THE RATIONALITY PROBLEM FOR FORMS OF $\overline{M}_{0,n}$

MATHIEU FLORENCE AND ZINOVY REICHSTEIN

Abstract. Let X be a del Pezzo surface of degree 5 defined over a field F. A theorem of Yu. I. Manin and P. Swinnerton-Dyer asserts that every Del Pezzo surface of degree 5 is rational. In this paper we generalize this result as follows. Recall that del Pezzo surfaces of degree 5 over a field F are precisely the F-forms of the moduli space $\overline{M}_{0,5}$ of stable curves of genus 0 with 5 marked points. Suppose $n \geqslant 5$ is an integer, and F is an infinite field of characteristic $\neq 2$. It is easy to see that every twisted F-form of $\overline{M}_{0,n}$ is unirational over F. We show that

(a) If n is odd, then every twisted F-form of $\overline{M}_{0,n}$ is rational over F.

(b) If n is even, there exists a field extension F/k and a twisted F-form X of $\overline{M}_{0,n}$ such that X is not retract rational over F.

1. Introduction

Let X be a del Pezzo surface of degree 5 defined over a field F. Yu. I. Manin [Man63, Theorem 3.15] showed that if X has an F-point, then X is rational over F. P. Swinnerton-Dyer [SD72] then proved that X always has an F-point; for alternative proofs of this assertion, see [SB92] and [Sko93]. In summary, one obtains the following result, published earlier by F. Enriques [E1897] (with an incomplete proof).

Theorem 1.1. (*Enriques, Manin, Swinnerton-Dyer*) *Every del Pezzo surface of degree* 5 *defined over a field F is F-rational. Equivalently, every F-form of $\overline{M}_{0,5}$ is F-rational.*

The purpose of this paper is to generalize this celebrated theorem as follows. As usual, we will denote the moduli space of smooth (respectively, stable) curves of genus g with n marked points by $M_{g,n}$ (respectively, $\overline{M}_{g,n}$). Recall that these moduli spaces are defined over the prime field. A *form* of a scheme X defined over a field F is an F-scheme Y, such that X and Y become isomorphic over the separable closure F^{sep}. We will use the terms "form", "F-form" and "twisted form" interchangeably throughout this paper. For a discussion of this notion and further references, see Section 2.

We now recall that $\overline{M}_{0,5}$ is a split del Pezzo surface of degree 5, and F-forms of $\overline{M}_{0,5}$ are precisely the del Pezzo surfaces of degree 5 defined over F. The main result of this paper is Theorem 1.2 below.

2010 *Mathematics Subject Classification.* 14E08, 14H10, 20G15, 16K50.

Key words and phrases. Rationality, moduli spaces of marked curves, twisted forms, Galois cohomology, Brauer group.

Florence was partially supported by the French National Agency (Project GeoLie ANR-15-CE40-0012).

Reichstein was partially supported by National Sciences and Engineering Research Council of Canada Discovery grant 253424-2017. Both authors are grateful to the Collaborative Research Group in Geometric and Cohomological Methods in Algebra at the Pacific Institute for the Mathematical Sciences for their support of this project.

Theorem 1.2. *Let $n \geqslant 5$ be an integer and F be an infinite field of characteristic $\neq 2$.*

(a) Assume n is odd. Then every F-form of $\overline{M}_{0,n}$ is rational over F.

(b) Assume n is even. If $\mathrm{Br}_2(F) \neq 0$, then there exists an F-form X of $\overline{M}_{0,n}$ such that X is not retract rational over F.

Several remarks are in order.

(1) If F is assumed to be an infinite field of characteristic different from 2, Theorem 1.1 is recovered from Theorem 1.2(a) by setting $n = 5$.

(2) In part (b), Br_2 denotes the 2-torsion of the Brauer group. The condition that $\mathrm{Br}_2(F) \neq 0$ is fairly mild; it is equivalent to the existence of a non-split quaternion algebra over F. In particular, $\mathrm{Br}_2(F) \neq 0$ if $F = k(t_1, t_2)$ where k is an arbitrary field of characteristic $\neq 2$ and t_1, t_2 are independent variables.

(3) The assumption that F is infinite is only used in part (a); see Section 6. In part (b) it is automatic. Indeed, by a theorem of J. Wedderburn, $\mathrm{Br}_2(F) = (0)$ for any finite field F; see e.g., [GS06, Remark 6.2.7].

(4) For $n \geqslant 5$, all F-forms of $\overline{M}_{0,n}$ are unirational over F; see [DR15, Theorem 6.1] or Proposition 4.4(a) below.

(5) It is natural to ask if similar rationality results hold for forms of $\overline{M}_{g,n}$ for $g \geqslant 1$. Theorems of J. Harris, D. Mumford, D. Eisenbud and G. Farkas [HM82, EH87, Fa00, Fa11], assert that $M_{g,0}$ is not unirational for any $g \geqslant 23$, and hence, neither is $M_{g,n}$ for any $n \geqslant 0$. Moreover, A. Logan [Lo03] exhibited an explicit integer $f(g)$, for each $1 \leqslant g \leqslant 22$ such that $M_{g,n}$ is not unirational as long as $n \geqslant f(g)$. Deciding for which of the finitely many remaining pairs (g, n) the moduli space $M_{g,n}$ is rational, stably rational or unirational, over $\mathbb{C}$ is a problem of ongoing interest; see, e.g., [CF07]. We have shown that in some cases (for small $n, g \geqslant 1$), every form of $\overline{M}_{g,n}$ is stably rational; see [FloR17].

The remainder of this paper will be devoted to proving Theorem 1.2.

2. Preliminaries on moduli spaces of curves and their twisted forms

The F-forms of a quasi-projective variety X are in a natural bijective correspondence with $H^1(F, \mathrm{Aut}(X))$; see [Se97, II.1.3]. Here $\mathrm{Aut}(X)$ is a functor which associates to the scheme S/F the abstract group $\mathrm{Aut}(X_S)$. This functor is not representable by an algebraic group defined over F in general. If it is, one usually says that $\mathrm{Aut}(X)$ is an algebraic group. In the case, where $\mathrm{Aut}(X)$ is an algebraic group, the bijective correspondence between $H^1(F, \mathrm{Aut}(X))$ and the set of F-forms of X (up to F-isomorphism) can be described explicitly by using the twisting operation. That is, to an $\mathrm{Aut}(X)$-torsor $\tau \colon Y \to \mathrm{Spec}(F)$, we associate the F-variety ${}^{\tau}X := (X \times Y)/\mathrm{Aut}(X)$, which is a twisted form of X. Up to F-isomorphism, ${}^{\tau}X$ depends only on the class α of τ in $H^1(F, \mathrm{Aut}(X))$; see [Se97, Section III.1.3]. By abuse of notation, we will sometimes write ${}^{\alpha}X$ in place of ${}^{\tau}X$. For the definition and basic properties of the twisting operation we refer the reader to [Flo08, Section 2] or [DR15, Section 3]. Conversely, to a twisted form X' of X defined over F, we associate the $\mathrm{Aut}(X)$-torsor $\mathrm{Isom}_F(X, X') \to \mathrm{Spec}(F)$.

The following recent result is the starting point for our investigation.

Theorem 2.1. *Let F be a field of characteristic $\neq 2$. If $2g + n \geqslant 5$, then the natural embedding $\mathrm{S}_n \to \mathrm{Aut}_F(\overline{M}_{g,n})$ is an isomorphism.*

In the case $g = 0$ and $F = \mathbb{C}$, Theorem 2.1 was proved by A. Bruno and M. Mella [BM13]. In the more general situation, where $F = \mathbb{C}$ but $g \geqslant 0$ is arbitrary, it is due to A. Massarenti [Mas14], and in full generality to B. Fantechi and A. Massarenti [FM17, Theorem A.2 and Remark A.4]. As an immediate consequence, we obtain the following.

Corollary 2.2. *Let F be a field of characteristic $\neq 2$, and g, n be non-negative integers such that $2g + n \geqslant 5$. Then every F-form of $\overline{M}_{g,n}$ is isomorphic to ${}^{\alpha}\overline{M}_{g,n}$ for some $\alpha \in H^1(F, \mathrm{S}_n)$.* $\square$

Remark 2.3. If S_n is the full automorphism group of the moduli space $M_{0,n}$ of smooth marked curves, then $\overline{M}_{0,n}$ can be replaced by $M_{0,n}$ in the statement of Theorems 1.2. The proof remains unchanged. In particular, by [Lin11, Section 4.10, Corollary 7], this is the case if F is a subfield of $\mathbb{C}$.

Remark 2.4. Recall that $\overline{M}_{g,n}$ is, by definition, the coarse moduli space of the functor which assigns to a scheme X, defined over F, the set of isomorphism classes of pairs (C, s), where $C \to X$ is a stable curve of genus g over X and $s = (s_1, \dots, s_n)$ is an n-tuple of disjoint sections $s_i \colon X \to C$. Equivalently, we may view s as a single closed embedding $s \colon X^{(n)} \hookrightarrow C$ (over X), where $X^{(n)} = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(F^n)$ is the disjoint union of n copies of X. To place our results into the context of moduli theory, we remark that if $2g + n \geq 5$, then every form of $\overline{M}_{g,n}$ admits a similar functorial interpretation. Suppose $\alpha \colon Y \to \mathrm{Spec}(F)$ is an S_n-torsor represented by an n-dimensional étale algebra E/F. Then ${}^{\alpha}\overline{M}_{g,n}$ is the coarse moduli space for the functor

$$X \mapsto \{\text{isomorphism classes of pairs } (C, s)\},$$

where $C \to X$ is a stable curve of genus g, and s is an embedding $X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E) \to C$ (over X). We will not use this functorial description of ${}^{\alpha}\overline{M}_{g,n}$ in the sequel.

3. Preliminaries on the Noether problem

Let G be a linear algebraic group, and $G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G, both defined over a field F. We will assume that this representation is generically free, i.e., there is a dense open subset $U \subset V$ such that the scheme-theoretic stabilizer of every point of U is trivial.

The following questions originated in the work of E. Noether. Here (R) stands for rationality, (SR) for stable rationality and (RR) for retract rationality.

Noether's problem (R): Is $F(V)^G$ rational over F?

Noether's problem (SR): Is $F(V)^G$ stably rational over F? That is, is there a field $E/F(V)^G$ such that E is rational over both $F(V)^G$ and F?

Noether's Problem (RR): Is $F(V)^G$ retract rational over F?

Recall that an irreducible variety Y defined over F is called *retract rational* if the identity map $Y \to Y$ factors through the affine space $\mathbb{A}^n_F$ for some $n \geqslant 1$:

$$\begin{tikzcd} Y \arrow[rr, "\mathrm{id}"] \arrow[dr, dashed, "i"'] & & Y \\ & \mathbb{A}^n_F . \arrow[ur, dashed, "j"'] & \end{tikzcd} \tag{3.1}$$

Here i and j are composable rational maps, i.e., the image of i and the domain of j intersect non-trivially. A finitely generated field extension L/F is called *retract rational* if some (and thus any) model Y of L/F is retract rational. Here by a model of L/F we mean an irreducible variety Y defined over F such that $F(Y) = L$.

Noether'r original paper [Noe13] only considered problem (R) (and only in the case, were G is a finite group and V is the regular representation of G). Subsequent attempts to solve problem (R) naturally led to problems (SR) and (RR). Note, in particular, that the answers to problems (SR) and (RR) depend only on the group G and not on the choice of generically free representation V. For this reason we will refer to these problems as *Noether's problems (SR) and (RR) for* G in the sequel. The answer to problem (R) may a priori depend on the choice of V.

Remark 3.2. (see [CTS08, Section 4.2]) Suppose G is a special group defined over F, i.e., $H^1(K, G) = \{1\}$ for every field extension K/F. Recall that a special group is always linear and connected; see [Se58, Theorem 4.4.1.1].

Let $\pi\colon V \dashrightarrow V/G$ be the rational quotient map. That is, V/G is any variety defined over F whose function field in $F(V)^G$, and π is induced by the inclusion of fields $F(V)^G \hookrightarrow F(V)$. If G is special, π has a rational section and thus V is birationally isomorphic to $V/G \times G$ over F. Consequently, Noether's problem (SR) has a positive solution for G if G is itself stably rational over F, and similarly for Noether's problem (RR).

Definition 3.3. We will say that a G-torsor α over a field K is r-trivial if it can be connected to the trivial torsor by a rational curve. In other words, α is r-trivial if there exists an open subset $C \subset \mathbb{A}^1$ defined over K, a G-torsor $Y \to C$, and K-points $p_1, p_2\colon \operatorname{Spec}(K) \to C$ such that $p_1^*(Y) \simeq \alpha$ and $p_2^*(Y)$ is split.

Note that our notion of r-triviality is a minor variant of the more commonly used notion of R-triviality, introduced by Manin [Man72]. A G-torsor α over K is called R-trivial if it can be connected to the trivial torsor by a chain of rational curves defined over K.

Lemma 3.4. *Suppose Noether's problem (RR) has a positive solution for an affine algebraic group G/K. Then every G-torsor $\alpha\colon X \to \operatorname{Spec}(K)$ is r-trivial, for every infinite field K containing F.*

Proof. There is a dense G-invariant open subset $V_0 \subset V$ which is the total space of a G-torsor $\pi\colon V_0 \to Y$; see [Se03, Section 5]. Here $\pi^*F(Y) = F(V)^G$. Recall that we are assuming Y is retract rational. After replacing Y by a dense open subset, we may further assume that Y is a locally closed subvariety of $\mathbb{A}^n$, $i\colon Y \dashrightarrow \mathbb{A}^n$ in (3.1) is the inclusion map, and $j\colon \mathbb{A}^n \dashrightarrow Y$ is regular on some dense open subset U of $\mathbb{A}^n$ containing Y.

It is well known that π is a versal torsor; once again, see [Se03, Section 5] or [DR15]. In particular, there is a K-point $p_1\colon \operatorname{Spec}(K) \to Y$ such that π restricts to α over p_1, i.e., $p_1^*(\pi) = \alpha$. Similarly, there is a point $p_2\colon \operatorname{Spec}(K) \to Y$ such that π splits over p_2. It now suffices to connect p_1 and p_2 by an affine rational curve $C \subset Y$, defined over K, which is smooth at p_1 and p_2. After removing a closed subset from C away from p_1 and p_2, we may assume that C is isomorphic to an open subset of $\mathbb{A}^1_K$. Then we obtain a torsor $T \to C$ with the desired properties by pulling back π to C.

To construct C, we first connect p_1 and p_2 by a rational curve C_0 in $\mathbb{A}^n$, then set $C := j(C_0)$. Note that since $j\colon U \to Y$ is the identity map on Y, the differential dj_p is surjective for every $p \in Y$. Hence, we can choose C_0 so that C is smooth at p_1 and p_2. $\square$

4. The Noether problem for a class of twisted groups

Let $G_0 := G(F^n/F) = (\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m$, where $\mathbb{G}_m$ is centrally embedded into $\mathrm{GL}_2 \times \mathbb{G}_m^n$ by $t \mapsto (t^{-1}\,\mathrm{Id}, t, \dots, t)$. The group G_0 and its twisted forms,

$$G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m, \tag{4.1}$$

where E/F is an étale algebra of degree n, will play a prominent role in the sequel.

Recall that $\overline{M}_{0,n}$ is S_n-equivariantly birationally isomorphic to $(\mathbb{P}^1)^n/\mathrm{PGL}_2$. In turn, $(\mathbb{P}^1)^n/\mathrm{PGL}_2$ is S_n-equivariantly birationally isomorphic to

$$(\mathbb{A}^2)^n/(\mathrm{GL}_2 \times (\mathbb{G}_m)^n)\,.$$

Here we identify $\mathbb{G}_m^n$ with the diagonal maximal torus in GL_n, and $(\mathbb{A}^2)^n$ with the affine space $\mathrm{Mat}_{2,n}$ of $2 \times n$ matrices. The group GL_2 acts on $\mathrm{Mat}_{2,n}$ via multiplication on the left, and the torus $\mathbb{G}_m^n$ acts via multiplication on the right. These two commuting linear actions give rise to a linear representation

$$\mathrm{GL}_2 \times \mathbb{G}_m^n \to \mathrm{GL}(\mathrm{Mat}_{2,n})\,.$$

One readily checks that the kernel of this representation is

$$H = \{(t^{-1}\,\mathrm{Id}, t, \dots, t) \in \mathrm{GL}_2 \times \mathbb{G}_m^n \mid t \in \mathbb{G}_m\} \simeq \mathbb{G}_m$$

and that the induced representation

$$\phi\colon G_0 = (\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m \to \mathrm{GL}(\mathrm{Mat}_{2,n})$$

is generically free (recall that we are assuming that $n \geqslant 5$ throughout). Now identify S_n with the subgroup of permutation matrices in GL_n, and let this group act on $\mathrm{Mat}_{2,n}$ linearly, via multiplication on the right. In summary,

$$\overline{M}_{0,n} \simeq (\mathbb{P}^1)^n/\mathrm{PGL}_2 \simeq \mathrm{Mat}_{2,n}/G_0, \tag{4.2}$$

where $\simeq$ denotes an S_n-equivariant birational isomorphism.

Let τ be an S_n-torsor over $\mathrm{Spec}(F)$. Since S_n normalizes $\mathbb{G}_m^n$ in GL_n, we can twist the group G_0 and the representation ϕ by τ and obtain a new group

$${}^\tau G_0 := {}^\tau(\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/{}^\tau H := G(E/F) \tag{4.3}$$

and a new representation ${}^\tau\phi\colon {}^\tau G_0 \to \mathrm{GL}({}^\alpha\,\mathrm{Mat}_{2,n})$ defined over F. Note that S_n acts trivially on H, and thus ${}^\tau H \simeq H \simeq \mathbb{G}_m$ over F. Moreover, by Hilbert's Theorem 90, ${}^\tau\,\mathrm{Mat}_{2,n}$ is isomorphic to $\mathrm{Mat}_{2,n}$ as an F-vector space. Explicitly,

$${}^\tau G_0 \simeq G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m\,,$$

${}^\tau\,\mathrm{Mat}_{2,n}$ is the affine space $\mathbb{A}(F^2 \otimes_F E)$, where GL_2 acts F-linearly on $F^2 \otimes_F E$ via multiplication on F^2 and $R_{E/F}(\mathbb{G}_m)$ acts via multiplication on E. We have thus proved the following:

Proposition 4.4. *Let F be a field, τ be an S_n-torsor over* $\mathrm{Spec}(F)$*, and E/F be the étale algebra associated to τ.*

(a) (cf. [DR15, Theorem 6.1]) ${}^{\tau}\overline{M}_{0,n}$ *is unirational.*

(b) ${}^{\tau}\overline{M}_{0,n}$ is rational over F if and only if Noether's problem (R) for the representation ${}^{\tau}\phi$ of the group $G(E/F)$ has a positive solution.

(c) ${}^{\tau}\overline{M}_{0,n}$ is stably rational over F if and only if Noether's problem (SR) for the group $G(E/F)$ has a positive solution.

(d) ${}^{\tau}\overline{M}_{0,n}$ is retract rational over F if and only if Noether's problem (RR) for the group $G(E/F)$ has a positive solution. □

5. The Galois cohomology of $G(E/F)$

Let E/F be a finite-dimensional étale algebra and $G := G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$ be the algebraic group we considered in the previous section; see (4.1).

Lemma 5.1. *Let $\mathcal{F}\colon$ **Fields**$_F \to$ **Sets** be the functor from the category of field extensions of F to the category of sets, defined as follows:*

$$\mathcal{F}(K) := \{\textit{isomorphism classes of quaternion } K\textit{-algebras } A \\ \textit{such that } A \textit{ is split by } E \otimes_F K\}.$$

Then the functors $\mathcal{F}$ and $H^1(*, G)$ are isomorphic.

Proof. Consider the the short exact sequence

$$1 \to R_{E/F}(\mathbb{G}_m) \to G \to \mathrm{PGL}_2 \to 1 \tag{5.2}$$

of algebraic groups and the associated long exact sequence

$$H^1(K, R_{E/F}(\mathbb{G}_m)) \to H^1(K, G) \xrightarrow{\alpha} H^1(K, \mathrm{PGL}_2) \xrightarrow{\delta} H^2(K, R_{E/F}(\mathbb{G}_m)) \tag{5.3}$$

of Galois cohomology sets. By Shapiro's Lemma,

$$H^1(K, R_{E/F}(\mathbb{G}_m)) \simeq H^1(K \otimes_F E, \mathbb{G}_m) = \{1\}\,,$$

and $H^2(K, R_{E/F}(\mathbb{G}_m)) \simeq H^2(K \otimes_F E, \mathbb{G}_m)$ is in a natural bijective correspondence with the Brauer group $\mathrm{Br}(K \otimes_F E)$. Thus the long exact sequence (5.3) simplifies to

$$\{1\} \to H^1(K, G) \xrightarrow{\alpha} H^1(K, \mathrm{PGL}_2) \xrightarrow{\delta} \mathrm{Br}(K \otimes_F E)\,. \tag{5.4}$$

Here $H^1(K, \mathrm{PGL}_2)$ is the set of isomorphism classes of quaternion algebras A/K. The connecting map δ takes an algebra A/K to $A \otimes_K (K \otimes_F E)$. By [Se97, Proposition 42], α is injective.[1] Hence, we can identify $H^1(K, G)$ with the kernel of δ, and the lemma follows. □

[1]Note that a priori the exact sequence (5.4) only tells us that α has trivial kernel. Injectivity is not automatic, since $H^1(K, R_{E/F}(\mathbb{G}_m))$ and $H^1(K, G)$ are pointed sets with no group structure.

Remark 5.5. When n is odd, Lemma 5.1 tells us that $H^1(K, G) = \{1\}$ for every field K/F. In other words, $G(E/F)$ is a special group. Using the short exact sequence (5.2) one readily checks that $G(E/F)$ is rational over F. By Remark 3.2, we conclude that the Noether problem (SR) for this group has a positive solution. In other words, every F-form of $\overline{M}_{0,n}$ is stably rational over F. This is a bit weaker than Theorem 1.2(a), which will be proved in the next section.

6. Proof of Theorem 1.2(a)

Suppose $n = 2s + 1 \geqslant 5$ is odd. Our goal is to show that ${}^{\tau}\overline{M}_{0,n}$ is rational over F for every infinite field F and every $\tau \in H^1(F, \mathrm{S}_n)$. Let E/F be the étale algebra representing τ. In view of Proposition 4.4(b), it suffices to show that Noether's problem (R) for the representation ${}^{\tau}\phi$ of the group $G(E/F)$ has a positive solution.

Recall that ${}^{\tau}\phi$ is the natural representation of $G(E/F)$ on $F^2 \otimes_F E$ of $G(E/F)$. The quotient $\mathbb{A}(F^2 \otimes_F E)/\mathrm{GL}_2$ is the Grassmannian $\mathrm{Gr}(2, E)$ (up to birational equivalence). Thus the quotient $\mathbb{A}(F^2 \otimes_F E)/G(E/F)$ is birational to the quotient $\mathrm{Gr}(2, E)/R_{E/F}(\mathbb{G}_m)$.

Note that the diagonal subgroup $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^n$ acts trivially on $\mathrm{Gr}(2, n)$. Hence,

$$\mathbb{G}_m = {}^{\alpha}(\mathbb{G}_m) \hookrightarrow {}^{\alpha}(\mathbb{G}_m^n) = R_{E/F}(\mathbb{G}_m)$$

acts trivially on $\mathrm{Gr}(2, E)$, and $R^0_{E/F}(\mathbb{G}_m) := R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$ acts faithfully on $\mathrm{Gr}(2, E)$. Our proof of the rationality of the quotient variety $\mathrm{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$ below is inspired by the arguments in [Flo13].

Fix an F-vector subspace W of E of dimension s, and define the rational map

$$\begin{aligned} f_W\colon\ \mathrm{Gr}(2, E) &\dashrightarrow \check{\mathbb{P}}(E) \\ V &\to V \cdot W, \end{aligned}$$

where $V \cdot W$ is the F-linear span of elements of the form $v \cdot w$ in E, as v ranges over V and w ranges over W. Here $v \cdot w$ stands for the product of v and w in E, and $\check{\mathbb{P}}(E)$ denotes the dual projective space to $\mathbb{P}(E)$. In other words, points of $\check{\mathbb{P}}(E)$ are $2s$-dimensional F-linear subspaces of E.

Lemma 6.1. *(a) The dual projective space $\check{\mathbb{P}}(E)_0$ has a point H whose orbit with respect to the natural action of $R^0_{E/F}(\mathbb{G}_m)$ is dense and whose stabilizer is trivial.*

(b) Suppose $W \in \mathrm{Gr}(s, E)$ is such that f_W is well defined (i.e., $\dim(V \cdot W) = 2s$ for general $V \in \mathrm{Gr}(2, E)$). Then f_W is equivariant with respect to the natural action of $R^0_{E/F}(\mathbb{G}_m)$ on $\mathrm{Gr}(2, E)$ and $\check{\mathbb{P}}(E)$.

(c) There exists $W \in \mathrm{Gr}(s, E)$ defined over F such that f_W is well defined and dominant.

Proof. The assertions of parts (a) and (b) can be checked after passing to the separable closure of F^{sep} of F. In other words, we may assume that $F = F^{sep}$. In this case E is the split algebra F^n, $R^0_{E/F}(\mathbb{G}_m) = \mathbb{G}_m^n/\mathbb{G}_m$, and $\check{\mathbb{P}}(E) = \check{\mathbb{P}}^{n-1}$.

(a) $(t_1, \ldots, t_n) \in \mathbb{G}_m^n/\mathbb{G}_m$ takes the hyperplane $H \in \check{\mathbb{P}}(E)$ given by $c_1x_1 + \cdots + c_nx_n = 0$ to the hyperplane given by $(t_1^{-1}c_1)x_1 + \cdots + (t_n^{-1}c_n)x_n = 0$. Thus any H with $c_1, \ldots, c_n \neq 0$ has a dense orbit in $\check{\mathbb{P}}(E)$ with trivial stabilizer. In fact, all such H lie in the same dense orbit; for future reference, we will denote this dense orbit by $\check{\mathbb{P}}(E)_0$.

(b) Given $t = (t_1, \dots, t_n) \in \mathbb{G}_m^n$, we see that

$$(tv) \cdot w = (t_1 a_1 b_1, \dots, t_n a_n b_n) = t(v \cdot w)\,.$$

for any $v = (a_1, \dots, a_n) \in V$ and $w = (b_1, \dots, b_n) \in W$. Hence, $(tV) \cdot W = t(V \cdot W)$, as desired.

(c) Recall that the eigenvalues of $a \in E$ are the eigenvalues of the multiplication map $E \to E$ given by $x \mapsto ax$. They are elements of F^{sep}. Under an isomorphism between $E \otimes_F F^{sep}$ and $(F^{sep})^n$ (over F^{sep}), a will be identified with an element of $(F^{sep})^n$ of the form $(\lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of a.

Choose $a \in E$ with distinct eigenvalues in F^{sep}. Elements of E with distinct eigenvalues form a Zariski open subvariety U of $\mathbb{A}(E)$ defined over F. Passing to F^{sep}, we see that $U \neq \emptyset$. Since F is assumed to be infinite, F-points are dense in U. We choose a to be one of these F-points, and set $W = \mathrm{span}_F(1, a, \dots, a^{s-1})$. We claim that for this choice of W, the rational map f_W is well defined and dominant.

First let us show that f_W is well defined. From the definition of $V \cdot W$ it is clear that $\dim(V \cdot W) \leqslant 2s$ for any $V \in \mathrm{Gr}(2, E)$ and that equality holds for V in a Zariski open subset of $\mathrm{Gr}(2, E)$. Thus in order to show that f_W is a well-defined rational map, it suffices to exhibit one element $V \in \mathrm{Gr}(2, E)$ such that $\dim(V \cdot W) = 2s$. We claim that $V = \mathrm{span}_F(1, a^s)$ has this property, i.e.,

$$V \cdot W = \mathrm{span}_F(1, a, \dots, a^{s-1}, a^s, \dots, a^{2s-1})$$

is a $2s$-dimensional subspace of E. It suffices to show that $1, a, \dots, a^{2s}$ are linearly independent over F. Passing to F^{sep}, we can write $a = (\lambda_1, \dots, \lambda_{2s+1})$, where $\lambda_1, \dots, \lambda_{2s+1}$ are distinct elements of F^{sep}. (Recall that $n = 2s+1$ throughout.) Since the $(2s+1) \times (2s+1)$ Vandermonde matrix

$$\begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_{2s+1} \\ \dots & \dots & \dots \\ \lambda_1^{2s} & \dots & \lambda_{2s+1}^{2s} \end{pmatrix}$$

is non-singular, we conclude that $1, a, \dots, a^{2s}$ are linearly independent over F^{sep} and hence, over F, as desired. This shows that f_W is well defined.

It remains to show that f_W is dominant. By part (b), the image of f_W is an $R^0_{E/G}(\mathbb{G}_m)$-invariant subvariety of $\check{\mathbb{P}}(E)$. In view of part (a), it suffices to show that this subvariety intersects the dense open orbit $\check{\mathbb{P}}(E)_0$. In fact, it suffices to show that $V \cdot W \in \check{\mathbb{P}}(E)_0$ for $V = \mathrm{span}_F(1, a^s)$, as above. To do this, we may pass to F^{sep} and thus identify $E \otimes_F F^{sep}$ with $(F^{sep})^n$. Then $a = (\lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are distinct non-zero elements of F^{sep}. Recall from part (a) that the complement of $\check{\mathbb{P}}(E)$ consists of hyperplanes of the form $c_1 x_1 + \dots + c_{2s} x_{2s} = 0$, where $c_i = 0$ for some i but $(c_1, \dots, c_{2s}) \neq (0, \dots, 0)$. It remains to show that the hyperplane $V \cdot W = \mathrm{span}_F(1, a, \dots, a^{s-1}, a^s, \dots, a^{2s-1})$ is not of this form. Indeed, assume the contrary. By symmetry we may assume that the equation of the hyperplane $\mathrm{span}(1, \dots, a^{2s-1})$ in E is $c_1 x_1 + \dots + c_{2s} x_{2s} = 0$, with $c_{2s+1} = 0$. Since $a^i \in V \cdot W$, this means that

$$c_1 \lambda_1^i + \dots + c_{2s} \lambda_{2s}^i = 0 \text{ for } i = 0, 1, \dots, 2s-1.$$

Since $\lambda_1, \dots, \lambda_{2s}$ are distinct, the $2s \times 2s$ Vandermonde matrix

$$\begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_{2s} \\ \dots & \dots & \dots \\ \lambda_1^{2s-1} & \dots & \lambda_{2s}^{2s-1} \end{pmatrix}$$

is non-singular. This implies that $c_1 = \dots = c_{2s} = 0$, a contradiction. We conclude that $V \cdot W \in \check{\mathbb{P}}(E)_0$, as desired. □

We are now ready to finish the proof of Theorem 1.2(a).

Let $W \in \mathrm{Gr}(s, E)$ be the s-dimensional F-vector subspace of E given by Lemma 6.1. Choose a dense open $R^0_{E/F}(\mathbb{G}_m)$-invariant subvariety $U \subset \mathrm{Gr}(2, E)$ defined over F such that $f_W \colon \mathrm{Gr}(2, E) \dashrightarrow \check{\mathbb{P}}(E)_0$ restricts to a regular map on U, and the rational quotient map $\mathrm{Gr}(2, E) \dashrightarrow \mathrm{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$ restricts to a $R^0_{E/F}(\mathbb{G}_m)$-torsor $\pi \colon U \to \mathrm{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$ (over a suitably chosen birational model of $\mathrm{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$). In summary, we obtain the following diagram of $R^0_{E/F}(\mathbb{G}_m)$-equivariant dominant rational maps:

$$\begin{tikzcd}
\mathrm{Gr}(2, E) \arrow[r, dashed, "f_W"] & \check{\mathbb{P}}(E) & \\
U \arrow[u, hook, "\text{open}"] \arrow[r, "f_W"] \arrow[d, "\pi"', "R^0_{E/F}(\mathbb{G}_m)\text{-torsor}"] & \check{\mathbb{P}}(E)_0 \arrow[u, hook', "\text{open}"'] \arrow[r, equal, "\sim"] & R^0_{E/F}(\mathbb{G}_m) \\
\mathrm{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m). & &
\end{tikzcd}$$

Now choose an F-point $H \in \check{\mathbb{P}}(E)_0$; this can be done because we are assuming that F is an infinite field. From the diagram, we see that $f_W^{-1}(H) \subset U$ is a section of π. In particular, $\mathrm{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$ is birationally isomorphic to $f_W^{-1}(H)$ over F. It thus remains to show that $f_W^{-1}(H)$ is rational over F.

Let Z be the F-vector subspace of E given by $Z = \{a \in E \mid a \cdot W \subset H\}$. Clearly $V \in U$ belongs to $\phi^{-1}(H)$ if and only if $V \subset Z$ or equivalently, $V \in \mathrm{Gr}(2, Z)$. Thus $f_W^{-1}(H) = \mathrm{Gr}(2, Z) \cap U$ is a dense open subset of $\mathrm{Gr}(2, Z)$. Clearly $f_W^{-1}(H)$ is non-empty. Since $\mathrm{Gr}(2, Z)$ is rational over F, we conclude that $f_W^{-1}(H)$ is also rational over F, as desired. □

7. Proof of Theorem 1.2(b)

We will deduce Theorem 1.2(b) from the following proposition.

Proposition 7.1. *Suppose F is a field of characteristic $\neq 2$ and $A = (a_1, a_2)$ is a quaternion division algebra over F, for some $a_1, a_2 \in F^*$. Set $a_3 = a_1 a_2$ and $E_i = F(\sqrt{a_i})$, for $i = 1, 2, 3$. Consider the étale F-algebra*

$$E = E_1^{n_1} \times E_2^{n_2} \times E_3^{n_3} \,,$$

for some $n_1, n_2, n_3 \geqslant 1$. Then Noether's problem (RR) has a negative solution for the group $G(E/F) = (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$.

Assuming that Proposition 7.1 is established, we can complete the proof of Theorem 1.2 as follows. By a theorem of A. S. Merkurjev [Mer81], $\mathrm{Br}_2(F)$ is generated, as an abelian group by classes of quaternion algebras. Since we are assuming that $\mathrm{Br}_2(F) \neq 0$, one of these classes, say, (a_1, a_2) is non-split. That is, (a_1, a_2) is a division algebra. Since we are assuming that $n \geqslant 6$ is even, we can choose $n_1, n_2, n_3 \geqslant 1$ so that $n_1 + n_2 + n_3 = n$. For example, we can take $n_1 = \dfrac{n}{2} - 2$, $n_2 = 1$ and $n_3 = 1$. By Proposition 7.1, Noether's problem (RR) has a negative solution for the group $G(E/F) = (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$. By Proposition 4.4(d), the F-form ${}^{\tau}\overline{M}_{0,n}$ of $\overline{M}_{0,n}$ is not retract rational over F, where $\tau \in H^1(K, \mathrm{S}_n)$ is the class of the étale algebra E/F. This completes the proof of Theorem 1.2(b). □

Proof of Proposition 7.1. Since E_1, E_2, E_3 are maximal subfields of A,

$$A \otimes_F E_i \simeq \mathrm{Mat}_2(E_i)$$

for $i = 1, 2, 3$. In other words, A is split by E/F. Thus by Lemma 5.1, A corresponds to a class in $H^1(F, G)$, where $G := G(E/F)$. Denote this class by α.

Our assumption that there exists a non-split quaternion algebra over F, implies that F is an infinite field; see Remark (3) in the Introduction. Thus Lemma 3.4 applies: it suffices to show that α is not r-trivial. Assume the contrary. Using Lemma 5.1 once again, we see that this means the following: there exists a quaternion algebra $A(t)$ over $F(t)$ such that

(a) $A(t)$ is split by $F(t) \otimes_F E$, and

(b) $A(t)$ is unramified at $t = 0$ and $t = 1$, $A(0)$ is split over F, and $A(1)$ is isomorphic to A.

Here $A(0)$ and $A(1)$ denote $A(t)$ specialized to the points $t = 0$ and $t = 1$. We now recall the Faddeev exact sequence

$$0 \to \mathrm{Br}(F) \to \mathrm{Br}(F(t)) \to \oplus_{\eta \in \mathbb{P}^1_F} H^1(F_\eta, \mathbb{Q}/\mathbb{Z})\,; \tag{7.2}$$

see e.g., [GS06, Corollary 6.4.6]. For $\eta \in \mathbb{P}^1_F$ denote the image of the Brauer class $[A(t)] \in \mathrm{Br}(F(t))$ in $H^1(F_\eta, \mathbb{Q}/\mathbb{Z})$ by α_η.

By property (a) above, $A(t)$ is split by $E_i(t) := F(t) \otimes_F E_i = F(t)(\sqrt{a_i})$ for $i = 1, 2, 3$. Note that $E_i(t)$ is a field extension of $F(t)$ of degree 2. Since $A(t)$ is a quaternion algebra over $F(t)$, $A(t)^{\otimes 2}$ is split over $F(t)$ and hence, $2\alpha_\eta = 0$ for every $\eta \in \mathbb{P}^1$. In particular, every α_η lies in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^1(F_\eta, \mathbb{Q}/\mathbb{Z})$.

We claim that α_η is the trivial class in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$ for every $\eta \in \mathbb{P}^1$. If we can prove this claim, then the Faddeev exact sequence (7.2) will tell us that $A(t)$ is constant, i.e., that $A(t)$ is isomorphic to $B \otimes_F F(t)$ over $F(t)$, for some quaternion algebra B defined over F. Consequently, $A(0)$ and $A(1)$ are both isomorphic to B over F and hence, are isomorphic to each other. Since $A(0)$ is split over F, and $A(1) \simeq A$ is a quaternion division algebra, this is a contradiction, and the proof of Proposition 7.1 will be complete.

It remains to prove the claim. Assume the contrary. Suppose $\alpha_\eta = (b)$, where (b) denotes the class of $b \in F_\eta^*$ in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$. Since we are assuming $\alpha_\eta \neq (0)$, b is not a square in F_η^*. On the other hand, since $A(t)$ splits over $F(t)(\sqrt{a_i})$, b becomes a square in $F_\eta(\sqrt{a_i})^*$ for $i = 1, 2, 3$. This is only possible if $F_\eta(\sqrt{a_i})$ is a field extension of F_η of degree 2 and $\sqrt{b} = f_i\sqrt{a_i}$ for some $f_i \in F_\eta^*$, where $i = 1, 2, 3$. Equivalently, $b = f_i^2 a_i$ or $(b) = (a_i)$ in $H^1(F_\eta, \mathbb{Z}/p\mathbb{Z})$. Since $a_1a_2a_3$ is a complete square in F^*, we conclude that

$$\alpha_\eta = (b) = (b) + (b) + (b) = (a_1) + (a_2) + (a_3) = (a_1a_2a_3) = 0$$

is the trivial class in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$, a contradiction. This completes the proof of the claim and thus of Proposition 7.1 and of Theorem 1.2(b). □

Acknowledgements

We are grateful to J.-L. Colliot-Thélène, A. Massarenti, M. Mella, M. Talpo and A. Vistoli for stimulating conversations. We also thank the anonymous referee for helpful comments.

References

[BM13] A. Bruno and M. Mella, *The automorphism group of* $\overline{M}_{0,n}$, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 949–968. MR3085097

[CF07] G. Casnati, C. Fontanari, *On the rationality of moduli spaces of pointed curves*, J. Lond. Math. Soc. (2) 75, no. 3: 582–596, 2007. MR2352722

[CTS08] J.-L. Colliot-Thélène, J.-J. Sansuc, The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group), in *Algebraic groups and homogeneous spaces*, 113–186, Tata Inst. Fund. Res. Stud. Math, Tata Inst. Fund. Res., Mumbai. MR2348904 (2008m:13010)

[DR15] A. Duncan, Z. Reichstein, *Versality of algebraic group actions and rational points on twisted varieties*, J. Algebraic Geom. **24** (2015), no. 3, 499–530. MR3344763

[EH87] D. Eisenbud, J. Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. 90 (1987), no. 2, 359–387. MR0910206 (88g:14027)

[E1897] F. Enriques, *Sulle irrazionalità da cui può farsi dipendere la resoluzione d'un' equazione algebrica* $f(x, y, z) = 0$ *con funzioni razionali di due parametri*, Math. Ann. 49 (1897), 1–23. JFM 28.0559.02

[FM17] B. Fantechi, A. Massarenti, *On the rigidity of moduli of curves in arbitrary characteristic*, Int. Math. Res. Not. **8**, (2017), 2431–2463. MR3658203

[Fa00] G. Farkas, The geometry of the moduli space of curves of genus 23, Math. Ann. **318** (2000), no. 1, 43–65. MR1785575 (2001f:14048)

[Fa11] G. Farkas, Birational aspects of the geometry of $\overline{\mathcal{M}}_g$, in *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, 57–110, Surv. Differ. Geom., 14, Int. Press, Somerville, MA. MR2655323 (2011g:14065).

[Flo08] M. Florence, *On the essential dimension of cyclic p-groups*, Invent. Math. **171** (2008), no. 1, 175–189. MR2358058

[Flo13] M. Florence, *Géométrie birationnelle équivariante des grassmanniennes*, J. Reine Angew. Math. **674** (2013), 81–98. MR3010547

[FloR17] M. Florence, Z. Reichstein, On the rationality problem for forms of moduli spaces of stable marked curves of positive genus, arXiv:1709.05696.

[GS06] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, 101, Cambridge Univ. Press, Cambridge, 2006. MR2266528 (2007k:16033)

[HM82] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88. MR0664324 (83i:14018)

[Lin11] В. Я. Лин, *Алгебраические функции, конфигурационные пространства, пространства Теихмюллера, и новые голоморфно-комбинаторные инварианты*, Функционал. анал. и его приложения, **45** (2011), но. 3, 55–78.
English translation: V. Ya. Lin, *Algebraic functions, configuration spaces, Teichmüller spaces, and new holomorphically combinatorial invariants*, in Funct. Anal. Appl. **45** (2011), no. 3, 204–224. MR2883239 (2012m:32005)

[Lo03] A. Logan, The Kodaira dimension of moduli spaces of curves with marked points. *Amer. J. Math.* 125, no. 1, 105–138, 2003. MR1953519

[Man63] Ю. И. Манин, *Рациональные поверхности над совершенными полями*, Inst. Hautes Études Sci. Publ. Math. No. 30 (1966), 55–113. MR0225780 (37 #1373)
English Translation: Yu. I. Manin, *Rational surfaces over perfect fields*, in American Mathematical Society Translations, Series 2, Vol. 84, Amer. Math. Soc., Providence, RI, 1969. MR0252145

[Man72] Ю. И. Манин, *Кубические формы: алгебра, геометрия, арифметика*, Издат. "Наука", Moscow, 1972. MR0360592
English Translation: Yu. I. Manin and M. Hazewinkel, *Cubic forms: algebra, geometry, arithmetic*, North-Holland, Amsterdam, 1974. MR0460349

[Mas14] A. Massarenti, *The automorphism group of* $\overline{M}_{g,n}$, J. Lond. Math. Soc. (2) **89** (2014), no. 1, 131–150. MR3174737

[Mer81] A. S. Merkurjev, *On the norm residue symbol of degree* 2, Dokl. Akad. Nauk SSSR **261** (1981), no. 3, 542–547. MR0638926

[Noe13] E. Noether, *Rationale Funktionenkörper*, Jahresber. Deutsch. Math.-Verein. **22** (1913) 316–319.

[SB92] N. I. Shepherd-Barron. The rationality of quintic Del Pezzo surfaces—a short proof. *Bull. London Math. Soc.*, 24(3):249–250, 1992. MR1157259

[Se58] J.-P. Serre, Espaces fibrés algébriques, in: *Anneaux de Chow et Applications*, Séminaire C. Chevalley, 1958, exposé 1. Reprinted in J.-P. Serre, *Exposés de séminaires* 1950–1999, deuxième édition, augmentée, Documents mathématiques **1**, Société mathématique de France 2008, 107–140. MR1867431

[Se97] J.-P. Serre, *Galois cohomology*, Springer-Verlag, Berlin, 1997. MR1999384

[Se03] J.-P. Serre, Cohomological invariants, Witt invariants, and trace forms. In *Cohomological invariants in Galois cohomology*, volume 28 of *Univ. Lecture Ser.*, pp. 1–100. Amer. Math. Soc., Providence, RI, 2003. Notes by Skip Garibaldi.

[Sko93] A. N. Skorobogatov. On a theorem of Enriques-Swinnerton-Dyer. *Ann. Fac. Sci. Toulouse Math. (6)*, 2(3):429–440, 1993. MR1260765

[SD72] H. P. F. Swinnerton-Dyer. Rational points on del Pezzo surfaces of degree 5. In *Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.)*, pages 287–290. Wolters-Noordhoff, Groningen, 1972. MR0376684

Institut de Mathématiques de Jussieu, Université Paris 6, place Jussieu, 75005 Paris, France
E-mail address: `mathieu.florence@imj-prg.fr`

Department of Mathematics, University of British Columbia, BC, Canada V6T 1Z2
E-mail address: `reichst@math.ubc.ca`