $\mu_x \in L_x$ such that*
1) $N_{L_x/F_x}(\mu_x) = \lambda$
2) $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_x^{\otimes 2})$
3) $ind(\alpha \otimes L_x) < ind(\alpha)$
4) for any branch (P, η) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ such that for generator σ of $Gal(L_P \otimes F_{P,\eta}/F_{P,\eta})$ there exists $\theta_{P,\eta} \in L_P \otimes F_{P,\eta}$ such that

$\phi_{P,\eta}(\mu_\eta)\mu_P^{-1} = \theta_{P,\eta}^{-\ell^d}\sigma(\theta_{P,\eta})^{\ell^d}$.
Then there exist a cyclic extension L of degree ℓ and $\mu \in L^*$ such that
- $N_{L/F}(\mu) = \lambda$ *and*
- $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$
- $ind(\alpha \otimes L) < ind(\alpha)$.

Proof. Let L/F, U_η, $\mathscr{P}$, θ_{U_η} and θ_P be as in (7.3). Since each L_x/F_x is cyclic or split, L/F is cyclic. Let σ be a generator of $\mathrm{Gal}(L/F)$. Let (P,η) be a branch. By (7.3), there exists $\gamma_{(P,\eta)} \in L \otimes F_{P,\eta}$ such that $\mu_{U_\eta}\mu_P^{-1} = \gamma_{P,\eta}^{-\ell^d}\sigma(\gamma_{P,\eta}^{\ell^d})$. Applying ([10, Theorem 3.6]) for the rational group GL_1, there exist $\gamma_{U_\eta} \in L \otimes F_{U_\eta}$ and $\gamma_P \in L \otimes F_P$ for every codimension zero point η of X_0 and $P \in \mathscr{P}$ such that for every branch (P,η), $\gamma_{P,\eta} = \gamma_{U_\eta}\gamma_P$.

Let $\mu'_{U_\eta} = \mu_{U_\eta}\gamma_{U_\eta}^{\ell^d}\sigma(\gamma_{U_\eta}^{-\ell^d}) \in L \otimes F_{U_\eta}$ and $\mu'_P = \mu_P\gamma_P^{-\ell^d}\sigma(\gamma_P^{\ell^d}) \in L \otimes F_P$. If (P,η) is a branch, then we have

$$\begin{aligned} \mu'_{U_\eta} &= \mu_{U_\eta}\gamma_{U_\eta}^{\ell^d}\sigma(\gamma_{U_\eta}^{-\ell^d}) \\ &= \mu_P\theta_{P,\eta}^{-\ell^d}\sigma(\theta_{P,\eta}^{\ell^d})\gamma_{U_\eta}^{\ell^d}\sigma(\gamma_{U_\eta}^{-\ell^d}) \\ &= \mu_P\gamma_P^{-\ell^d}\sigma(\gamma_P^{\ell^d}) \\ &= \mu'_P \in L \otimes F_{P,\eta}. \end{aligned}$$

Hence, by ([9, Proposition 6.3]), there exists $\mu \in L$ such that $\mu = \mu'_{U_\eta}$ and $\mu = \mu'_P$ for every codimension zero point η of X_0 and $P \in \mathscr{P}$. Clearly $N_{L/F}(\mu) = \lambda$ over F. Let $P \in \mathscr{P}$. Since $\alpha \cdot (\mu_P) = 0$ and $\alpha \cdot (\gamma_P^{\ell^d}) = 0$, $\alpha \cdot (\mu) = 0 \in H^3(L \otimes F_P, \mu_n^{\otimes 2})$. Similarly $\alpha \cdot (\mu) = 0 \in H^3(L \otimes F_{U_\eta}, \mu_n^{\otimes 2})$ for every codimension zero point η of X_0. Hence, by ([12, Theorem 3.1.5]), $\alpha \cdot (\mu) = 0$ in $H^3(L, \mu_n^{\otimes 2})$. □

Proposition 7.5. *Suppose that for each codimension zero point η of X_0 there exist a field (not necessarily cyclic) or split extension L_η/F_η of degree ℓ, $\mu_\eta \in F_\eta$ and for every closed point P of X_0 there exist a cyclic or split extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that for every point x of X_0*
1) $N_{L_x/F_x}(\mu_x) = \lambda$
2) $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_x^{\otimes 2})$
3) $ind(\alpha \otimes L_x) < ind(\alpha)$
4) for any branch (P,η) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ such that for a generator σ of Gal$(L_P \otimes F_{P,\eta}/F_{P,\eta})$ there exists $\theta_{P,\eta} \in L_P \otimes F_{P,\eta}$ such that $\phi_{P,\eta}(\mu_\eta)\mu_P^{-1} = \theta_{P,\eta}^{-\ell^d}\sigma(\theta_{P,\eta})^{\ell^d}$.
Then there exist a field extension N/F of degree coprime to ℓ, a field extension L/F of degree ℓ and $\mu \in (L \otimes N)^*$ such that
- $N_{L\otimes N/N}(\mu) = \lambda$ *and*
- $\alpha \cdot (\mu) = 0 \in H^3(L \otimes N, \mu_n^{\otimes 2})$
- $ind(\alpha \otimes L) < ind(\alpha)$.

Proof. Let L/F, U_η, $\mathscr{P}$, θ_{U_η}, θ_P and $\gamma_{P,\eta}$ be as in (7.3). Since L/F is a degree ℓ extension, there exists a field extension N/F of degree coprime to ℓ such that $L \otimes N/N$ is a cyclic extension.

Let $\mathscr{Y}$ be the integral closure of $\mathscr{X}$ in N and Y_0 the reduced special fibre of $\mathscr{Y}$. Let $\phi : Y_0 \to X_0$ be the induced morphism. Let $y \in Y_0$ and $x = \phi(y) \in X_0$. Then the inclusion $F \subset N$, induces an inclusion $F_x \subset N_y$. Let $L'_y = L \otimes_F F_x \otimes_{F_x} N_y$. Since $L \otimes N/N$ is a cyclic extension of degree ℓ, L'_y/N_y is either cyclic or split extension of degree ℓ.

Let $\eta' \in Y_0$ be a codimension zero point. Then $\eta = \phi(\eta') \in X_0$ is a codimension zero point. Then $F_\eta \subset FL_{\eta'}$, $L\otimes F_{U_\eta} \subset L\otimes F_\eta$ and $\theta_{U_\eta} \in L\otimes F_\eta$. Let $\mu_{\eta'} = \theta_{U_\eta}\otimes 1 \in L\otimes F_\eta \otimes_{F_\eta} N_{\eta'} = L'_{\eta'}$.

Let $Q \in Y_0$ be a closed point and $P = \phi(Q) \in X_0$. Then P is a closed point of X_0 and $F_P \subset N_Q$. Suppose that $P \in U_\eta$ for some codimension zero point η of X_0. Then $F_{U_\eta} \subset F_P$, $L\otimes F_{U_\eta} \subset L\otimes F_P$ and $\theta_{U_\eta} \in L\otimes F_P$. Let $\mu_Q = \theta_{U_\eta}\otimes 1 \in L\otimes F_P \otimes_{F_P} N_Q = L'_Q$. Suppose that P is not in U_η for any codimension zero point η of X_0. Let $\mu_Q = \theta_P \otimes 1 \in L\otimes F_P \otimes_{F_P} N_Q$.

Let $y \in Y_0$ and $x = \phi(x) \in X_0$. Since $N_{L_x/F_x}(\mu_x) = \lambda$, $\alpha\cdot(\mu_x) = 0 \in H^3(L_x, \mu_n^{\otimes 2})$ and $\mathrm{ind}(\alpha\otimes F_x) < \mathrm{ind}(\alpha)$, it follows that $N_{L'_y/N_y}(\mu_y) = \lambda$, $\alpha\cdot(\mu_y) = 0 \in H^3(L'_y, \mu_n^{\otimes 2})$ and $\mathrm{ind}(\alpha\otimes L'_y) < \mathrm{ind}(\alpha)$.

Let (Q, η') be a branch in Y_0 and $P = \phi(Q)$, $\eta = \phi(\eta')$. Then (P, η) is a branch in X_0. The isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ induces an isomorphism $\phi'_{Q,\eta'} : L'_{\eta'}\otimes N_{Q,\eta'} \to L'_Q \otimes N_{Q,\eta'}$. By the choice of $\mu_{\eta'}$ and μ_Q it follows that for any generator σ of $\mathrm{Gal}(L'_Q\otimes N_{Q,\eta'}/N_{Q,\eta'})$ there exists $\theta_{Q,\eta'}$ such that $\phi'_{Q,\eta'}(\mu_{\eta'})\mu_Q^{-1} = \theta_{Q,\eta'}^{-\ell d}\sigma(\theta_{Q,\eta'})^{\ell d}$. Thus, by (7.4), there exists a cyclic extension L'/N and $\mu' \in L'$ such that $N_{L'/N}(\mu') = \lambda$, $\mathrm{ind}(\alpha\otimes L') < \mathrm{ind}(\alpha\otimes N)$ and $\alpha\cdot(\mu') = 0 \in H^3(L', \mu_n^{\otimes 2})$. By the construction we have $L' = L\otimes N$. □

8. Types of points, special points and type 2 connections

Let F, $\alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha\cdot(\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\mathscr{X}$ and X_0 be as in (§7). Further assume that
- $\mathscr{X}$ is regular such that $\mathrm{ram}_{\mathscr{X}}(\alpha)\cup \mathrm{supp}_{\mathscr{X}}(\lambda)\cup X_0$ is a union of regular curves with normal crossings.
- the intersection of any two distinct irreducible curves in X_0 is at most one closed point.

We fix the following notation.
- $\mathscr{P}$ is the set of points of intersection of distinct irreducible curves in X_0.
- $\mathscr{O}_{\mathscr{X},\mathscr{P}}$ is the semi-local ring at the points of $\mathscr{P}$ on $\mathscr{X}$.
- if a codimension zero point η of X_0 contains a closed point $P \in \mathscr{P}$, then $\pi_\eta \in \mathscr{O}_{\mathscr{X},\mathscr{P}}$ is a prime defining η on $\mathscr{O}_{\mathscr{X},\mathscr{P}}$.

Let η be a codimension zero point of X_0. For the rest of this paper, let (E_η, σ_η) denote the lift of the residue of α at η. Since $\alpha \in H^2(F, \mu_n)$ with n a power of ℓ, $[E_\eta : F_\eta]$ is a power of ℓ. If α is unramified at η, then $E_\eta = F_\eta$ and let $M_\eta = F_\eta$. If α is ramified at η, then $E_\eta \neq F_\eta$ and there is a unique subextension of E_η of degree ℓ and we denote it by M_η.

Remark 8.1. Let η be a codimension zero point of X_0. Suppose α is ramified at η. Since $\mathrm{ind}(\alpha\otimes F_\eta) = \mathrm{ind}(\alpha\otimes E_\eta)[E_\eta : F_\eta]$ (cf. 4.2) and $M_\eta \subset E_\eta$, it follows that $\mathrm{ind}(\alpha\otimes M_\eta) < \mathrm{ind}(\alpha)$.

We divide the codimension zero points η of X_0 as follows:

Type 1: $\nu_\eta(\lambda)$ is coprime to ℓ and $\mathrm{ind}(\alpha\otimes F_\eta) = \mathrm{ind}(\alpha)$
Type 2: $\nu_\eta(\lambda)$ is coprime to ℓ and $\mathrm{ind}(\alpha\otimes F_\eta) < \mathrm{ind}(\alpha)$
Type 3: $\nu_\eta(\lambda) = r\ell$, $r\alpha\otimes E_\eta \neq 0$ and $\mathrm{ind}(\alpha\otimes F_\eta) = \mathrm{ind}(\alpha)$
Type 4: $\nu_\eta(\lambda) = r\ell$, $r\alpha\otimes E_\eta \neq 0$ and $\mathrm{ind}(\alpha\otimes F_\eta) < \mathrm{ind}(\alpha)$
Type 5: $\nu_\eta(\lambda) = r\ell$, $r\alpha\otimes E_\eta = 0$ and $\mathrm{ind}(\alpha\otimes F_\eta) = \mathrm{ind}(\alpha)$

Type 6: $\nu_\eta(\lambda) = r\ell$, $r\alpha \otimes E_\eta = 0$ and $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$.

Let P be a closed point of $\mathscr{X}$. Suppose P is the point of intersection of two distinct codimension zero points η_1 and η_2 of X_0. We say that the point P is a
1) **special point of type I** if η_1 is of type 1 and η_2 is of type 2,
2) **special point of type II** if η_1 is of type 1 and η_2 is of type 4,
3) **special point of type III** if η_1 is of type 3 or 5 and η_2 is of type 4,
4) **special point of type IV** if η_1 is of type 1, 3 or 5 and η_2 is of type 5 with $M_{\eta_2} \otimes F_{P,\eta_2}$ not a field.

Lemma 8.2. *Suppose that η_1 and η_2 are two distinct codimension zero points of X_0 and P a point of intersection of η_1 and η_2. Suppose that α is ramified at η_1. Let (E_{η_1}, σ_1) be the lift of residue of α at η_1. If $E_{\eta_1} \otimes F_{P,\eta_1}$ is not a field, then $ind(\alpha \otimes F_P) < ind(\alpha)$.*

Proof. Suppose that $E_{\eta_1} \otimes F_{P,\eta_1}$ is not a field. Since E_{η_1}/F_{η_1} is a cyclic extension, $E_{\eta_1} \otimes F_{P,\eta_1} \simeq \prod E_{\eta_1,P}$ with $[E_{\eta_1,P} : F_{P,\eta_1}] < [E_{\eta_1} : F_{\eta_1}]$. We have $(E_{\eta_1}, \sigma_1, \pi_{\eta_1}) \otimes F_{P,\eta_1} = (E_{\eta_1,P}, \sigma_1, \pi_{\eta_1})$ (cf. §2).

Write $\alpha \otimes F_{\eta_1} = \alpha_1 + (E_{\eta_1}, \sigma_1, \pi_{\eta_1})$ as in (4.1). Then $\alpha \otimes F_{P,\eta_1} = \alpha_1 \otimes F_{P,\eta_1} + (E_{\eta_1,P}, \sigma_1, \pi_{\eta_1})$. By (4.2), we have $\mathrm{ind}(\alpha \otimes F_{\eta_1}) = \mathrm{ind}(\alpha_1 \otimes E_{\eta_1})[E_{\eta_1} : F_{\eta_1}]$. We have

$$\begin{aligned} \mathrm{ind}(\alpha \otimes F_{P,\eta_1}) &\leq \mathrm{ind}(\alpha_1 \otimes E_{\eta_1,P})[E_{\eta_1,P} : F_{P,\eta_1}] \\ &\leq \mathrm{ind}(\alpha_1 \otimes E_{\eta_1})[E_{\eta_1,P} : F_{P,\eta_1}] \\ &< \mathrm{ind}(\alpha_1 \otimes E_{\eta_1})[E_{\eta_1} : F_{\eta_1}] \\ &= \mathrm{ind}(\alpha \otimes F_{\eta_1}). \end{aligned}$$

Thus, by (5.8), $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. □

Lemma 8.3. *Let $\eta \in X_0$ be a point of codimension zero and P a closed point on η. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up at P and γ the exceptional curve in $\mathscr{X}_P$. If $E_\eta \otimes F_{P,\eta}$ is not a field or η is of type 2, 4 or 6, then γ is of type 2, 4 or 6.*

Proof. If $E_\eta \otimes F_{P,\eta}$ is not a field, then by (8.2), $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. If η is of type 2, 4 or 6, then $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$ and hence by (5.8), $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. Since $F_P \subset F_\gamma$, we have $\mathrm{ind}(\alpha \otimes F_\gamma) \leq \mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. Hence γ is of type 2, 4 or 6. □

Lemma 8.4. *Let η_1 and η_2 be two distinct codimension zero points of X_0 intersecting at a closed point P. Suppose that η_1 is of type 1 or 2 and η_2 is of type 2. Then there exists a sequence of blow-ups $\psi : \mathscr{X}' \to \mathscr{X}$ such that if $\tilde{\eta}_i$ are the strict transforms of η_i, then*
1) $\psi : \mathscr{X}' \setminus \psi^{-1}(P) \to \mathscr{X} \setminus \{P\}$ is an isomorphism
2) $\psi^{-1}(P)$ is the union of irreducible regular curves $\gamma_1, \cdots, \gamma_m$
3) $\tilde{\eta}_1 \cap \gamma_1 = \{P_0\}$, $\gamma_i \cap \gamma_{i+1} = \{P_i\}$, $\gamma_m \cap \tilde{\eta}_2 = \{P_m\}$, $\tilde{\eta} \cap \gamma_i = \emptyset$ for all $i > 1$, $\tilde{\eta}_2 \cap \gamma_i = \emptyset$ for all $i < m$, $\tilde{\eta}_1 \cap \tilde{\eta}_2 = \emptyset$, $\gamma_i \cap \gamma_j = \emptyset$ for all $j \neq i+1$,
4) γ_1 and γ_m are of type 6 and γ_i, $1 < i < m$ are of type 2, 4 or 6,
5) $\psi^{-1}(P)$ has no special points.

Proof. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P and γ the exceptional curve in $\mathscr{X}_P$. Let $\tilde{\eta}_i$ be the strict transform of η_i. Then $\tilde{\eta}_1$ intersects γ only at one point P_0 and $\tilde{\eta}_2$ intersects γ at only one point P_1. Since η_2 is of type 2, by (8.3), γ is of type 2, 4 or 6 and hence P_1 is not a special point.

Let $s_1 = \nu_{\eta_1}(\lambda)$, $s_2 = \nu_{\eta_2}(\lambda)$. Then $\nu_\gamma(\lambda) = s_1 + s_2$. Suppose $s_1 + s_2 = \ell^{d+1} r_0$ for some integer r_0, where $\ell^d = \mathrm{ind}(\alpha)$. Since $\ell^d\alpha = 0$, $\ell^d r_0 \alpha = 0$. Thus, γ is of type 6. Hence P_0 is not a special point and $\mathscr{X}_P$ has all the required properties.

Suppose $s_1 + s_2 = \ell^t r_0$ with $t \le d$ and r_0 coprime to ℓ. Then, blow-up the points P_0 and P_1 and let γ_1 and γ_2 be the exceptional curves in this blow-up. Then we have $\eta_{\gamma_1}(\lambda) = 2s_1 + s_2$ and $\eta_{\gamma_2}(\lambda) = s_1 + 2s_2$. If $2s_1 + s_2$ is not of the form $\ell^{d+1} r_1$ for some $r_1 \ge 1$, then blow-up, the point of intersection of the strict transform of η_1 and γ_1. If $s_1 + 2s_2$ is not of the form $\ell^{d+1} r_2$ for some $r_2 \ge 1$, then blow-up, the point of intersection of the strict transform of η_2 and γ_2. Since s_1 and s_2 are coprime to ℓ, there exist i and j such that $is_1 + s_2 = \ell^{d+1} r$ and $s_1 + js_2 = \ell^{d+1} r'$ for some $r, r' \ge 1$. Thus, we get the required finite sequence of blow-ups. □

Proposition 8.5. *There exists a regular proper model of F with no special points.*

Proof. Let $P \in \mathscr{P}$. Then there exist two codimension zero points η_1 and η_2 of X_0 intersecting at P.

Suppose that P is a special point of type I. Let $\psi : \mathscr{X}' \to \mathscr{X}$ be a sequence of blow-ups as in (8.4). Then there are no special points in $\psi^{-1}(P)$. Since there are only finitely many special points in $\mathscr{X}$, replacing $\mathscr{X}$ by a finite sequence of blow ups at all special points of type I, we assume that $\mathscr{X}$ has no special points of type I.

Suppose P is a special point of type II. Without loss of generality we assume that, η_1 is of type 1 and η_2 is of type 4. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P and γ the exceptional curve in $\mathscr{X}_P$. Since η_2 is of type 4, by (8.3), γ is of type 2, 4 or 6. Since η_1 is of type 1 and η_2 is of type 4, $\nu_{\eta_1}(\lambda)$ is coprime to ℓ and $\nu_{\eta_2}(\lambda)$ is divisible by ℓ. Since $\nu_\gamma(\lambda) = \nu_{\eta_1}(\lambda) + \nu_{\eta_2}(\lambda)$, $\nu_\gamma(\lambda)$ is coprime to ℓ and hence γ is of type 2. Let $\tilde{\eta}_i$ be the strict transform of η_i in $\mathscr{X}_P$. Then $\tilde{\eta}_i$ and γ intersect at only one point Q_i. Since γ is of type 2, Q_1 is a special point of type I and Q_2 is not a special point. Thus, as above, by replacing $\mathscr{X}$ by a sequence of blow-ups of $\mathscr{X}$, we assume that $\mathscr{X}$ has no special points of type I or II.

Suppose P is a special point of type III. Without loss of generality assume that η_1 is of type 3 or 5 and η_2 of type 4. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P, γ, $\tilde{\eta}_i$, and Q_i be as above. Since η_2 is of type 4, by (8.3), γ is of type 2, 4 or 6. Since $\nu_{\eta_1}(\lambda)$ and $\nu_{\eta_2}(\lambda)$ are divisible by ℓ, $\nu_\gamma(\lambda) = \nu_{\eta_1}(\lambda) + \nu_{\eta_2}(\lambda)$ is divisible by ℓ. Thus γ is of type 4 or 6. Hence Q_2 is not a special point. By (5.7), $\alpha \otimes F_P = (E_P, \sigma, u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2})$ for some cyclic extension E_P/F_P and $u \in \hat{A}_P$ a unit and at least one of d_i is coprime to ℓ (in fact equal to 1). In particular, $\alpha \otimes F_P$ is split by the extension $F_P(\sqrt[m]{u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}})$, where m is the degree of E_P/F_P which is a power of ℓ. Suppose $d_1 + d_2$ is coprime to ℓ. Since $\nu_\gamma(\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}) = d_1 + d_2$, $F_P(\sqrt[m]{u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}})$ is totally ramified at γ. Thus, by (4.3), γ is of type 6. Hence Q_1 is not a special point. Suppose that $d_1 + d_2$ is divisible by ℓ. Let π_γ be a prime defining γ at Q_1. Then, we have $u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2} = w_1\pi_{\eta_1}^{d_1}\pi_\gamma^{d_1+d_2}$ for some unit w_1 at Q_1. Since one of d_i is coprime to ℓ and $d_1 + d_2$ is divisible by ℓ, d_i are not divisible by ℓ. In particular $2d_1 + d_2$ is coprime to ℓ. Let $\mathscr{X}_{Q_1}$ be the blow-up of $\mathscr{X}_P$ at Q_1 and γ' be the generic point of the exceptional curve in $\mathscr{X}_{Q_1}$. Then $\nu_{\gamma'}(u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}) = \nu_{\gamma'}(w_1\pi_{\eta_1}^{d_1}\pi_\gamma^{d_1+d_2}) = 2d_1 + d_2$. Since $2d_1 + d_2$ is coprime to ℓ, once again by (4.3), γ' is of type 6. In particular no point on the exceptional curve in $\mathscr{X}_{Q_1}$ is a special point. Thus, replacing $\mathscr{X}$ by a sequence of blow-ups, we assume that $\mathscr{X}$ has no special points of type I, II or III.

Suppose P is a special point of type IV. Without loss of generality assume that, η_1 is of type 1, 3 or 5 and η_2 is of type 5, with $M_{\eta_2} \otimes F_{P,\eta_2}$ not a field. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P and γ, $\tilde{\eta}_i$, Q_i be as above. Since $M_{\eta_2} \otimes F_{P,\eta_2}$ is not a field, by (8.3), γ is of type 2, 4 or 6. If γ is of type 6, then Q_1 and Q_2 are not special points. Suppose γ is of type 2 or 4. Then Q_1 and Q_2 are special points of type I, II or III. Thus, as above, by replacing $\mathscr{X}$ by a sequence of blow-ups of $\mathscr{X}$, we assume that $\mathscr{X}$ has no special points. □

Let η and η' be two codimension zero points of X_0 (may not be distinct). We say that there is a **type 2 connection** from η to η' if one of the following holds
- one of η or η' is of type 2
- there exist distinct codimension zero points $\eta_1, \cdots, \eta_n$ of X_0 of type 2 such that η intersects η_1, η' intersects η_n, η_i intersects η_{i+1} for all $1 \leq i \leq n-1$, η does not intersect η_i for $i > 1$, η' does not intersect η_i for $i < n$ and η_i does not intersect η_j for $j \neq i+1$.

Proposition 8.6. *There exists a regular proper model $\mathscr{X}$ of F such that*
1) $\mathscr{X}$ has no special points
2) if η_1 and η_2 are two (not necessarily distinct) codimension zero points of X_0 with η_1 of type 3 or 5 and η_2 of type 3, 4 or 5, then there is no type 2 connection between η_1 and η_2.

Proof. Let $\mathscr{X}$ be a regular proper model with no special points (8.5). Let $m(\mathscr{X})$ be the number of type 2 connections between a point of type 3 or 5 and a point of type 3, 4 or 5. We prove the proposition by induction on $m(\mathscr{X})$. Suppose $m(\mathscr{X}) \geq 1$. We show that there is a sequence of blow-ups $\mathscr{X}'$ of $\mathscr{X}$ with no special points and $m(\mathscr{X}') < m(\mathscr{X})$.

Let η be a codimension zero point of X_0 of type 3 or 5 and η' a codimension zero point of X_0 of types 3, 4 or 5. Suppose η and η' have a type 2 connection. Then there exist $\eta_1, \cdots, \eta_n$ codimension zero points of X_0 of type 2 with η intersecting η_1, η' intersecting η_n and η_i intersecting η_{i+1} for $i = 1, \cdots, n-1$.

Suppose $n = 1$. Let Q be the point of the intersection of η and η_1. Let $\mathscr{X}_Q \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at Q and γ the exceptional curve in $\mathscr{X}_Q$. Since η_1 is of type 2, by (8.3), γ is of type 2, 4 or 6. Since η is of type 3 or 5 and η_1 is of type 2, ℓ divides $\nu_\eta(\lambda)$ and ℓ does not divide $\nu_{\eta_1}(\lambda)$. Since $\nu_\gamma(\lambda) = \nu_\eta(\lambda) + \nu_{\eta_1}(\lambda)$, $\nu_\gamma(\lambda)$ is not divisible by ℓ and hence γ is of type 2. Let $\tilde{\eta}$ and $\tilde{\eta}_1$ be the strict transform of η and η_1 in $\mathscr{X}_Q$. Since γ is a point of type 2, the points of intersection of $\tilde{\eta}$ and $\tilde{\eta}_1$ with γ are not special points. Hence $\mathscr{X}_Q$ has no special points. By replacing $\mathscr{X}$ by $\mathscr{X}_Q$ we assume that $n \geq 2$ and $\mathscr{X}$ has no special points.

Let P be the point of intersection of η_1 and η_2. Let $\mathscr{X}'$ be as in (8.4). Then $\mathscr{X}'$ has no special points and all the exceptional curves in $\mathscr{X}'$ are of type 2, 4 or 6 and the exceptional curves which intersect the strict transforms of η_1 and η_2 are of type 6. In particular the number of type 2 connections between the strict transforms of η and η' is one less than the number of type 2 connections between η and η'. Since all the exceptional curves in $\mathscr{X}'$ are of type 2, 4 or 6, $m(\mathscr{X}') = m(\mathscr{X}) - 1$. Thus, by induction, we have a regular proper model with required properties. □

Lemma 8.7. *Let $\mathscr{X}$ be as in (8.6) and X_0 the special fibre of $\mathscr{X}$. Let η be a codimension zero point of X_0 of type 2 and η' a codimension zero point of X_0 of type 3 or 5. Suppose there is a type 2 connection from η to η'. If there is a type 2 connection from η to a type 3 or 5 point η'', then $\eta' = \eta''$. Further, if $\eta_1, \cdots, \eta_n$ are* *codimension zero points of X_0 of type 2 giving a type 2 connection from η to η' and $\gamma_1, \cdots, \gamma_m$ codimension zero points of X_0 of type 2 giving another type 2 connection from η to η', then $n = m$ and $\eta_i = \gamma_i$ for all i.*

Proof. Suppose η'' is a codimension zero point of X_0 of type 3 or 5 with type 2 connection to η. Since η is of type 2 and there is a type 2 connection from η' to η''. Since no two points of type 3 or 5 have a type 2 connection (cf. 8.6), $\eta' = \eta''$. Suppose $\gamma_1, \cdots, \gamma_m$ is of type 2 connection from η to η'. If η_i is not equal to γ_i, then we will have type 2 connection from η' to η' and hence a contradiction to the choice of $\mathscr{X}$ (cf. 8.6). Thus $n = m$ and $\eta_i = \gamma_i$ for all i. □

Let η be a codimension zero point of X_0 of type 2 and η' be a codimension zero point of X_0 of type 3 or 5. Suppose there is a type 2 connection $\eta_1, \cdots, \eta_n$ from η to η'. Then, by (8.7), η' and η_n are uniquely defined by η. We call this point of intersection of η_n with η' as **the point of type 2 intersection of η and η'**. Once again note that such a closed point is uniquely defined by η.

9. Choice of L_P and μ_P at closed points

Let F, $\alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\mathscr{X}$ and X_0 be as in (§7 and §8). Throughout this section we assume that $\mathscr{X}$ has no special points and if η_1 and η_2 are two (not necessarily distinct) codimension zero points of X_0 with η_1 is of type 3 or 5 and η_2 is of type 3, 4 or 5, then there is no type 2 connection between η_1 and η_2.

Let η be a codimension zero point of X_0 of type 5. Then we call η of **type 5a** if α is unramified at η and of **type 5b** if α is ramified at η. Suppose η is of type 5b. Then α is ramified and hence M_η is the unique subextension of E_η of degree ℓ, where (E_η, σ_η) is the lift of the residue of α.

For the rest of the paper we assume that κ is a finite field.

Lemma 9.1. *Let η be a codimension zero point of X_0 of type 5b. Then ind$(\alpha \otimes M_\eta) <$ ind(α) and there exists $\mu_\eta \in M_\eta$ such that $N_{M_\eta/F_\eta}(\mu_\eta) = \lambda$ and $\alpha \cdot (\mu_\eta) = 0 \in H^3(M_\eta, \mu_n^{\otimes 2})$.*

Proof. Since η is of type 5b, α is ramified at η, $\nu_\eta(\lambda) = r\ell$ and $r\alpha \otimes E_\eta = 0$. By (8.1), ind$(\alpha \otimes M_\eta) <$ ind(α). Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma, \pi_\eta)$ as in (4.1) and $\lambda = \theta_\eta \pi_\eta^{r\ell}$ where θ_η is a unit at η and π_η is a parameter at η. Let β_0 be the image of α' in $H^2(\kappa(\eta), \mu_n)$. Since $\alpha' \otimes E_\eta = \alpha \otimes E_\eta$ and $r\alpha \otimes E_\eta = 0$, $r\beta_0 \otimes E(\eta) = 0$. Let θ_0 be the image of θ_η in $\kappa(\eta)$. Then, by (3.5), there exists $\mu_0 \in M_\eta(\eta)$, such that $N_{M_\eta(\eta)/\kappa(\eta)}(\mu_0) = \theta_0$ and $r\beta_0 \otimes M_\eta(\eta) = (E_\eta(\eta), \sigma, \mu_0)$. Thus, by (4.8), there exists $\mu_\eta \in M_\eta$ with the required properties. □

Lemma 9.2. *Let $P \in \mathscr{P}$, η_1 and η_2 be codimension zero points of X_0 containing P. Suppose that η_1 and η_2 are of type 5. Then there exist a cyclic field extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that*
1) $N_{L_P/F_P}(\mu_P) = \lambda$,
2) *ind*$(\alpha \otimes L_P) <$ *ind*(α),
3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$,
4) *if η_i is of type 5a, then $L_P \otimes F_{P,\eta_i}/F_{P,\eta_i}$ is an unramified field extension,*
5) *if η_i is of type 5b, then $L_P \otimes F_{P,\eta_i} \simeq M_{\eta_i} \otimes F_{P,\eta_i}$.*

Proof. Since $\mathscr{X}$ has no special points, P is not a special point of type IV. Since η_1 and η_2 are of type 5 intersecting at P, $M_{\eta_1} \otimes F_{P,\eta_1}$ and $M_{\eta_2} \otimes F_{P,\eta_2}$ are fields. Suppose

η_i is of type 5a. If $\alpha \otimes F_{P,\eta_i} = 0$, then let $L_{P,\eta_i}/F_{P,\eta_i}$ be any cyclic unramified field extension with λ a norm and $\mu_{\eta_i} \in L_{P,\eta_i}$ with $N_{L_{P,\eta_i}/F_{P,\eta_i}}(\mu_{\eta_i}) = \lambda$. If $\alpha \otimes F_{P,\eta_i} \neq 0$, then let $L_{P,\eta_i}/F_{P,\eta_i}$ be a cyclic unramified field extension of degree ℓ and μ_{η_i} be as in (4.10). Suppose η_i is of type 5b. Let $L_{P,\eta_i} = M_{\eta_i} \otimes F_{P,\eta_i}$ and $\mu_{\eta_i} \in M_{\eta_i}$ be as in (9.1). Then, by choice $L_{P,\eta_i}/F_{P,\eta_i}$ are unramified field extensions. By applying (6.4) to L_{P,η_i} and μ_{η_i}, there exist a cyclic field extension L_P/F_P and $\mu_P \in L_P$ with required properties. □

Lemma 9.3. *Let η be a codimension zero point of X_0 of type 3 and P a closed point on the closure of η. Then, there exists a cyclic field extension $L_{P,\eta}/F_{P,\eta}$ of degree ℓ such that if $\alpha \otimes E_\eta \otimes F_{P,\eta} \neq 0$, then ind$(\alpha \otimes E_\eta \otimes L_{P,\eta}) <$ ind$(\alpha \otimes E_\eta \otimes F_{P,\eta})$.*

Proof. Since E_η/F_η is a cyclic unramified field extension of degree a power of ℓ, $E_\eta \otimes F_{P,\eta} \simeq \prod E_{P,\eta}$ for some cyclic field extension $E_{P,\eta}/F_{P,\eta}$ of degree a power of ℓ. Let $E(\eta)_P$ be the residue field of $E_{P,\eta}$. Then $E(\eta)_P/\kappa(\eta)_P$ is a cyclic extension of degree a power of ℓ. Note that either $E(\eta)_P = \kappa(\eta)_P$ or $E(\eta)_P/\kappa(\eta)_P$ is a cyclic extension of degree a positive power of ℓ. If $E(\eta)_P = \kappa(\eta)_P$, then let $L(\eta)_P/\kappa(\eta)_P$ be any cyclic extension of degree ℓ. Suppose $E(\eta)_P \neq \kappa(\eta)_P$. Since $E(\eta)_P/\kappa(\eta)_P$ is a cyclic extension of degree a positive power of ℓ, there is only one subextension of $E(\eta)_P$ which is cyclic over $\kappa(\eta)_P$ of degree ℓ. Since $\kappa(\eta)_P$ is a local field containing primitive ℓ^{th} root of unity, there are at least 2 non-isomorphic cyclic field extensions of $\kappa(\eta)_P$ of degree ℓ. Thus there exists a cyclic field extension $L(\eta)_P/\kappa(\eta)_P$ of degree ℓ which is not isomorphic to a subfield of $E(\eta)_P$. Let $L_{P,\eta}/F_{P,\eta}$ be the unramified extension of degree ℓ with residue field $L(\eta)_P$.

Suppose $\alpha \otimes E_\eta \otimes F_{P,\eta} \neq 0$. Then $\alpha \otimes E_{P,\eta} \neq 0$. Since $E_{P,\eta}$ and $L_{P,\eta}$ are cyclic field extensions of $F_{P,\eta}$ and $L_{P,\eta}$ is not isomorphic to a subfield of $E_{P,\eta}$, $E_{P,\eta} \otimes L_{P,\eta}$ is a field and $[E_{P,\eta} \otimes L_{P,\eta} : E_{P,\eta}] = [L_{P,\eta} : F_{P,\eta}] = \ell$. In particular $E(\eta)_P \otimes L(\eta)_P$ is a field and $[E(\eta)_P \otimes L(\eta)_P : E(\eta)_P] = \ell$. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in (4.1). Since $\alpha \otimes E_\eta = \alpha' \otimes E_\eta$, $\alpha \otimes E_\eta$ is unramified at η. Let β_P be the image of $\alpha \otimes E_\eta \otimes F_{P,\eta}$ in $H^2(E(\eta)_P, \mu_n)$. Since $\alpha \otimes E_{P,\eta} \neq 0$ and $E_{P,\eta}$ is a completely discretely valued field with residue field $E(\eta)_P$, $\beta_P \neq 0$. Since $E(\eta)_P$ is a local field, $\text{ind}(\beta_P \otimes E(\eta)_P \otimes L(\eta)_P) < \text{ind}(\beta_P)$ and hence $\text{ind}(\alpha \otimes E_\eta \otimes L_{P,\eta}) = \text{ind}(\alpha \otimes E_{P,\eta} \otimes L_{P,\eta}) < \text{ind}(\alpha \otimes E_{P,\eta}) = \text{ind}(\alpha \otimes E_\eta \otimes F_{P,\eta})$. □

Lemma 9.4. *Let $P \in \mathscr{P}$, η_1 and η_2 be codimension zero points of X_0 containing P. Suppose that η_1 is of type 2 and η_2 is of type 5 or 6. Then there exist $\mu_i \in F_P$, $1 \leq i \leq \ell$, such that*
1) $\mu_1 \cdots \mu_\ell = \lambda$,
2) $\nu_{\eta_1}(\mu_1) = \nu_{\eta_1}(\lambda)$, $\nu_{\eta_1}(\mu_i) = 0$ *for* $i \geq 2$,
3) $\nu_{\eta_2}(\mu_i) = \nu_{\eta_2}(\lambda)/\ell$ *for all* $i \geq 1$,
4) $\alpha \cdot (\mu_i) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$.

Proof. Since η_1 is of type 2 and η_2 is of type 5 or 6, we have $\lambda = w\pi_{\eta_1}^{r_1}\pi_{\eta_2}^{r_2\ell}$ with r_1 coprime to ℓ and $r_2\alpha \otimes E_{\eta_2} = 0$. Hence, by (6.7), there exists $\theta \in F_P$ such that $\alpha \cdot (\theta) = 0$, $\nu_{\eta_1}(\theta) = 0$ and $\nu_{\eta_2}(\theta) = r_2$. For $i \geq 2$, let $\mu_i = \theta$ and $\mu_1 = \lambda\theta^{1-\ell}$. Then μ_i have the required properties. □

Lemma 9.5. *Let $P \in \mathscr{P}$, η_1 and η_2 be codimension zero points of X_0 containing P. Suppose that η_1 and η_2 are of type 5 or 6. Then there exist $\mu_i \in F_P$, $1 \leq i \leq \ell$, such that*

1) $\mu_1 \cdots \mu_\ell = \lambda$,
2) $\nu_{\eta_j}(\mu_i) = \nu_{\eta_j}(\lambda)/\ell$ *for all* $i \geq 0$ *and* $j = 1, 2$,
4) $\alpha \cdot (\mu_i) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$.

Proof. Since η_1 and η_2 are of type 5 or 6, by (6.8), there exists $\theta \in F_P$ such that $\alpha \cdot (\theta) = 0$ and $\nu_{\eta_i}(\theta) = \nu_{\eta_i}(\lambda)/\ell$ for $i = 1, 2$. For $i \geq 2$, let $\mu_i = \theta \in F_P$ and $\mu_1 = \lambda\theta^{1-\ell} \in F_P$. Then μ_i have the required properties. □

Lemma 9.6. *Let* $P \in \mathscr{P}$, η_1 *be a codimension zero point of* X_0 *of type 3 and* η_2 *a codimension zero point of* X_0 *of type 5. Suppose* η_1 *and* η_2 *intersect at* P. *Then there exist a cyclic field extension* L_P/F_P *of degree* ℓ *and* $\mu_P \in L_P$ *such that*
1) $N_{L_P/F_P}(\mu_P) = \lambda$
2) $ind(\alpha \otimes L_P) < ind(\alpha)$
3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$
4) $L_P \otimes F_{P,\eta_i}/F_{P,\eta_i}$ *is an unramified field extension*
5) *if* $\lambda \in F_P^{*\ell}$ *and* $\alpha \otimes E_{\eta_1} \otimes F_{P,\eta_1} \neq 0$, *then* $ind(\alpha \otimes (E_{\eta_1} \otimes F_{P,\eta_1}) \otimes (L_P \otimes F_{P,\eta_1})) < ind(\alpha \otimes E_{\eta_1} \otimes F_{P,\eta_1})$
6) *if* η_2 *is of type 5b, then* $L_P \otimes F_{P,\eta_2} \simeq M_{\eta_2} \otimes F_{P,\eta_2}$.

Proof. Suppose $\lambda \notin F_P^{*\ell}$. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$. Then $N_{L_P/F_P}(\mu_P) = \lambda$ and by (6.2) 2) and 3) are satisfied. Since η_i is of type 3 or 5, $\nu_{\eta_i}(\lambda)$ is divisible by ℓ and hence 4) is satisfied. Since $\lambda \notin F_P^{*\ell}$, the case 5) does not arise. Suppose that η_2 is of type 5b. Since $\mathscr{X}$ has no special points, $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field. Since λ is a norm from M_{η_2} (9.1), by (2.6), we have $L_P \otimes F_{P,\eta_2} \simeq M_{\eta_2} \otimes F_{P,\eta_2}$.

Suppose that $\lambda \in F_P^{*\ell}$. Let L_{P,η_1} be as in (9.3) and $\mu_{P,\eta_1} = \sqrt[\ell]{\lambda}$. Write $\alpha \otimes F_{\eta_1} = \alpha_1 \otimes (E_{\eta_1}, \sigma_1, \pi_{\eta_1})$ as in (4.1). Then by (4.2), we have $\mathrm{ind}(\alpha \otimes F_{\eta_1}) = \mathrm{ind}(\alpha \otimes E_{\eta_1})[E_{\eta_1} : F_{\eta_1}]$. Since η_1 is of type 3, $\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha \otimes F_{\eta_1})$ and $r_1\alpha \otimes E_{\eta_1} \neq 0$, where $\nu_{\eta_1}(\lambda) = r_1\ell$. In particular $\alpha \otimes E_{\eta_1} \neq 0$. By the choice of L_{P,η_1} as in (9.3), we have either $\alpha \otimes E_{\eta_1} \otimes F_{P,\eta_1} = 0$ or $\mathrm{ind}(\alpha \otimes E_{\eta_1} \otimes L_{P,\eta_1}) < \mathrm{ind}(\alpha \otimes E_{\eta_1} \otimes F_{P,\eta_1})$. Thus $\mathrm{ind}(\alpha \otimes E_{\eta_1} \otimes F_{P,\eta_1}) < \mathrm{ind}(\alpha \otimes E_{\eta_1})$. We have $\mathrm{ind}(\alpha \otimes L_{P,\eta_1}) \leq \mathrm{ind}(\alpha \otimes E_{\eta_1} \otimes L_{P,\eta_1})[E_{\eta_1} \otimes L_{P,\eta_1} : L_{P,\eta_1}] < \mathrm{ind}(\alpha \otimes E_{\eta_1})[E_{\eta_1} : F_{\eta_1}] = \mathrm{ind}(\alpha)$. Since L_{P,η_1} is a field and $\mathrm{cores}_{L_{P,\eta_1}/F_{P,\eta_1}}(\alpha \cdot (\mu_{P,\eta_1})) = \alpha \cdot (\lambda) = 0$, by (4.6), $\alpha \cdot (\mu_{P,\eta_1}) = 0 \in H^3(L_{P,\eta_1}, \mu_n^{\otimes 2})$. Since $\mathscr{X}$ has no special points, $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field. Let $L_{P,\eta_2} = M_{\eta_2} \otimes F_{P,\eta_2}$ and $\mu_{P,\eta_2} = \sqrt[\ell]{\lambda}$. Then $N_{L_{P,\eta_2}/F_{P,\eta_2}}(\mu_{P,\eta_2}) = \lambda$ and by (9.1), $\mathrm{ind}(\alpha \otimes L_{P,\eta_2}) < \mathrm{ind}(\alpha)$ and $\alpha \cdot (\mu_{P,\eta_2}) = 0$. Then, by (6.4), there exist L_P and μ_P with required properties. □

Lemma 9.7. *Let* $P \in \mathscr{P}$, η_1 *and* η_2 *be codimension zero points of* X_0 *of type 3, 4 or 6. Suppose* η_1 *and* η_2 *intersect at* P. *Then there exist a cyclic field extension* L_P/F_P *of degree* ℓ *and* $\mu_P \in L_P$ *such that*
1) $N_{L_P/F_P}(\mu_P) = \lambda$
2) $ind(\alpha \otimes L_P) < ind(\alpha)$
3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$
4) $L_P \otimes F_{P,\eta_i}/F_{P,\eta_i}$ *is an unramified field extension,*
5) *if* η_i *is of type 3,* $\lambda \in F_P^{*\ell}$ *and* $\alpha \otimes E_{\eta_i} \otimes F_{P,\eta_i} \neq 0$, *then* $ind(\alpha \otimes (E_{\eta_i} \otimes F_{P,\eta_i}) \otimes (L_P \otimes F_{P,\eta_i})) < ind(\alpha \otimes E_{\eta_i} \otimes F_{P,\eta_i})$.

Proof. Suppose $\lambda \notin F_P^{*\ell}$. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$. Then $N_{L_P/F_P}(\mu_P) = \lambda$ and by (6.2), 2) and 3) are satisfied. Since η_i are of type 3, 4 or 6, $\nu_{\eta_i}(\lambda)$ is divisible by ℓ and hence 4) is satisfied. Since $\lambda \notin F_P^{*\ell}$, 5) does not arise.

Suppose that $\lambda \in F_P^{*\ell}$. If η_i is of type 3, then let L_{P,η_1} be as in (9.3) and $\mu_{P,\eta_1} = \sqrt[\ell]{\lambda}$. Then, as in (9.6), $N_{L_P/F_P}(\mu_P) = \lambda$, $\text{ind}(\alpha \otimes L_{P,\eta_i}) < \text{ind}(\alpha)$ and $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$. Suppose that η_i is of type 4 or 6. Let $L_{P,\eta_i}/F_{P,\eta_i}$ be a cyclic unramified field extension of degree ℓ and μ_{P,η_i} be as in (4.10).

Then, by (6.4), there exist L_P and μ_P with required properties. □

Proposition 9.8. *Let $P \in \mathscr{P}$. Then there exist a cyclic field extension or split extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that*
1) $N_{L_P/F_P}(\mu_P) = \lambda$
2) $ind(\alpha \otimes L_P) < ind(\alpha)$
3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$
Further, suppose η is a codimension zero point of X_0 containing P.
4) If η is of type 1, then $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$.
5) Suppose η is of type 2 with a type 2 connection to a type 5 point η'. Let Q be the type 2 intersection of η and η'. If $M_{\eta'} \otimes F_{Q,\eta'}$ is not a field, then $L_P = \prod F_P$ and $\mu_P = (\theta_1, \cdots, \theta_\ell)$ with $\theta_i \in F_P$, $\nu_\eta(\theta_1) = \nu_\eta(\lambda)$ and $\nu_\eta(\theta_i) = 0$ for $i \geq 2$.
6) Suppose η is of type 2 with a type 2 connection to a type 5 point η'. Let Q be the type 2 intersection of η and η'. If $M_{\eta'} \otimes F_{Q,\eta'}$ is a field, then $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$.
7) Suppose η is of type 2 and there is no type 2 connection from η to any type 5 point. Then $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$.
8) If η is of type 3, then $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension and further if $\lambda \in F_P^{*\ell}$ and $\alpha \otimes E_\eta \otimes F_{P,\eta} \neq 0$, then $ind(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_P \otimes F_{P,\eta})) < ind(\alpha \otimes E_\eta \otimes F_{P,\eta})$.
9) If η is of type 4, then $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension.
10) If η is of type 5a, then $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension.
11) If η is of type 5b, then $L_P \otimes F_{P,\eta} \simeq M_\eta \otimes F_{P,\eta}$ and if $L_P = \prod F_P$, then $\mu_P = (\theta_1, \cdots, \theta_\ell)$ with $\nu_\eta(\theta_i) = \nu_\eta(\lambda)/\ell$.
12) If η is of type 6, then either $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension or $L_P = \prod F_P$, with $\mu_P = (\theta_1, \cdots, \theta_\ell)$ and $\nu_\eta(\theta_i) = \nu_\eta(\lambda)/\ell$.

Proof. Let η_1 and η_2 be two codimension zero points intersecting at P. By the choice of $\mathscr{X}$, X_0 is a union of regular curves with normal crossings and hence there are no other codimension zero points of X_0 passing through P.

Case I. Suppose that either η_1 or η_2, say η_1, is of type 1. Then $\nu_{\eta_1}(\lambda)$ is coprime to ℓ and hence $\lambda \notin F_P^{*\ell}$. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$. Then, by (6.2), L_P and μ_P satisfy 1), 2) and 3). By choice 4) is satisfied. Since $\mathscr{X}$ has no special points, η_2 is not of type 2 or 4. Thus 5), 6), 7) and 9) do not arise. Suppose η_2 is of type 3, 5 or 6. Then $\nu_{\eta_2}(\lambda)$ is divisible by ℓ and hence $L_P \otimes F_{P,\eta_2}/F_{P,\eta_2}$ is an unramified field extension. Thus 8), 10) and 12) are satisfied. Suppose η_2 is of type 5b. Since $\mathscr{X}$ has no special points and η_1 is of type 1, $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field. Since λ is a norm from the extension M_{η_2}/F_{η_2} (9.1) and $\lambda \notin F_{P,\eta_2}^{*\ell}$, by (2.6), $M_{\eta_2} \otimes F_{P,\eta_2} \simeq F_{P,\eta_2}(\sqrt[\ell]{\lambda})$ and hence 11) is satisfied.

Case II. Suppose neither η_1 nor η_2 is of type 1. Suppose either η_1 or η_2 is of type 2, say η_1 is of type 2. Then $\nu_{\eta_1}(\lambda)$ is coprime to ℓ and hence $\lambda \notin F_P^{*\ell}$.

Suppose that η_1 has type 2 connection to a codimension zero point η' of X_0 of type 5. Let Q be the closed point on η' which is the type 2 intersection point of η_1 and η'. By the choice of $\mathscr{X}$, η_2 is of type 2, 5 or 6. Note that if η_2 is also of type 2, then Q is also the point of type 2 intersection of η_2 and η'. Thus if both η_1 and η_2 are of type 2, η' and Q do not depend on whether we start with η_1 or η_2.

Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ is not a field. Let $L_P = \prod F_P$. Suppose η_2 is of type 2. Then, let $\mu_P = (\lambda, 1, \cdots, 1) \in L_P = \prod F_P$. Suppose η_2 is of type 5. Then by the assumption on $\mathscr{X}$, $\eta_2 = \eta'$, $Q = P$. Thus $M_{\eta_2} \otimes F_{P,\eta_2} = M_{\eta'} \otimes F_{Q,\eta'}$ is not a field and hence η_2 is of type 5b. Let $\mu_i \in F_P$ be as in (9.4) and $\mu_P = (\mu_1, \cdots, \mu_\ell)$. Suppose η_2 is of type 6. Let $\mu_i \in F_P$ be as in (9.4) and $\mu_P = (\mu_1, \cdots, \mu_\ell) \in L_P$. Then, L_P and μ_P satisfy 1) and 3). Since η_1 is of type 2, $\mathrm{ind}(\alpha \otimes F_{\eta_1}) < \mathrm{ind}(\alpha)$ and hence, by (5.8), $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$ and 2) is satisfied. Since neither η_1 nor η_2 is of type 1, the case 4) does not arise. By choice L_P satisfies 5). Since there is only one type 5 point with a type 2 connection to η_1 or η_2, the case 6) does not arise. Clearly the case 7) does not arise. Since η_2 is not of type 3, 4 or 5a, the cases 8), 9) and 10) do not arise. By choice of L_P and μ_P, 11) and 12) are satisfied.

Suppose $M_{\eta'} \otimes F_{Q,\eta'}$ is a field. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$. Since $\lambda \notin F_P^{*\ell}$, by (6.2), L_P and μ_P satisfy 1), 2) and 3). As above the cases 4), 5), 7) and 8) do not arise. By choice 6) is satisfied. Suppose η_2 is of type 5. Then $\eta_2 = \eta'$, $Q = P$ and $\nu_{\eta_2}(\lambda)$ is divisible by ℓ and hence 9) is satisfied. Suppose η_2 is of type 5b. Since $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field, as in case I, $M_{\eta_2} \otimes F_{P,\eta_2} \simeq L_P \otimes F_{P,\eta_2}$ and hence 10) is satisfied. Since $\lambda \notin F_P^{*\ell}$ and if η_2 is of type 6, $\nu_{\eta_2}(\lambda)$ is divisible by ℓ, $L_P \otimes F_{P,\eta_2}/F_{P,\eta_2}$ is an unramified field extension and hence 11) is satisfied.

Suppose that neither η_1 nor η_2 have a type 2 connection to a point of type 5. In particular η_2 is not of type 5. Then, let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$. Then, by (6.2), L_P and μ_P satisfy 1), 2) and 3). Since neither η_1 nor η_2 is of type 1, the case 4) does not arise. Since neither η_1 nor η_2 has type 2 connection to a point of type 5, 5) and 6) do not arise. By the choice of L_P and μ_P, 7) is satisfied. If η_2 is of type 4 or 6, $\nu_{\eta_2}(\lambda)$ is divisible by ℓ, 8), 9) and 12) are satisfied. Since neither η_1 nor η_2 is of type 5, 10) and 11) do not arise.

Case III. Suppose neither of η_i is of type 1 or 2. Suppose that one of the η_i, say η_1, is of type 3. Since $\mathscr{X}$ has no special points, η_2 is not of type 4 and hence η_2 is of type 3, 5 or 6. If η_2 is of type 5, let L_P and μ_P be as in (9.6). If η_2 is of type 3 or 6, let L_P and μ_P be as in (9.7). Then, 1), 2), 3), 8), 9), 10), 11) and 12) are satisfied and other cases do not arise.

Case IV. Suppose neither of η_i is of type 1, 2 or 3. Suppose that one of the η_i, say η_1, is of type 4. Since $\mathscr{X}$ has no special points, η_2 is not of type 5. Hence η_2 is of type 4 or 6. Let L_P and μ_P be as in (9.7). Then L_P and μ_P have the required properties.

Case V. Suppose neither of η_i is of type 1, 2, 3 or 4. Suppose that one of the η_i is of type 5, say η_1 is of type 5. Then η_2 is of type 5 or 6. Suppose that η_2 is of type 5. Since $\mathscr{X}$ has no special points, $M_{\eta_i} \otimes F_{P,\eta_i}$ are fields for $i = 1, 2$. Let L_P and μ_P be as in (9.2). Then L_P and μ_P have the required properties. Suppose that η_2 is of type 6. Suppose $M_{\eta_1} \otimes F_{P,\eta_1}$ is a field. Let $L_{P,\eta_1} = M_{\eta_1} \otimes F_{P,\eta_1}$ and $\mu_{\eta_1} \in M_{\eta_1}$ with $N_{M_{\eta_1}/F_{\eta_1}}(\mu_{\eta_1}) = \lambda$ (cf., 9.1). Let L_P and μ_P be as in (6.5) with $L_P \otimes F_{P,\eta_1} \simeq L_{P,\eta_1}$.

Then L_P and μ_P have the required properties. Suppose that $M_{\eta_1} \otimes F_{P,\eta_1}$ is not a field. Let $L_P = \prod F_P$ and $\mu_i \in F_P$ be as in (9.5) and $\mu_P = (\mu_1, \cdots, \mu_\ell) \in L_P$. Then L_P and μ_P have the required properties.

Case VI. Suppose neither of η_i is of type 1, 2, 3, 4 or 5. Then, η_1 and $t_P\eta_2$ are of type 6. Let L_P and μ_P be as in (9.7). Then L_P and μ_P have the required properties. $\square$

10. Choice of L_η and μ_η at codimension zero points.

Let F, $n = \ell^d$, $\alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha \neq 0$, $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\mathscr{X}$, X_0 and $\mathscr{P}$ be as in (§7, §8 and §9). Assume that $\mathscr{X}$ has no special points and there is no type 2 connection between a codimension zero point of X_0 of type 3 or 5 and a codimension zero point of X_0 of type 3, 4 or 5. Further we assume that for every closed point P of X_0, the residue field $\kappa(P)$ at P is a finite field. Let P be a closed point P of X_0. Then there exists $t_P \geq d$ such that there is no primitive $\ell^{t_P^{\text{th}}}$ root of unity in $\kappa(P)$.

For a codimension zero point η of X_0, let $\mathscr{P}_\eta = \eta \cap \mathscr{P}$.

Proposition 10.1. *Let η be a codimension zero point of X_0 of type 1. For each $P \in \mathscr{P}_\eta$, let (L_P, μ_P) be chosen as in (9.8) and $L_\eta = F_\eta(\sqrt[\ell]{\lambda})$ and $\mu_\eta = \sqrt[\ell]{\lambda} \in L_\eta$. Then*
1) $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$
3) *$ind(\alpha \otimes L_\eta) < ind(\alpha)$*
4) *for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} = 1.$$

Proof. By choice, we have $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$. Since η is of type 1, $\nu_\eta(\lambda)$ is coprime to ℓ and hence by (4.7), L_η and μ_η satisfies 2) and 3). Let $P \in \mathscr{P}_\eta$. Since η is of type 1, by the choice of L_P and μ_P (cf. 9.8(4)), we have $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$. Hence L_η and μ_η satisfy 4). $\square$

Lemma 10.2. *Let η be a codimension zero point of X_0. For each $P \in \mathscr{P}_\eta$, let $\theta_P \in F_P$ with $\alpha \cdot (\theta_P) = 0 \in H^3(F_{P,\eta}, \mu_n^{\otimes 2})$. Suppose $\nu_\eta(\theta_P) = 0$ for all $P \in \mathscr{P}_\eta$. Then there exists $\theta_\eta \in F_\eta$ such that*
1) $\alpha \cdot (\theta_\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$
2) *for $P \in \mathscr{P}_\eta$, $\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{\ell^{2t_P}}$.*

Proof. Let $\pi_\eta \in F_\eta$ be a parameter. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in (4.1). Let $E(\eta)$ be the residue field of E_η. Since $\alpha \cdot (\theta_P) = 0 \in H^3(F_{P,\eta}, \mu_n^{\otimes 2})$ and $\nu_\eta(\theta_P) = 0$, by (4.7), we have $(E(\eta) \otimes \kappa(\eta)_P, \sigma_0, \overline{\theta}_P) = 0 \in H^2(\kappa(\eta)_P, \mu_n)$, where $\overline{\theta}_P$ is the image of $\theta_P \in \kappa(\eta)_P$. Hence $\overline{\theta}_P$ is a norm from $E(\eta) \otimes \kappa(\eta)_P$ for all $P \in \mathscr{P}_\eta$. For $P \in \mathscr{P}_\eta$, let $\tilde{\theta}_P \in E(\eta) \otimes \kappa(\eta)_P$ with $N_{E(\eta)\otimes\kappa(\eta)_P/\kappa(\eta)_P}(\tilde{\theta}_P) = \overline{\theta}_P$. By weak approximation, there exists $\tilde{\theta} \in \kappa(\eta)$ which is sufficiently close to $\tilde{\theta}_P$ for all $P \in \mathscr{P}_\eta$. Let $\theta_0 = N_{E(\eta)/\kappa(\eta)}(\tilde{\theta}) \in \kappa(\eta)$. Then θ_0 is sufficiently close to $\overline{\theta}_P$ for all $P \in \mathscr{P}_\eta$. In particular, $\theta_0^{-1}\overline{\theta}_P \in \kappa(\eta)_P^{\ell^{2t_P}}$. Let $\theta_\eta \in F_\eta$ have image θ_0 in $\kappa(\eta)$. Then $(E_\eta, \sigma_\eta, \theta_\eta) = 0$ and hence, by (4.7), $\alpha \cdot (\theta_\eta) = 0$. Since $\theta_0^{-1}\overline{\theta}_P \in \kappa(\eta)_P^{\ell^{2t_P}}$ and $F_{P,\eta}$ is a complete discretely valuated field with residue field $\kappa(\eta)_P$, it follows that $\theta_\eta^{-1}\theta_P \in F_{P,\eta}^{\ell^{2t_P}}$. $\square$

Proposition 10.3. *Let η be a codimension zero point of X_0 of type 2. Suppose there is a type 2 connection between η and a codimension zero point η' of X_0 of type 5. Let Q be the point of type 2 intersection of η and η'. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ is not a field. For each $P \in \mathscr{P}_\eta$, let $\mu_P = (\theta_1^P, \cdots, \theta_\ell^P) \in L_P = \prod F_P$ be as in (9.8). Let $L_\eta = \prod F_\eta$. Then there exists $\mu_\eta = (\theta_1^\eta, \cdots, \theta_\ell^\eta) \in L_\eta$ such that*
1) $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$
3) $ind(\alpha \otimes L_\eta) < ind(\alpha)$,
4) $\mu_P^{-1}\mu_\eta \in (L_\eta \otimes F_{P,\eta})^{\ell^{2t_P}}$ for all $P \in \mathscr{P}_\eta$.

Proof. Let $i \geq 2$. By choice (cf. 9.8(5)), we have $\nu_\eta(\theta_i^P) = 0$ and $\alpha \cdot (\theta_i^P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$ for all $P \in \mathscr{P}_\eta$. By (10.2), there exists $\theta_i^\eta \in F_\eta$ such that $\alpha \cdot (\theta_i^\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$ and $(\theta_i^P)^{-1}\theta_i^\eta \in F_{P,\eta}^{\ell^{2t_P}}$ for all $P \in \mathscr{P}_\eta$. Let $\theta_1^\eta = \lambda(\theta_2^\eta \cdots \theta_\ell^\eta)^{-1}$. Then $\theta_1^\eta \cdots \theta_\ell^\eta = \lambda$ and $(\theta_1^P)^{-1}\theta_1^\eta \in F_{P,\eta}^{\ell^{2t_P}}$. Since $\alpha \cdot (\lambda) = 0$ and $\alpha \cdot (\theta_i^\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$ for $i \geq 2$, we have $\alpha \cdot (\theta_1) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$. Hence $L_\eta = \prod F_\eta$ and $\mu_\eta = (\theta_1^\eta, \cdots, \theta_\ell^\eta) \in L_\eta$. Since η is of type 2, $\text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha)$ and hence L_η, μ_η have the required properties. □

Proposition 10.4. *Let η be a codimension zero point of X_0 of type 2. For each $P \in \mathscr{P}_\eta$, let (L_P, μ_P) be chosen as in (9.8). Suppose one of the following holds.*
- *There is a type 2 connection between η and codimension zero point η' of X_0 of type 5 with Q the point of type 2 intersection of η and η' and $M_{\eta'} \otimes F_{Q,\eta'}$ is a field.*
- *There is no type 2 connection between η and any codimension zero point of X_0 of type 5.*

Let $L_\eta = F_\eta(\sqrt[\ell]{\lambda})$ and $\mu_\eta = \sqrt[\ell]{\lambda}$. Then
1) $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$
3) $ind(\alpha \otimes L_\eta) < ind(\alpha)$
4) for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} = 1.$$

Proof. Since $\nu_\eta(\lambda)$ is coprime to ℓ, by (4.7), $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$ and $\text{ind}(\alpha \otimes L_\eta) < \text{ind}(\alpha)$. Clearly $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$. By the choice of (L_P, μ_P) (cf. 9.8), for $P \in \mathscr{P}_\eta$, we have $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$. Thus L_η and μ_η have the required properties. □

Lemma 10.5. *Let η be a codimension zero point of X_0 of type 3, 4 or 5a. Let $P \in \eta$. Suppose there exists $L_{P,\eta}/F_{P,\eta}$ a degree ℓ unramified field extension and $\mu_{P,\eta} \in L_{P,\eta}$ such that*
1) $N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = \lambda$,
2) $ind(\alpha \otimes L_{P,\eta}) < ind(\alpha)$,
3) $\alpha \cdot (\mu_{P,\eta}) = 0 \in H^3(L_{P,\eta}, \mu_n^{\otimes 2})$,
4) If η is of type 3, $\lambda \in F_P^{*\ell}$ and $\alpha \otimes E_\eta \otimes F_{P,\eta} \neq 0$, then $ind(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) < ind(\alpha \otimes E_\eta \otimes F_{P,\eta})$.
Then $ind(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) < ind(\alpha)/[E_\eta : F_\eta]$.

Proof. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in (4.1). Then, by (4.2), $\text{ind}(\alpha \otimes F_\eta) = \text{ind}(\alpha' \otimes E_\eta)[E_\eta : F_\eta] = \text{ind}(\alpha \otimes E_\eta)[E_\eta : F_\eta]$. Let $t = [E_\eta : F_\eta]$ and β be the image of α' in $H^2(\kappa(\eta), \mu_n)$.

Suppose η is of type 4. Then $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$ and hence $\mathrm{ind}(\alpha \otimes E_\eta) = \mathrm{ind}(\alpha \otimes F_\eta)/t < \mathrm{ind}(\alpha)/t$. We have $\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) \leq \mathrm{ind}(\alpha \otimes E_\eta) < \mathrm{ind}(\alpha)/t$.

Suppose that η is of type 5a. Then α is unramified at η and hence $E_\eta = F_\eta$ and $t = 1$. The lemma is clear if $\alpha \otimes F_{P,\eta} = 0$. Suppose $\alpha \otimes F_{P,\eta} \neq 0$. Then $\beta \neq 0$. Since $L_{P,\eta}$ is a an unramified field extension, the residue field $L_P(\eta)$ of $L_{P,\eta}$ is a field extension of $\kappa(\eta)_P$ of degree ℓ. Since $\kappa(\eta)_P$ is a local field and $\mathrm{ind}(\beta)$ is divisible by ℓ, $\mathrm{ind}(\beta \otimes L_P(\eta)) < \mathrm{ind}(\beta)$ ([3, p. 131]). In particular $\mathrm{ind}(\alpha \otimes L_{P,\eta}) < \mathrm{ind}(\alpha)$.

Suppose that η is of type 3. Then $r\alpha \otimes E_\eta \neq 0$ and hence $\alpha' \otimes E_\eta = \alpha \otimes E_\eta \neq 0$. In particular $\mathrm{ind}(\alpha \otimes F_\eta) > t$ and $\beta \otimes E(\eta) \neq 0$. If $\alpha \otimes E_\eta \otimes F_{P,\eta} = 0$, then $\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) = 1 < \mathrm{ind}(\alpha)/t$. Suppose that $\alpha \otimes E_\eta \otimes F_{P,\eta} \neq 0$. Suppose $\lambda \in F_P^{*\ell}$. Then, by the choice of $L_{P,\eta}$, $\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) < \mathrm{ind}(\alpha \otimes E_\eta \otimes F_{P,\eta}) \leq \mathrm{ind}(\alpha \otimes E_\eta) = \mathrm{ind}(\alpha)/t$. Suppose $\lambda \not\in F_P^{*\ell}$. Then $\lambda \not\in F_{P,\eta}^{*\ell}$. Since $L_{P,\eta}$ is a field extension of degree ℓ and λ is a norm from $L_{P,\eta}$, by (2.6), $L_{P,\eta} \simeq F_{P,\eta}(\sqrt[\ell]{\lambda})$. Since η is of type 3, $\nu_\eta(\lambda) = r\ell$ and $\lambda = \theta_\eta \pi_\eta^{r\ell}$ with $\theta_\eta \in F_\eta$ a unit at η. Let $\overline{\theta}_\eta$ be the image of θ_η in $\kappa(\eta)$. Then $\overline{\theta}_\eta \not\in \kappa(\eta)_P^\ell$ and $L_P(\eta) = \kappa(\eta)_P(\sqrt[\ell]{\overline{\theta}_\eta})$. Since $\alpha \cdot (\lambda) = 0$, by (4.7), $r\ell\alpha' = (E_\eta, \sigma_\eta, \theta_\eta)$ and hence $r\ell\beta = (E(\eta), \sigma_0, \overline{\theta}_\eta)$. Thus, by (3.3), $\mathrm{ind}(\beta \otimes E(\eta)_P \otimes L_P(\eta)) < \mathrm{ind}(\beta \otimes E(\eta))$. Thus

$$\begin{aligned}
\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) &= \mathrm{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) \\
&= \mathrm{ind}(\beta \otimes E(\eta)_P \otimes L_P(\eta)) \\
&< \mathrm{ind}(\beta \otimes E(\eta)) = \mathrm{ind}(\alpha' \otimes E_\eta) \\
&= \mathrm{ind}(\alpha \otimes E_\eta) = \mathrm{ind}(\alpha)/t.
\end{aligned}$$

□

Proposition 10.6. *Let η be a codimension zero point of X_0 of type 3, 4 or 5a. For each $P \in \mathscr{P}_\eta$, let (L_P, μ_P) be chosen as in (9.8). Then there exist a field extension L_η/F_η of degree ℓ and $\mu_\eta \in L_\eta$ such that*

1) $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$
3) *$\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$*
4) *for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}.$$

Proof. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in (4.1). By (4.7), $r\ell\alpha' = (E_\eta, \sigma_\eta, \theta_\eta)$. Let β be the image of α' in $H^2(\kappa(\eta), \mu_n)$ and $E(\eta)$ the residue field of E_η. Then $r\ell\beta = (E(\eta), \sigma_0, \theta_0) \in H^2(\kappa(\eta), \mu_n)$, where σ_0 is the automorphism of $E(\eta)$ induced by σ_η and θ_0 is the image of θ_η in $\kappa(\eta)$.

Let S be a finite set of places of $\kappa(\eta)$ containing the places given by closed points of $\mathscr{P}_\eta$ and places ν of $\kappa(\eta)$ with $\beta \otimes \kappa(\eta)_\nu \neq 0$. For each $\nu \in S$, we now give a cyclic field extension $L_\nu/\kappa(\eta)_\nu$ of degree ℓ and $\mu_\nu \in L_\nu$ satisfying the conditions of (3.1) with $E_0 = E(\eta)$ and $d = \mathrm{ind}(\alpha)/t$.

Let $\nu \in S$. Then ν is given by a closed point P of η. If $P \in \mathscr{P}$, let $L_{P,\eta} = L_P \otimes F_{P,\eta}$ and $\mu_{P,\eta} = \mu_P \otimes 1 \in L_{P,\eta}$. Suppose $P \not\in \mathscr{P}$. Suppose that $\lambda \not\in F_P^{*\ell}$. Then $\lambda \not\in F_{P,\eta}^{*\ell}$. Let $L_{P,\eta} = F_{P,\eta}(\sqrt[\ell]{\lambda})$ and $\mu_{P,\eta} = \sqrt[\ell]{\lambda}$. Suppose that $\lambda \in F_P^{*\ell}$. If η is of type 3, then let $L_{P,\eta}/F_{P,\eta}$ be a cyclic unramified field extension of degree ℓ as in (9.3) and $\mu_{P,\eta} = \sqrt[\ell]{\lambda}$. If η is of type 4 or 5a, then let $L_{P,\eta}/F_{P,\eta}$ be a cyclic unramified field extension of degree ℓ as in (4.10) and $\mu_{P,\eta} = \sqrt[\ell]{\lambda}$.

Since $L_{P,\eta}/F_{P,\eta}$ is an unramified field extension of degree ℓ, the residue field $L_P(\eta)$ is a degree ℓ field extension of $\kappa(\eta)_P$. Let $L_\nu = L_P(\eta)$. We have $\nu_\eta(\lambda) = r\ell$ for some integer r and $\lambda = \theta_\eta\pi_\eta^{r\ell}$ for some parameter π_η at η and $\theta_\eta \in F_\eta$ a unit at η. Further π_η is a parameter in $L_{P,\eta}$. Since $N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = \lambda$, $\mu_{P,\eta} = \theta_{P,\eta}\pi_\eta^r$ for some $\theta_{P,\eta} \in L_P \otimes F_{P,\eta}$ which is a unit at η. Let μ_ν be the image of $\theta_{P,\eta}$ in $L_\nu = L_P(\eta)$. Then $N_{L_\nu/\kappa(\eta)_\nu}(\mu_\nu) = \theta_0$. Since the corestriction map $H^2(L_\nu, \mu_n) \to H^2(\kappa(\eta)_\nu, \mu_n)$ is injective, $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$. Let $t = [E_\eta : F_\eta]$. By (10.5), we have $\text{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes L_{P,\eta}) < \text{ind}(\alpha)/t$. Since $\alpha \otimes E_\eta = \alpha' \otimes E_\eta$, we have $\text{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes L_{P,\eta}) < \text{ind}(\alpha)/\ell^d$. Since $\text{ind}(\beta \otimes E_0 \otimes L_\mu) = \text{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_P \otimes F_{P,\eta}))$, $\text{ind}(\beta \otimes E_0 \otimes L_\nu) < \text{ind}(\alpha)/t$.

Since $\kappa(\eta)$ is a global-field, by (3.1), there exist a field extension $L_0/\kappa(\eta)$ of degree ℓ and $\mu_0 \in L_0$ such that

1) $N_{L_0/k}(\mu_0) = \theta_0$
2) $r\beta \otimes L_0 = (E(\eta) \otimes L_0, \sigma_0 \otimes 1, \mu_0)$
3) $\text{ind}(\beta \otimes E(\eta) \otimes L_0) < \text{ind}(\alpha)/t$
4) $L_0 \otimes \kappa(\eta)_P \simeq L_P(\eta)$ for all $P \in \mathscr{P}_\eta$
5) μ_0 is close to $\overline{\theta}_P$ for all $P \in \mathscr{P}_\eta$.

Then, by (4.8), there exist a field extension L_η/F_η of degree ℓ and $\mu \in L_\eta$ such that

- residue field of L_η is L_0,
- μ a unit in the valuation ring of L_η,
- $\overline{\mu} = \mu_0$,
- $N_{L_\eta/F_\eta}(\mu) = \theta_\eta$,
- $\alpha \cdot (\mu\pi_\eta^r) \in H^3(L_\eta, \mu_n^{\otimes 2})$ is unramified.

Since L_η is a complete discretely valued field with residue field L_0 a global field, $H^3_{nr}(L_\eta, \mu_n^{\otimes 2}) = 0$ ([27, p. 85]) and hence $\alpha \cdot (\mu\pi_\eta^r) = 0$. Since L_η/F_η is unramified and $\alpha \otimes L_\eta = \alpha' \otimes L_\eta + (E_\eta \otimes L_\eta, \sigma_\eta, \pi_\eta)$, $\text{ind}(\alpha \otimes L_\eta) \leq \text{ind}(\alpha' \otimes E_\eta \otimes L_\eta)[E_\eta \otimes L_\eta : L_\eta] = \text{ind}(\beta \otimes E(\eta) \otimes L_0)t < \text{ind}(\alpha)$. Thus L_η and $\mu_\eta = \mu\pi_\eta^r \in L_\eta$ have the required properties. □

Proposition 10.7. *Let η be a codimension zero point of X_0 of type 5b. Let (E_η, σ_η) be the residue of α at η and M_η be the unique subfield of E_η with M_η/F_η a cyclic extension of degree ℓ. For each $P \in \mathscr{P}_\eta$, let L_P and μ_P be as in (9.8). Then there exists $\mu_\eta \in M_\eta$ such that*

1) $N_{M_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(M_\eta, \mu_n^{\otimes 2})$
3) *$ind(\alpha \otimes M_\eta) < ind(\alpha)$*
4) *for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : M_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}.$$

Proof. Let $E(\eta)$ and $M(\eta)$ be the residue fields of E_η and M_η at η. Since η is of type 5b, $M(\eta)$ is the unique subfield of $E(\eta)$ with $M(\eta)/\kappa(\eta)$ a cyclic field extension of degree ℓ. Let π_η be a parameter at η. Since η is of type 5, $\nu_\eta(\lambda) = r\ell$ and $\lambda = \theta_\eta\pi_\eta^{r\ell}$ for some $\theta_\eta \in F$ a unit at η. Let $\overline{\theta}_\eta$ be the image of θ_η in $\kappa(\eta)$. Let $P \in \mathscr{P}_\eta$. Suppose $M_\eta \otimes F_{P,\eta}$ is a field. Since $N_{M_\eta \otimes F_{P,\eta}/F_{P,\eta}}(\mu_P) = \lambda = \theta_\eta\pi_\eta^{r\ell}$, we have $\mu_P = \mu'_P\pi_\eta^r$ with $\mu'_P \in M_\eta \otimes F_{P,\eta}$ a unit at η and $N_{M_\eta \otimes F_{P,\eta}/F_{P,\eta}}(\mu'_P) = \theta_\eta$. Suppose $M_\eta \otimes F_{P,\eta}$ is not a field. Then, by the choice of μ_P (cf. 9.8(10)), we have $\mu_P = \mu'_P\pi_\eta^r$, where $\mu'_P = (\theta'_1, \cdots, \theta'_\ell) \in M_\eta \otimes F_{P,\eta'} = \prod F_{P,\eta}$ with each $\theta'_i \in F_{P,\eta}$ is a unit at η. Let $\overline{\mu'}_P$ be the image of μ'_P in the residue field $M(\eta) \otimes \kappa(\eta)_P$ of $M_\eta \otimes F_{P,\eta}$ at η. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in (4.1). Let β be the image of α' in $H^2(\kappa(\eta), \mu_n)$. Since

$\alpha \cdot (\lambda) = 0$, by (4.7), $r\ell\beta = (E(\eta), \sigma_\eta, \overline{\theta}_\eta)$. Since $\alpha \cdot (\mu_P) = 0$ in $H^3(M_\eta \otimes F_{P,\eta}, \mu_n^{\otimes 2})$, once again by (4.7), $r\beta \otimes \kappa(\eta)_P = (E(\eta) \otimes M(\eta) \otimes \kappa(\eta)_P, \sigma_\eta, \overline{\mu'}_P)$. Since $\kappa(\eta)$ is a global field, by (3.6), there exists $\mu'_\eta \in M(\eta)$ such that
1) $N_{M(\eta)/\kappa(\eta)}(\mu'_\eta) = \overline{\theta}_\eta$
2) $r\beta \otimes M(\eta) = (E(\eta) \otimes M(\eta), \sigma_\eta, {\mu'}_\eta)$
3) $\overline{\mu'}_P$ is close to μ'_η for all $P \in \mathscr{P}_\eta$.
Since M_η is complete, there exists $\tilde{\mu_\eta} \in M_\eta$ such that $N_{M_\eta/F_\eta}(\tilde{\mu'_\eta}) = \theta_\eta$ and the image of $\tilde{\mu'_\eta}$ in $M(\eta)$ is μ'_η. Let $\mu_\eta = \tilde{\mu'_\eta}\pi_\eta^r$. Since M_η/F_η is of degree ℓ, $\mathrm{ind}(\alpha \otimes M_\eta) < \mathrm{ind}(\alpha \otimes F_\eta)$ (cf. 8.1). Thus μ_η has the required properties. ☐

Proposition 10.8. *Let η be a codimension zero point of X_0 of type 6. For each $P \in \mathscr{P}_\eta$, let L_P and μ_P be as in (9.8). Then there exist a field extension L_η/F_η of degree ℓ and $\mu_\eta \in L_\eta$ such that*
1) $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$
3) *$\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$*
4) *for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : M_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}.$$

Proof. Let $P \in \mathscr{P}_\eta$. Suppose $L_P \otimes F_{P,\eta}$ is a field. Let $L_P(\eta)$, $\overline{\theta}_P \in L_P(\eta)$, $\theta_0 \in \kappa(\eta)$ and β be as in the proof of (10.6). Then, as in the proof of (10.6), we have $N_{L_P(\eta)/\kappa(\eta)_P}(\overline{\theta}_P) = \theta_0$ and $\mathrm{ind}(\beta \otimes E_0 \otimes L_P(\eta)) < \mathrm{ind}(\alpha)/[E_\eta : F_\eta]$. As in the proof of (10.7), we have $r\beta \otimes L_P(\eta) = (E_0 \otimes L_P(\eta), \sigma_0 \otimes 1, \overline{\theta}_P)$.

If L_P/F_P is not a field, by choice (cf. 9.8(11)), we have $\mu_P = (\theta_1\pi_\eta^r, \cdots, \theta_\ell\pi_\eta^r)$. Since $\alpha \cdot (\mu_P) = 0$ in $H^3(L_P, \mu_n^{\otimes}) = \prod H^3(F_P, \mu_n^{\otimes 2})$, we have $\alpha \cdot (\theta_i\pi_\eta^r) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$. Thus, by (4.7), we have $r\beta \otimes \kappa(\eta)_P = (E_0, \sigma_0 \otimes 1, \overline{\theta}_i)$ for all i. Since $L_P(\eta) = \prod \kappa(\eta)_P$ and $\overline{\theta}_P = (\overline{\theta}_1, \cdots, \overline{\theta}_\ell)$, we have $r\beta \otimes L_P(\eta) = (E_0 \otimes L_P(\eta), \sigma_0 \otimes 1, \overline{\theta}_P)$.

As in the proof of (10.6), we construct L_η and μ_η with the required properties. ☐

Lemma 10.9. *Let η be a codimension one point of X_0 and P a closed point on η. Suppose there exist $\theta_\eta \in F_\eta$ such that $\alpha \cdot (\theta_\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$. Then there exists $\theta_P \in F_P$ such that $\alpha \cdot (\theta_P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$, $\nu_\eta(\theta_P) = \nu_\eta(\theta_\eta)$ and $\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{*\ell^{2t_P}}$.*

Proof. Let π be a prime representing η at P. Since η is regular on $\mathscr{X}$, there exists a prime δ at P such that the maximal ideal at P is generated by π and δ. Since $F_{P,\eta}$ is a complete discrete valued field with π as a parameter, $\theta_\eta = w\pi^s$ for some $w \in F_\eta$ unit at η. Since the residue field $\kappa(\eta)_P$ of $F_{P,\eta}$ is a complete discrete valued field with $\overline{\delta}$ as a parameter, we have $\overline{w} = \overline{u}\overline{\delta}^r$ for some $u \in F_P$ unit at P. Let $\theta_P = u\delta^r\pi^s$. Then clearly $\nu_\eta(\theta_\eta) = \nu_\eta(\theta_P)$ and $\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{\ell^{2t_P}}$. Since $\alpha \cdot (\theta_P)$ is unramified at P except possibly at π and δ and $\alpha \cdot (\theta_P) = \alpha \cdot (\mu_P) = 0 \in H^3(F_{P,\eta}, \mu_n^{\otimes 2})$, by (5.5), $\alpha \cdot (\theta_P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$. ☐

11. The main theorem

Theorem 11.1. *Let K be a local field with residue field κ and F the function field of a curve over K. Let D be a central simple algebra over F of exponent n, α its class in $H^2(F, \mu_n)$, and $\lambda \in F^*$. If $\alpha \cdot (\lambda) = 0$ and n is coprime to $\mathrm{char}(\kappa)$, then λ is a reduced norm from D^*.*

Proof. As in the proof of (4.12), we assume that $n = \ell^d$ for prime ℓ with $\ell \neq$char(κ) and F contains a primitive $\ell^{\rm th}$ root of unity. We prove the theorem by induction on ind(D).

Suppose that ind$(D) = 1$.Then D is a matrix algebra and hence every element of F is a reduced norm. Assume that ind$(D) > 1$.

Without loss of generality we assume that K is algebraically closed in F. Let X be a smooth projective geometrically integral curve over K with $K(X) = F$. Let R be the ring of integers in K and κ its residue field. Let $\mathscr{X}$ be a regular proper model of F over R such that the union of ram$_{\mathscr{X}}(\alpha)$, supp$_{\mathscr{X}}(\lambda)$ and the special fibre X_0 of $\mathscr{X}$ is a union of regular curves with normal crossings. By (8.6), we assume that $\mathscr{X}$ has no special points, and there is no type 2 connection between a codimension zero point of X_0 of type 3, or 5 and codimension zero point of X_0 of type 3, 4 or 5.

Let $\mathscr{P}$ be the set of nodal points of X_0. For each $P \in \mathscr{P}$, let L_P and μ_P be as in (9.8). Let η be a codimension zero point of X_0 and $\mathscr{P}_\eta = \mathscr{P} \cap \eta$. Let L_η and μ_η be as in 10.1, 10.3, 10.4, 10.6, 10.7 or 10.8 depending on the type of η. Then L_η/F_η is an extension of degree ℓ and $\mu_\eta \in L_\eta$ such that

1) $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$
2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$
3) ind$(\alpha \otimes L_\eta) <$ ind(α)
4) for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}.$$

Let $P \in \mathscr{X}$ be a closed point with $P \notin \mathscr{P}$. Then there is a unique codimension zero point η of X_0 with $P \in \eta$. We give a choice of a cyclic or split extension L_P/F_P of degree ℓ and $\mu_P \in L_P^*$ such that

1) $N_{L_P/F_P}(\mu_P) = \lambda$,
2) ind$(\alpha \otimes L_P) <$ ind(α),
3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$,
4) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}.$$

Suppose that η is of type 1. Then, by the choice of L_η and μ_η (10.1), $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda}$ have the required properties.

Suppose that η is of type 2. Suppose that there is a type 2 connection to a codimension zero point η' of X_0 of type 5. Let Q be the point of type 2 intersection η and η'. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ not a field. Then, by choice (cf. 10.3), we have $L_\eta = \prod F_\eta$ and $\mu_\eta = (\theta_1, \cdots, \theta_\ell)$. Since $\alpha \cdot (\mu_\eta) = 0$, we have $\alpha \cdot (\theta_i) = 0$. For each i, $2 \leq i \leq \ell$, by (10.9), there exists $\theta_i^P \in F_P$ such that $\alpha \cdot (\theta_i^P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$ and $\theta_i^{-1}\theta_i^P \in F_{P,\eta}^{*\ell^{2t_P}}$. Let $\theta_1^P = \lambda(\theta_2^P \cdots \theta_\ell^P)^{-1}$. Then $L_P = \prod F_P$ and $\mu_P = (\theta_1^P, \cdots, \theta_\ell^P)$ have the required properties. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ is a field or there is no type 2 connection from η to any point of type 5. Then, by the choice (10.4), we have $L_\eta = F_\eta(\sqrt[\ell]{\lambda})$ and $\mu_\eta = \sqrt[\ell]{\lambda}$. Hence $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = \sqrt[\ell]{\lambda} \in L_P$ have the required properties.

Suppose that η is not of type 1 or 2. Then, by choice L_η/F_η is an unramified field extension of degree ℓ or the split extension of degree ℓ. Let $\hat{A}_P$ be the completion of the local ring at P and π a prime in $\hat{A}_P$ defining η at P. Since $P \notin \mathscr{P}$ and ram$_{\mathscr{X}}(\alpha)$ is union of regular curves with normal crossings, there exists a prime $\delta \in \hat{A}_P$ such that α is unramified on $\hat{A}_P$ except possibly at π and δ. Further, $\lambda = w\pi^r\delta^s$ for some

unit $u \in \hat{A}_P$. Since η is not of type 1 or 2, $\nu_\eta(\lambda) = r$ is divisible by ℓ. Thus, by (6.5), there exist a cyclic extension L_P/F_P and $\mu_P \in L_P$ such that
1) $L_P \otimes F_{P,\eta} \simeq L_\eta \otimes F_{P,\eta}$,
2) $\mathrm{ind}(\alpha \otimes L_P) < \mathrm{ind}(\alpha)$,
3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$,
4) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}.$$

Thus for every $x \in X_0$, we have chosen an extension L_x/F_x of degree ℓ and $\mu_x \in L_x$ such that
1) $N_{L_x/F_x}(\mu_x) = \lambda$
2) $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_x^{\otimes 2})$
3) $\mathrm{ind}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$
4) for any branch (P, η), there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and $\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}$. Further if P is a closed point of X_0, then L_P/F_P is cyclic or the split extension.

Let (P, η) be a branch. Since $\kappa(P)$ has no $\ell^{2t_P^{\mathrm{th}}}$ primitive root of unity and $\kappa(\eta)_P$ is a complete discretely valued field with residue field $\kappa(P)$, $\kappa(\eta)_P$ has no $\ell^{2t_P^{\mathrm{th}}}$ primitive root of unity. Since $F_{P,\eta}$ is a complete discretely valued field with residue field $\kappa(\eta)_P$, $F_{P,\eta}$ has no $\ell^{2t_P^{\mathrm{th}}}$ primitive root of unity. Since $\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^{2t_P}}$ and $t_P \geq d$, by (2.8), for a generator σ of $\mathrm{Gal}(L_P \otimes F_{P,\eta}/F_{P,\eta})$, there exists $\theta_{P,\eta} \in L_P \otimes F_{P,\eta}$ such that $\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} = \theta_{P,\eta}^{-\ell^d} \sigma(\theta_{P,\eta})^{\ell^d}$.

By (7.5), there exist extensions L/F of degree ℓ, N/F of degree coprime to ℓ, and $\mu \in L \otimes N$ such that
- $N_{L \otimes N/F}(\mu) = \lambda$ and
- $\alpha \cdot (\mu) = 0 \in H^3(L \otimes N, \mu_n^{\otimes 2})$
- $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.

Since $L \otimes N$ is also a function field of a curve over a p-adic field, by induction hypotheses, μ is a reduced norm from $D \otimes L \otimes N$ and hence $\lambda = N_{L \otimes N/N}(\mu)$ is a reduced norm from D. Since $N_{N/F}(\lambda) = \lambda^{[N:F]}$, $\lambda^{[N:F]}$ is a norm from D. Since $[N : F]$ is coprime to ℓ, λ is a reduced norm from D. □

Corollary 11.2. *Let K be a local field with residue field κ and F the function field of a curve over K. Let Ω be the set of divisorial discrete valuations of F. Let D be a central simple algebra over F of index coprime to char(κ) and $\lambda \in F$. If λ is a reduced norm from $D \otimes F_\nu$ for all $\nu \in \Omega$, then λ is a reduced norm from D.*

Proof. Since λ is a reduced norm from F_ν for all $\nu \in \Omega_F$, $\alpha \cdot (\lambda) = 0$ in $H^3(F_\nu, \mu_n^{\otimes 2})$ for all $\nu \in \Omega$. Thus, by ([16, Proposition 5.2]), $\alpha \cdot (\lambda) = 0$ in $H^3(F, \mu_n^{\otimes 2})$ and by (11.1), λ is a reduced norm from D. □

Acknowledgments: We thank Nivedita for the wonderful conversations during the preparation of the paper. The first author is partially supported by National Science Foundation grants DMS-1401319 and DMS-1463882 and the third author is partially supported by National Science Foundation grants DMS-1301785 and DMS-1463882.

References

[1] A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloq. Publ., Vol. 24, Amer. Math. Soc., Providence, RI, 1961, revised printing.

[2] M. Artin, *Algebraic approximation of structures over complete local rings*. Publ. Math. IHES, **36** (1969), 23–58.

[3] J.W.S Cassels and A. Fröhlich, *Algebraic Number Theory*, Thomson Book Company Inc, Washington, D.C, 1967.

[4] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, in K-Theory and Algebraic Geometry : Connections with Quadratic Forms and Division Algebras, AMS Summer Research Institute, Santa Barbara 1992, ed. W. Jacob and A. Rosenberg, Proceedings of Symposia in Pure Mathematics **58**, Part I (1995) 1-64.

[5] J.-L. Colliot-Thélène, R. Parimala, V. Suresh, *Patching and local global principles for homogeneous spaces over function fields of p-adic curves*, Commentari Math. Helv. **87** (2012), 1011–1033.

[6] B. Fein and M. Schacher, $\mathbb{Q}(t)$ *and* $\mathbb{Q}((t))$*-Admissibility of Groups of Odd Order*, Proceedings of the AMS, **123** (1995), 1639–1645.

[7] M. Fried and M. Jarden, *Field Arithmetic*, A Series of Modern Surveys in Mathematics, Volume 11, Springer.

[8] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol.101, Cambridge University Press, Cambridge, 2006.

[9] D. Harbater and J. Hartmann, *Patching over fields*, Israel J. Math. **176** (2010), 61–107.

[10] D. Harbater, J. Hartmann and D. Krashen, *Applications of patching to quadratic forms and central simple algebras*, Invent. Math. **178** (2009), 231–263.

[11] D. Harbater, J. Hartmann and D. Krashen, *Local-global principles for torsors over arithmetic curves*, American Journal of Mathematics, **137** (2015), 1559–1612

[12] D. Harbater, J. Hartmann and D. Krashen, *Local-global principles for Galois cohomology*, Commentarii Mathematici Helvetici, **89**, (2014), 215–253.

[13] D. Harbater, J. Hartmann and D. Krashen, *Refinements to patching and applications to field invariants*,

[14] Y. Hu, *Hasse Principle for Simply Connected Groups over Function Fields of Surfaces*, J. Ramanujan Math. Soc. **29** (2014), no. 2, 155–199.

[15] B. Jacob and A. Wadsworth, *Division algebras over Henselian fields*, J. Algebra **128** (1990), 126–179.

[16] K. Kato, *A Hasse principle for two-dimensional global fields, J. reine angew. Math.* **366** (1986), 142-181.

[17] F. Lorenz, *Algebra Volume II: Fields with Structure, Algebras and Advanced Topics*, Universitytext, Springer (2008).

[18] M.-A. Knus, A.S. Merkurjev, M. Rost and J.-P. Tignol, *The Book of Involutions*, A.M.S, Providence RI, 1998.

[19] A.S. Merkurjev and A.A. Suslin, *The norm residue homomorphism of degree 3*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), 339-356; translation in Math. USSR-Izv. **36** (1991), 349-367.

[20] J.S.Milne, *Étale Cohomology*, Princeton University Press, Princeton, New Jersey (1980).

[21] R. Parimala and V. Suresh, *Period-index and u-invariant questions for function fields over complete discretely valued fields*, Invent. math. 197 (2014), 215–235.

[22] R. Parimala and V. Suresh, *On the u-invariant of function fields of curves over complete discretely valued fields*, Adv. Math. 280 (2015), 729–742.

[23] R. Preeti, *Classification theorems for Hermitian forms, the Rost kernel and Hasse principle over fields with* $cd_2(k) \leq 3$, J. Algebra **385** (2013), 294–313.

[24] J. Riou, *Classes de Chern, morphismes de Gysin, pureté absolue* In: *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schemas quasi-excellents.* Sém. à l'École polytechnique 2006–2008, dirigé par L. Illusie, Y. Laszlo et F. Orgogozo. *Astérisque* **363-364**, 2014, pp. 301–349.

[25] D. J. Saltman, *Division algebras over p-adic curves*, J. Ramanujan Math. Soc. **12** (1997), 25–47.

[26] J-P. Serre, *Local fields*, Springer-Verlag, New York, 1979.

[27] J-P. Serre, *Galois Cohomology*, Springer-Verlag, New York, 1997.

[28] J-P. Serre, Cohomological invariants, Witt invariants and trace forms, in *Cohomological Invariants in Galois Cohomology*, Skip Garibaldi, Alexander Merkurjev, Jean-Pierre Serre, University Lecture Series **28**, Amer. Math. Soc. 2003.

[29] B. Surendranath Reddy and V. Suresh, *Admissibility of groups over function fields of p-adic curves*, Adv. in Math. **237** (2013) 316-330.

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