ON THE RATIONALITY PROBLEM FOR FORMS OF MODULI SPACES OF STABLE MARKED CURVES OF POSITIVE GENUS

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Abstract. Let $M_{g,n}$ (respectively, $\overline{M}_{g,n}$) be the moduli space of smooth (respectively stable) curves of genus g with n marked points. Over the field of complex numbers, it is a classical problem in algebraic geometry to determine whether or not $M_{g,n}$ (or equivalently, $\overline{M}_{g,n}$) is a rational variety. Theorems of J. Harris, D. Mumford, D. Eisenbud and G. Farkas assert that $M_{g,n}$ is not unirational for any $n \geqslant 0$ if $g \geqslant 22$. Moreover, P. Belorousski and A. Logan showed that $M_{g,n}$ is unirational for only finitely many pairs (g, n) with $g \geqslant 1$. Finding the precise range of pairs (g, n), where $M_{g,n}$ is rational, stably rational or unirational, is a problem of ongoing interest.

In this paper we address the rationality problem for twisted forms of $\overline{M}_{g,n}$ defined over an arbitrary field F of characteristic $\neq 2$. We show that all F-forms of $\overline{M}_{g,n}$ are stably rational for $g = 1$ and $3 \leqslant n \leqslant 4$, $g = 2$ and $2 \leqslant n \leqslant 3$, $g = 3$ and $1 \leqslant n \leqslant 14$, $g = 4$ and $1 \leqslant n \leqslant 9$, $g = 5$ and $1 \leqslant n \leqslant 12$.

1. Introduction

Let $M_{g,n}$ (respectively $\overline{M}_{g,n}$) be the moduli space of smooth (respectively stable) curves of genus g with n marked points. Recall that these moduli spaces are defined over the prime field ($\mathbb{Q}$ in characteristic zero and $\mathbb{F}_p$ in characteristic p). The purpose of this paper is to address the rationality problem for twisted forms of $\overline{M}_{g,n}$. Recall that a *form* of a scheme X defined over a field F is another scheme Y, also defined over F, such that X and Y become isomorphic over the separable closure F^{sep}. We will use the terms "form", "twisted form" and "F-form" interchangeably throughout this paper. Forms of $\overline{M}_{g,n}$ are of interest because they shed light on the arithmetic geometry of $\overline{M}_{g,n}$, and because they are coarse moduli spaces for natural moduli problems in their own right; see [FR17, Remark 2.4].

This paper is a sequel to [FR17], where we considered twisted forms of $\overline{M}_{0,n}$. The main results of [FR17] can be summarized as follows.

Theorem 1.1. *Let F be a field of characteristic $\neq 2$ and $n \geqslant 5$ be an integer. Then*

(a) all F-forms of $\overline{M}_{0,n}$ are unirational.

(b) If n is odd, all F-forms of $\overline{M}_{0,n}$ are rational.

2010 *Mathematics Subject Classification.* 14E08, 14H10, 14G27, 14H45.

Key words and phrases. Stable curves, marked curves, moduli spaces, rationality problem, group actions, twisted varieties, Weil restriction.

Florence was partially supported by the French National Agency (Project GeoLie ANR-15-CE40-0012).

Reichstein was partially supported by National Sciences and Engineering Research Council of Canada grant No. 253424-2017.

(c) If n is even, then there exist fields E/F and E-forms of $\overline{M}_{0,n}$ that are not stably rational (or even retract rational) over E.

In the present paper we will study the rationality problem for forms of $\overline{M}_{g,n}$ in the case, where $g \geqslant 1$. Here the the rationality problem for the usual (split) moduli space $\overline{M}_{g,n}$ (or equivalently, for $M_{g,n}$) over the field of complex numbers is already highly non-trivial. Theorems of J. Harris, D. Mumford, D. Eisenbud [HM82, EH87] and G. Farkas [Fa11] assert that if $g \geqslant 22$, then $M_{g,0}$ is not unirational (and hence, neither is $M_{g,n}$ for any $n \geqslant 0$). Moreover, work of P. Belorousski [Bel98] (for $g = 1$) and A. Logan [Lo03] (for $g \geqslant 2$) tells us that $M_{g,n}$ is unirational for only finitely many pairs (g, n) with $g \geqslant 1$. Finding the precise range of pairs (g, n), where $M_{g,n}$ is rational, stably rational or unirational, is a problem of ongoing interest. In particular, over $\mathbb{C}$, $M_{g,n}$ is known to be rational for $1 \leqslant n \leqslant r_g$ and not unirational for $n \geqslant n_g$, where

g	1	2	3	4	5
r_g	10	12	14	15	12
n_g	11	-	-	16	15

,

see [Lo03] and [CF07]. Surprisingly, we have not been able to find specific values for n_2 and n_3 in the literature, even though Logan showed that they exist; see [Lo03, Theorem 2.4]. The main result of the present paper is as follows.

Theorem 1.2. *Let F be a field of characteristic $\neq 2$. Then every F-form of $\overline{M}_{g,n}$ is stably rational over F if*

$g = 1$ *and* $3 \leqslant n \leqslant 4$ (Theorems 4.1 and 5.1),
$g = 2$ *and* $2 \leqslant n \leqslant 3$ (Theorems 6.1 and 7.1),
$g = 3$ *and* $1 \leqslant n \leqslant 14$ (Theorem 8.1),
$g = 4$ *and* $1 \leqslant n \leqslant 9$ (Theorem 9.1),
$g = 5$ *and* $1 \leqslant n \leqslant 12$ (Theorem 10.1).

Several remarks are in order.

(1) (Stable) rationality of every form of $\overline{M}_{g,n}$ is a priori much stronger than (stable) rationality of $\overline{M}_{g,n}$ itself. For example, $\overline{M}_{1,1} \simeq \mathbb{P}^1$ is obviously rational, but its forms are conic curves which are not unirational in general.

(2) Theorem 1.2 also holds for $(g, n) = (1, 2)$ (respectively, $(2, 1)$), provided $\mathrm{char}(F) = 0$ (respectively, $\mathrm{char}(F) \neq 2, 3$); see Remark 2.11.

(3) By [DR15, Theorem 6.1(b)], every F-form of $\overline{M}_{1,n}$ is unirational for $3 \leqslant n \leqslant 9$.

(4) In some cases, with additional work, our proofs can be modified to establish rationality of all forms of $\overline{M}_{g,n}$, rather than just stable rationality. To keep our arguments as uniform and transparent as possible, we will be satisfied with establishing stable rationality in the present paper.

Roughly speaking, the approach taken in [CF07] and [BCF09] to prove the rationality of $M_{g,n}$ over $\mathbb{C}$ for small g and n, is to find an intrinsic map of a general curve of genus g with n marked points to some projective space $\mathbb{P}(V)$, thus reducing the rationality problem for $\overline{M}_{g,n}$ to the rationality problem for a quotient variety of the form

$X/\mathrm{PGL}(V)$. The $\mathrm{PGL}(V)$-variety X is studied separately in each case; the rationality of $X/\mathrm{PGL}(V)$ is often established using the so-called "no-name lemma" [Do85]. Our proof of Theorem 1.2 proceeds along similar lines, with an important caveat: in order to establish (stable) rationality of all twisted forms of $\overline{M}_{g,n}$, our geometric constructions need to be S_n-equivariant, i.e., symmetric in the n marked points on the curve. This gives rise to an $\mathrm{S}_n \times \mathrm{PGL}(V)$-equivariant birational isomorphism between $\overline{M}_{g,n}$ and a rational quotient of the form $X/\mathrm{PGL}(V)$, where X is a suitable variety. A theorem of B. Fantechi and A. Massarenti [FM14] (see Theorem 2.9 and Corollary 2.10 below) then allows us to conclude that every F-form of $\overline{M}_{g,n}$ is birationally isomorphic to ${}^{T}X/\mathrm{PGL}(V)$, for a suitable S_n-torsor $T \to \mathrm{Spec}(F)$. Here ${}^{T}X$ denotes the twist of X by T; see Section 2b. To prove stable rationality for quotients of this form we appeal to Proposition 3.6, a convenient variant of the "no-name lemma" developed in Section 3. While our argument follows the same general pattern for all pairs (g,n) covered by Theorem 1.2, our proof relies on case-by-case analysis carried out in Sections 4-10. In each case our argument takes advantage of some numerical coincidence, such as the degree a certain line bundle on a general marked curve being prime to the dimension of its space of global section. Why such numerical coincidences occur in every case and why the situation of Theorem 1.1(c), where some forms of $\overline{M}_{0,n}$ are stably rational and others are not, does not reproduce itself for any of the pairs (g,n) covered by Theorem 1.2, remains a mystery to us.

2. Preliminaries

All algebraic groups in this paper will be assumed to be affine, and all algebraic varieties to be quasi-projective.

2a. **Group actions and rational quotients.** Consider the action of an algebraic group G on an integral algebraic variety X defined over a field F. A rational quotient for this action is, by definition, and F-variety Y such that $F(Y) = F(X)^G$. Clearly Y is unique up to birational isomorphism. The natural inclusion $F(Y) \hookrightarrow F(X)$ induces a rational map $\pi\colon X \dashrightarrow Y$ which is called *the rational quotient map*. By a theorem of Rosenlicht [Ro56, Theorem 2], there exists a dense open subvariety $Y_0 \subset Y$ and a G-invariant dense open subvariety $X_0 \subset X$ such that $\pi_{|X_0}\colon X_0 \to Y_0$ is regular, and the preimage of any $y \in Y_0$ is a single G-orbit in X_0; see also [BGR17, Section 7].

We say that the action of G on X is *generically free* if there exists a dense open subset $U \subset X$ such that the scheme-theoretic stabilizer $G_x = \{1\}$ for every $x \in U$. For a generically free action, Rosenlicht's theorem can be strengthened as follows: the open subvarieties $Y_0 \subset Y$ and $X_0 \subset X$ can be chosen so that $\pi_{|X_0}\colon X_0 \to Y_0$ is, in fact, a G-torsor; see [BF03, Theorem 4.7].

2b. **Twisting.** Let G be an algebraic group defined over a field F, X be an F-variety endowed with a G-action, and $P \to \mathrm{Spec}(F)$ be a G-torsor. The twisted variety ${}^{P}X$ is defined as ${}^{P}X := (X \times P)/G$. Here $X \times P$ is, in fact, a G-torsor over $(X \times P)/G$; in particular, $X \times P \to (X \times P)/G$ is a geometric quotient and a rational quotient (as in Section 2a above). A G-equivariant morphism of F-varieties $f\colon Y \to X$ gives rise to a G-equivariant morphism $f \times \mathrm{id}\colon X \times P \to Y \times P$ which descends to an F-morphism ${}^{P}f\colon {}^{P}X \to^{P} Y$. Similarly a G-equivariant rational map $f\colon X \dashrightarrow Y$ of F-varieties induces a rational map ${}^P f\colon {}^P X \dashrightarrow {}^P Y$. For the basic properties of the twisting operation we refer the reader to [Flo08, Section 2] or [DR15, Section 3]. In particular, we will repeatedly use the following facts in the sequel.

Lemma 2.1. *Let G be an algebraic group defined over a field F and $f\colon X \to Y$ be a G-equivariant morphism of F-varieties and $P \to \operatorname{Spec}(F)$ be a G-torsor.*

(a) If $X = Z \times Y$ and f is projection to the first factor, then ${}^P X \simeq_F {}^P Z \times {}^P Y$ and ${}^P f$ is also projection to the first factor.

(b) If f is an open (respectively closed) immersion, then so is ${}^P f$.

(c) If f is dominant (respectively birational, respectively an isomorphism), then so is ${}^P f$.

(d) If f is a vector bundle of rank r, then so is ${}^P f$. In particular, in this case ${}^P X$ is rational over ${}^P Y$. Here we are assuming that G acts on X by vector bundle automorphisms. That is, for any $g \in G$ and $y \in Y$, g restricts to a linear map between the fibers $f^{-1}(y)$ and $f^{-1}(g(y))$.

Proof. (a)-(c) See [DR15, Corollary 3.4].

(d) The first assertion is a consequence of Hilbert's Theorem 90. The second assertion follows from the first, since the vector bundle ${}^P \pi\colon {}^P X \to {}^P Y$ becomes trivial after passing to some dense Zariski open subset of ${}^P Y$. □

The F-forms of a variety X are in a natural bijective correspondence with $H^1(F, \operatorname{Aut}(X))$; see [Se97, II.1.3]. (Recall that all varieties in this paper are assumed to be quasi-projective.) Here $\operatorname{Aut}(X)$ is a functor which associates to the scheme S/F the abstract group $\operatorname{Aut}(X_S)$. In general this functor is not representable by an algebraic group defined over F. If it is, one usually says that $\operatorname{Aut}(X)$ is an algebraic group. In this case the bijective correspondence between $H^1(F, \operatorname{Aut}(X))$ (which may be viewed as a set of $\operatorname{Aut}(X)$-torsors $P \to \operatorname{Spec}(F)$) and the set of F-forms of X (up to F-isomorphism) can be described explicitly as follows. An $\operatorname{Aut}(X)$-torsor $P \to \operatorname{Spec}(F)$ corresponds to the twisted variety ${}^P X$, and a twisted form Y of X corresponds to the isomorphism scheme $P = \operatorname{Isom}_F(X, Y)$, which is naturally an $\operatorname{Aut}(X)$-torsor over $\operatorname{Spec}(F)$; see [Se97, Section III.1.3], [Sp98, Section 11.3].

2c. **Étale algebras.** An étale algebra A/F is a commutative F-algebra of the form $F_1 \times \cdots \times F_r$, where each A_i is a finite separable field extension of F. n-dimensional étale algebras over A are F-forms of the split étale algebra $A = F \times \cdots \times F$ (n times). The automorphism group of this split algebra is the symmetric group S_n, permuting the n factors of F. Thus n-dimensional étale algebras over F are in a natural bijective correspondence with the Galois cohomology set $H^1(F, \mathrm{S}_n)$; see, e.g., [Ser03, Examples 2.1 and 3.2].

2d. **Weil restriction.** Let A be an étale algebra over F and $X \to \operatorname{Spec}(A)$ be a variety defined over A. The Weil restriction (or Weil transfer) of X to F is, by definition an F-variety $R_{A/F}(X)$ satisfying

$$\operatorname{Mor}_F(Y, R_{A/F}(X)) \simeq \operatorname{Mor}_A(Y_A, X) \tag{2.2}$$

where $Y_A := Y \times_{\mathrm{Spec}(F)} \mathrm{Spec}(A)$, $\mathrm{Mor}_F(Y, Z)$ denotes the set of F-morphisms $Y \to Z$, and $\simeq$ denotes an isomorphism of functors (in Y). For generalities on this notion we refer the reader to [BLR90, Section 7.6]. For a brief summary, see [Ka00, Section 2]. In particular, it is shown in [BLR90, Theorem 4] that if X is quasi-projective over A, then $R_{A/F}(X)$ exists. Note that uniqueness of $R_{A/F}(X)$ follows from (2.2) by Yoneda's lemma.

The following properties of Weil restriction will be helpful in the sequel.

Lemma 2.3. *Let A/F be an étale algebra and X be a (quasi-projective) variety defined over A. Then*

(a) If X is an algebraic group over A, then $R_{A/F}(X)$ is naturally an algebraic group over F.

(b) Let V be a free A-module of finite rank, and $X = \mathbb{A}_A(V)$ be the associated affine space. Then $R_{A/F}(X) = \mathbb{A}_F(V)$, where we view V as an F-vector space.

(c) If X and Y are birationally isomorphic over A, then $R_{A/F}(X)$ and $R_{A/F}(Y)$ are birationally isomorphic over F.

(d) If X is a rational variety over A, then $R_{A/F}(X)$ is rational over F.

Proof. (a) The structure of an algebraic group on X is given by the identity element $1 : \mathrm{Spec}(F) \to X$, the multiplication map $\mu\colon X \times X \to X$ and the inverse $i\colon X \to X$, satisfying certain commutative diagrams, such as

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mathrm{id}\times\mu} & X \times X \\ {\scriptstyle \mu\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ X \times X & \xrightarrow{\mu} & X \end{array}$$

for associativity. Since the Weil transfer is functorial with respect to morphisms and direct products (see [BLR90, p. 192] or [Ka00, Proposition 1.1]), we see that 1, μ and i induce the identity $R_{A/F}(1)$, multiplication $R_{A/F}(\mu)$ and inverse $R_{A/F}(i)$ on $R_{A/F}(X)$. Applying Weil restriction to the appropriate commutative diagrams, we see that $R_{A/F}(X)$ is an algebraic group with respect to $R_{A/F}(1)$, $R_{A/F}(\mu)$ and $R_{A/F}(i)$.

(b) follows directly from (2.2). For details, see [Ka00, Lemma 1.2].

(c) Since X and Y are birationally isomorphic, there exists a variety U defined over A and open immersions $i\colon U \hookrightarrow X$ and $j\colon U \hookrightarrow Y$. In fact, after replacing U by an open subvariety, we may assume that U is quasi-projective (we may even assume that U is affine). Since Weil restriction commutes with open immersions, i and j induce open immersions of $R_{A/F}(U)$ into $R_{A/F}(X)$ and $R_{A/F}(Y)$, respectively, and part (c) follows.

(d) By our assumption, X is birationally isomorphic to $Y = \mathbb{A}^d$ over A, where d is the dimension of X. By part (c), $R_{A/F}(X)$ and $R_{A/F}(Y)$ are birationally isomorphic over F, and by part (b), $R_{A/F}(Y)$ is an affine space over F. $\square$

In the special case, where X is defined over F, the Weil transfer $R_{A/F}(X_A)$ can be explicitly described as follows. The symmetric group S_n acts on the n-fold direct product X^n by permuting the factors. If $P \to \mathrm{Spec}(F)$ is a S_n-torsor, and A/F is the étale algebra of degree n representing the class of P in $H^1(F, \mathrm{S}_n)$, then $R_{A/F}(X_A) = {}^P(X^n)$; see, e.g., [DR15, Proposition 3.2].

2e. **Special groups.** An algebraic group G defined over a field F is called *special* if $H^1(E, G) = \{1\}$ for every field extension E/F. In other words, every G-torsor $P \to \mathrm{Spec}(E)$ is split. Special groups were introduced by J.-P. Serre [Se58], who showed, among other things, that every special group is affine and connected. A direct product of special groups is again special. The general group GL_n is special by Hilbert's Theorem 90. If A/F is an étale algebra and G is a special group over F then the Weil restriction $R_{A/F}(G_A)$ is a special group over F by Shapiro's Lemma. In particular, $R_{A/F}(\mathbb{G}_m)$ is special.

Over an algebraically closed field of characteristic 0, special groups were classified by A. Grothendieck [Gro58]. For a partial classification of special reductive groups over an arbitrary field due to M. Huruguen, see [Hu16].

2f. **Twisted groups and their actions.** Let H and G be algebraic group. Assume G acts on H by group automorphisms; denote this action by

$$(g, h) \mapsto {}^g h\,. \tag{2.4}$$

Let $P \to \mathrm{Spec}(F)$ be a G-torsor.

Lemma 2.5. *${}^P H$ is naturally an algebraic group.*

Proof. We argue as in the proof of Lemma 2.3(a). The structure of an algebraic group on H is given by the G-equivariant morphisms,

$$1\colon \mathrm{Spec}(F) \to H, \;\; \mu\colon H \times H \to H \;\text{ and }\; i\colon H \to H,$$

the identity, multiplication and inverse. To check that

$${}^P 1\colon \mathrm{Spec}(F) \to {}^P H, \;\; {}^P\mu\colon {}^P H \times {}^P H \to {}^P H \;\text{ and }\; {}^P i\colon {}^P H \to {}^P H$$

satisfy the properties of the identity, the multiplication and the inverse for ${}^P H$, encode the group axioms for H into commutative diagrams and twist each diagram by P. $\square$

Proposition 2.6. *(a) Suppose G and H both act on an algebraic variety X defined over F, and these actions skew-commute, in the sense that*

$$g \cdot (h \cdot x) = {}^g h \cdot (g \cdot x) \tag{2.7}$$

for any $g \in G$, $h \in H$ and $x \in X$. Let $P \to \mathrm{Spec}(F)$ be a G-torsor. Then

(a) the H-action on X naturally induces an ${}^P H$-action on ${}^P X$.

(b) Furthermore, assume that $f\colon X \to Y$ is a G-equivariant H-torsor. (Here H acts trivially on Y.) Then ${}^P f$ is an ${}^P H$-torsor. In particular, if ${}^P H$ is special and rational, then ${}^P X$ is rational over ${}^P Y$.

Note that the term "skew-commute" defined in part (a), is somewhat ambiguous, because it depends on the action (2.4) of G on H. In most cases the action (2.4) is clear from the context, and making an explicit reference to it complicates the terminology. We hope this ambiguity will not cause any confusion for the reader.

Proof. (a) Condition (2.7) tell us that the H-action map $f\colon H \times X \to X$ is G-equivariant. Twisting this map by P and remembering that ${}^P(H \times X) = {}^P H \times {}^P X$ by Lemma 2.1(a), we obtain a morphism

$${}^P f\colon {}^P H \times {}^P X \to {}^P X\,.$$

It remains to show that ${}^P f$ is a group action. Since f is a group action, the diagram

$$\begin{tikzcd} H \times H \times X \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\mu\times\mathrm{id}"'] & H \times X \arrow[d, "f"] \\ H \times X \arrow[r, "f"] & X \end{tikzcd}$$

commutes. Here $\mu\colon H \times H \to H$ is the multiplication map. By (2.7), this entire diagram is G-equivariant. Twisting it by P we obtain the commutative diagram

$$\begin{tikzcd} {}^P H \times {}^P H \times {}^P X \arrow[r, "\mathrm{id}\times {}^P f"] \arrow[d, "{}^P\mu\times\mathrm{id}"'] & {}^P H \times {}^P X \arrow[d, "{}^P f"] \\ {}^P H \times {}^P X \arrow[r, "{}^P f"] & {}^P X \end{tikzcd}$$

which tells us that ${}^P f$ is a group action.

(b) To check that ${}^P f$ is an H-torsor, we may pass to the separable closure F^{sep} of F. Over F^{sep}, P becomes split. Hence, ${}^P H$ becomes isomorphic to H, and ${}^P f\colon {}^P X \to {}^P Y$ becomes the same as $f\colon X \to Y$, which we know is a H-torsor.

If ${}^P H$ is special, then ${}^P f$ splits over a dense open subset of ${}^P Y$. Thus ${}^P X$ is birationally isomorphic to ${}^P H \times_F {}^P Y$. If ${}^P H$ is also rational over F, this tells us that ${}^P X$ is rational over ${}^P Y$, as desired. $\square$

2g. **Automorphism of marked curves.** The following well-known result will be repeatedly used in the sequel. Proofs can be found, e.g., in [Ha77, Corollary 4.4.7] for $g = 1$ and [Ha77, Exercise V.1.11] for $g \geqslant 2$.

Proposition 2.8. *Suppose $2g+n \geqslant 5$. Then $\mathrm{Aut}(C, p_1, \dots, p_n) = \{1\}$ for a general point $(C, p_1, \dots, p_n)$ of $\overline{M}_{g,n}$ (or equivalently, of $M_{g,n}$).* $\square$

2h. **Automorphisms and forms of $\overline{M}_{g,n}$.** The following theorem is the starting point of our investigation.

Theorem 2.9. (A. Massarenti [Mas13], B. Fantechi and A. Massarenti [FM14]) *Let F be a field of characteristic $\neq 2$. If $g, n \geqslant 1$, $(g, n) \neq (2, 1)$ and $2g + n \geqslant 5$, then the natural embedding $\mathrm{S}_n \to \mathrm{Aut}_F(\overline{M}_{g,n})$ is an isomorphism.*

Using the bijective correspondence between F-forms of X and $\mathrm{Aut}(X)$-torsors $P \to \mathrm{Spec}(F)$ described at the end of Section 2b, we obtain the following.

Corollary 2.10. *For F, g, n as in Theorem 2.9 every F-form of $\overline{M}_{g,n}$ is F-isomorphic to ${}^P\overline{M}_{g,n}$ for some S_n-torsor $P \to \mathrm{Spec}(F)$.* $\square$

Remark 2.11. Theorem 1.2 also holds in the following cases.

(a) $g = 2$ and $n = 1$, and $\mathrm{char}(F) = 0$,

(b) $g = 1$ and $n = 2$ and $\mathrm{char}(F) \neq 2$ or 3.

In case (a), $\overline{M}_{2,1}$ has no non-trivial automorphisms by [FM14, Theorem 1] and hence, no non-split forms. On the other hand, the split form of $\overline{M}_{2,1}$ is known to be rational; see [CF07].

In case (b), the automorphism group of $\overline{M}_{1,2}$ is non-trivial; however, it is special see [FM14, Proposition 2.4]. As a consequence, $\overline{M}_{1,2}$ has no non-split forms (see [DR15, Remark 6.4]) and the split form of $\overline{M}_{1,2}$ is rational (see [CF07]).

Remark 2.12. We do not know if $\overline{M}_{g,n}$ can be replaced by $M_{g,n}$ in the statement of Theorem 2.9. If so, then $\overline{M}_{g,n}$ can also be replaced by $M_{g,n}$ in the statements of Theorems 1.2. The proof remains unchanged.

3. Anti-versal group actions

Recall that an action of an algebraic group G on an integral variety X defined over a field F is (weakly) *versal* if ${}^P X(E) \neq \emptyset$ for every field extension E/F, and for every G-torsor $P \to \mathrm{Spec}(E)$; see [DR15]. We will now consider group actions that are far from being versal in the following sense.

Definition 3.1. We will say that the action of G on a geometrically integral F-variety X is *anti-versal* if ${}^P X(E) = \emptyset$ for any field extension E/F, and any non-split G-torsor $P \to \mathrm{Spec}(E)$.

Example 3.2. The G-action on $X = G$ by translations is anti-versal. Indeed, by the definition of the twisting operation, ${}^P X \simeq P$ for any G-torsor $P \to \mathrm{Spec}(E)$, and $P(E) \neq \emptyset$ if and only if P is split.

Example 3.3. The natural action of $\mathrm{PGL}(V)$ on $X = \mathbb{P}(V)$ is anti-versal, for any finite-dimensional F-vector space V. Indeed, if E/F is a field extension and $P \to \mathrm{Spec}(E)$ is a PGL_n-torsor, then ${}^P X$ is the Brauer-Severi variety associated to P. This variety has an E-point if and only if P is split over E.

Lemma 3.4. *(a) If $X \longrightarrow Y$ is a G-equivariant morphism between geometrically integral G-varieties, and the G-action on Y is anti-versal, then so is the G-action on X.*

(b) If the actions of G on X and H on Y are anti-versal, then so is the product action of $G \times H$ on $X \times Y$.

(c) Suppose that A/F is an étale algebra, and that X is an F-variety, equipped with an anti-versal action of the group $G := \mathrm{PGL}_n$. If $[A : F]$ is prime to n, then the induced PGL_n-action on the Weil transfer $R_{A/F}(X)$ is anti-versal.

Proof. Let E/F be a field extension and $P \to \mathrm{Spec}(E)$ be a G-torsor.

(a) Twisting the morphism $X \to Y$ by P, we obtain a morphism ${}^P X \to {}^P Y$. If ${}^P X$ has an E-point, then so does ${}^P Y$. Since Y is anti-versal, P is split, as desired.

(b) Let E/F be a field extension. A $G \times H$-torsor over $\mathrm{Spec}(E)$ is of the form $P \times_E U$, where $P \to \mathrm{Spec}(E)$ is a G-torsor and $U \to \mathrm{Spec}(E)$ is an H-torsor. By Lemma 2.1(a)

$$ {}^{P \times U}(X \times Y) \simeq_F {}^{P \times U} X \times {}^{P \times U} Y \simeq_F {}^P X \times {}^U Y . $$

If this twisted variety has an E-point, then ${}^P X(E) \neq \emptyset$ and ${}^U Y(E) \neq \emptyset$. Since both X and Y are anti-versal, we conclude that both P and U are split, and hence, so is $P \times_E U$.

(c) First note that ${}^P R_{A/F}(X)$ is naturally isomorphic to $R_{A/F}({}^P X)$. Thus ${}^P R_{/F}(X)$ has an E-point if and only if X has an $E \otimes_F A$-point; see (2.2). Since the PGL_n-action on

X is anti-versal, this tells us that P is split by $E \otimes_F A$. On the other hand, since $[A:F]$ is prime to n, this implies that P is split over E, as desired. $\square$

Example 3.5. Lemma 3.4(a) implies that if a G-variety Y is anti-versal, then $Z \times Y$ (with diagonal G-action) is also anti-versal for any G-variety Z. In particular, taking $Y = G$, with translation G-action, as in Example 3.2, we see that $Z \times G$ is anti-versal for any G-variety Z.

Proposition 3.6. *Consider a generically free action of an algebraic group G on an integral variety X defined over a field F. Suppose there exists a G-equivariant rational map $X \dashrightarrow Y$ such that the G-action on Y is anti-versal. Then*

(a) the rational quotient map $q: X \dashrightarrow X/G$ has a rational section. In other words, there exists a G-equivariant birational isomorphism $X \simeq (X/G) \times G$.

(b) If the group G is rational over F, then X is rational over X/G.

Proof. (b) is an immediate consequence of (a). To prove (a), let X_0 be the domain of the rational map $X \dashrightarrow Y$. Note that X_0 is a G-invariant open subvariety of X; in particular, the G-action on X_0 is generically free. By Lemma 3.4(a), the G-action on X_0 is anti-versal. After replacing X by X_0, we may assume that the G-action on X is anti-versal.

Recall that the function field $E = F(X/G)$ of X/G is, by definition, the field of G-invariant functions on X. That is, $E = F(X)^G$. Since the action of G on X is generically free, the rational quotient map

$$q : X \dashrightarrow X/G$$

gives rise to a G-torsor over the generic point of X/G; see Section 2a. Denote this torsor by $P \to \mathrm{Spec}(E)$. The twist ${}^P X$ is the generic fiber of the natural map

$$(X \times X)/G \longrightarrow X/G,$$

induced by the first projection. This map has a section, induced by the diagonal embedding $X \longrightarrow X \times X$. Hence, the twist ${}^P X$ has an E-point. By the definition of anti-versality, this means that P is trivial. Equivalently, q has a rational section, and $X \simeq (X/G) \times G$, as claimed. $\square$

The following lemma supplies us with a family of anti-versal actions for $G = \mathrm{PGL}_n$, generalizing Example 3.3. These examples will be used in place of Y in our subsequent applications of Proposition 3.6.

Lemma 3.7. *Let V be an n-dimensional F-vector space and $\rho\colon \mathrm{GL}(V) \to \mathrm{GL}(W)$ be a finite-dimensional representation of $\mathrm{GL}(V)$ over a field F, such that $\rho(t \cdot \mathrm{Id}_V) = t^d \cdot \mathrm{Id}_W$. Suppose $1 \leqslant m < n$. If d and m are both coprime to n, then the action of $\mathrm{PGL}(V)$ on the Grassmannian $\mathrm{Gr}(m, W)$ induced by ρ is anti-versal. In particular, if d and n are coprime, then the $\mathrm{PGL}(V)$-action on $\mathrm{PGL}(W)$ is anti-versal.*

Proof. The second assertion is obtained from the first by setting $m = 1$. To prove the first assertion, consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "\times d"] & \mathrm{GL}(V) \arrow[r] \arrow[d, "\rho"] & \mathrm{PGL}(V) \arrow[r] \arrow[d, "\bar{\rho}"] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathrm{GL}(W) \arrow[r] & \mathrm{PGL}(W) \arrow[r] & 1
\end{tikzcd}$$

and the associated connecting maps

$$\begin{array}{ccccc} [P] & \in & H^1(E, \mathrm{PGL}(V)) & \xrightarrow{\partial} & H^2(E, \mathbb{G}_m) \\ \downarrow{\scriptstyle \bar{\rho}^*} & & \downarrow{\scriptstyle \bar{\rho}^*} & & \downarrow{\scriptstyle \times d} \\ [U] & \in & H^1(E, \mathrm{PGL}(W)) & \xrightarrow{\partial} & H^2(E, \mathbb{G}_m). \end{array}$$

Let E/F be a field extension, $P \to \mathrm{Spec}(E)$ be a $\mathrm{PGL}(V)$-torsor, $[P]$ be the class of P in $H^1(E, \mathrm{PGL}(V))$ and $U \to \mathrm{Spec}(E)$ be the $\mathrm{PGL}(W)$-torsor representing the image of $[P]$ in $H^1(E, \mathrm{PGL}(W))$. Since P is a $\mathrm{PGL}(V)$-torsor and $\dim(V) = n$, we have

$$n \cdot \partial([P]) = 0 \text{ in } \mathrm{Br}(E). \tag{3.8}$$

Now observe that the twised variety ${}^P\,\mathrm{Gr}(m, W) = {}^U\,\mathrm{Gr}(m, W)$ is the generalized Brauer-Severi variety $\mathrm{BS}(m, A)$, where A is the central simple algebra whose Brauer class is $\partial([U]) = d \cdot \partial([P])$. Our goal is to show that if $\mathrm{BS}(m, A)$ has an E-point, then P is split. Indeed, assume that $\mathrm{BS}(m, A)$ has an E-point. Then by [Ar81, (3.4)], the index of A divides m. Thus $m \cdot \partial([U]) = 0$ or equivalently,

$$md \cdot \partial([P]) = 0 \text{ in } \mathrm{Br}(E). \tag{3.9}$$

Since md and n are coprime, (3.8) and (3.9) tell us that $\partial([P]) = 0$ in $\mathrm{Br}(E)$. Equivalently, P is split over E, as desired. □

4. Forms of $\overline{M}_{1,3}$

In this section we will prove the following.

Theorem 4.1. *Let F be a field of characteristic $\neq 2$. Then every F-form of $\overline{M}_{1,3}$ is stably rational.*

For the rest of this section, V will denote a 3-dimensional F-vector space, and $W_3 := \mathrm{S}^3(V^*)$ will denote the 10-dimensional space of cubic forms on V. Consider the closed subvariety X of $\mathbb{P}(W_3) \times \mathbb{P}(V)^3$ given by

$$X := \{(C, p_1, p_2, p_3) \,|\, p_1, p_2, p_3 \in C, \text{ and } p_1,\ p_2,\ p_3 \text{ are collinear}\}.$$

Here points of $\mathbb{P}(W_3)$ are viewed as cubic curves in $\mathbb{P}(V) = \mathbb{P}^2$. The group S_3 acts on X by permuting p_1, p_2, p_3; this action commutes with the natural action of $\mathrm{PGL}(V)$ on X.

Lemma 4.2. *(a) The rational quotient $X/\mathrm{PGL}(V)$ is S_3-equivariantly birationally isomorphic to $M_{1,3}$.*

(b) The $\mathrm{PGL}(V)$-action on X is generically free.

Proof. (a) The natural map $f\colon X \dashrightarrow M_{1,3}$ sending the quadruple (C, p_1, p_2, p_3) (with C smooth) to its isomorphism class is clearly S_3-equivariant and factors through the rational quotient $X/\mathrm{PGL}(V)$. We claim that the induced S_3-equivariant rational map $\overline{f}\colon X/\mathrm{PGL}(V) \dashrightarrow M_{1,3}$ is a birational isomorphism.

To prove this claim, we will construct the inverse $\overline{h}\colon M_{1,3} \to X/\mathrm{PGL}(V)$ to $\overline{f}$ as follows. Given a point $(C, p_1, p_2, p_3) \in M_{1,3}$, consider the invertible sheaf $\mathcal{O}_C(p_1 + p_2 + p_3)$. By the Riemann-Roch theorem, the space of global sections $H^0(C, \mathcal{O}_C(p_1 + p_2 + p_3))$ is 3-dimensional. Identifying this space with V^* we obtain an embedding

$$h \colon C \hookrightarrow \mathbb{P}(V)$$

of C into $\mathbb{P}(V) = \mathbb{P}^2$ as a curve of degree 3. The element of V^* corresponding to $1 \in H^0(C, \mathcal{O}_C(p_1 + p_2 + p_3))$ cuts out a projective line $L \subset \mathbb{P}(V)$ passing through $h(p_1)$, $h(p_2)$ and $h(p_3)$. This shows that $(h(C), h(p_1), h(p_2), h(p_3)) \in X$. Note that $(h(C), h(p_1), h(p_2), h(p_3)) \in X$ is not intrinsically defined by (C, p_1, p_2, p_3), because it depends on the isomorphism we have chosen between V^* with $H^0(C, \mathcal{O}_C(p_1 + p_2 + p_3))$. On the other hand, the image of $(h(C), h(p_1), h(p_2), h(p_3))$ in $X/\mathrm{PGL}(V)$ does not depend on this choice. Using the universal property of $M_{1,3}$ one readily checks that there is a rational map $\overline{h} \colon M_{1,3} \dashrightarrow X/\mathrm{PGL}(V)$ which sends (C, p_1, p_2, p_3) to the image of $(h(C), h(p_1)h(p_2), h(p_3)) \in X$ in $X/\mathrm{PGL}(V)$.

It remains to show that $\overline{f} \colon X/\mathrm{PGL}(V) \dashrightarrow M_{1,3}$ and $\overline{h} \colon M_{1,3} \dashrightarrow X/\mathrm{PGL}(V)$ are mutually inverse rational maps. The composition $\overline{f}\,\overline{h} \colon M_{1,3} \dashrightarrow M_{1,3}$ is clearly the identity map: $\overline{h}$ embeds (C, p_1, p_2, p_3) into $\mathbb{P}(V) = \mathbb{P}^2$, and $\overline{f}$ "forgets" this embedding and returns (C, p_1, p_2, p_3) as an abstract curve with three marked points. On the other hand, to see that $\overline{h}\,\overline{f} \colon X/\mathrm{PGL}(V) \dashrightarrow X/\mathrm{PGL}(V)$ is the identity map, note that if C is a smooth cubic curve in $\mathbb{P}(V)$ with three distinct marked collinear points p_1, p_2 and p_3, then $\mathcal{O}_C(p_1+p_2+p_3) = \mathcal{O}_C(1)$. In other words, we can identify the space of global sections $H^0(C, \mathcal{O}_C(p_1 + p_2 + p_3))$ with V^* so that $h(C, p_1, p_2, p_3) = (C, p_1, p_2, p_3)$.

(b) Suppose $H \subset \mathrm{PGL}(V)$ is the stabilizer subgroup of $(C, p_1, p_2, p_3) \in X$ in general position. One readily checks that the natural homomorphism $H \to \mathrm{Aut}(C, p_1, p_2, p_3)$ is injective. Since the map f defined in part (a) is dominant, we may assume that $\mathrm{Aut}(C, p_1, p_2, p_3) = \{1\}$ is the trivial group; see Proposition 2.8. Thus $H = \{1\}$ as well. □

Proof of Theorem 4.1. By Corollary 2.10, every F-form of $\overline{M}_{1,3}$ is isomorphic to ${}^P\overline{M}_{1,3}$ for some S_3-torsor $P \to \mathrm{Spec}(F)$. By Lemma 4.2(a), ${}^P\overline{M}_{1,3}$ is birationally isomorphic to ${}^PX/\mathrm{PGL}(V)$. It thus remains to show that ${}^PX/\mathrm{PGL}(V)$ is stably rational over F. We will fix the S_3-torsor P for the rest of the proof.

Claim 1: ${}^PX$ is rational over ${}^PX/\mathrm{PGL}(V)$.

We will deduce Claim 1 from Proposition 3.6(b), with $G = \mathrm{PGL}(V)$ and $Y = \mathbb{P}(V^*)$. To apply Proposition 3.6, it suffices to check that

(i) $\mathrm{PGL}(V)$ is rational,

(ii) $\mathrm{PGL}(V)$-action on ${}^PX$ is generically free,

(iii) the $\mathrm{PGL}(V)$-action on $\mathbb{P}(V^*)$ is anti-versal, and

(iv) there exists a $\mathrm{PGL}(V)$-equivariant rational map ${}^PX \dashrightarrow \mathbb{P}(V^*)$.

(i) is obvious. To prove (ii), note that the property of being generically free is geometric, i.e., can be checked after passing to the separable closure of our base field F. When we pass to the separable closure of F, ${}^PX$ becomes isomorphic to X. Thus (ii) follows from Lemma 4.2(b). (iii) follows from Lemma 3.7 with $W = V^*$, and $d = m = 1$. To prove (iv), we begin with the $\mathrm{S}_3 \times \mathrm{PGL}(V)$-equivariant rational map $X \dashrightarrow \mathbb{P}(V^*)$ sending the quadruple $(C, p_1, p_2, p_3) \in X$ to the unique line $L \subset \mathbb{P}(V)$ passing through p_1, p_2 and p_3.

Here S_3 acts trivially on $\mathbb{P}(V^*)$. Twisting by P, we obtain a desired $\mathrm{PGL}(V)$-equivariant rational map

$$ {}^P X \dashrightarrow \mathbb{P}(V^*) \, . $$

This completes the proof of Claim 1.

Claim 2: ${}^P X$ is stably rational over F.

To prove Claim 2, let $T \to \mathbb{P}(V^*)$ be the tautological line bundle, whose fiber over the line $\{l = 0\}$ in $\mathbb{P}(V^*)$ consists of vectors $v \in V$ such that $l(v) = 0$. Let $(T^3)_0$ be the dense open subset of $T^3 := T \times_{\mathbb{P}(V^*)} T \times_{\mathbb{P}(V^*)} T$ consisting of triples (v_1, v_2, v_3) which impose independent conditions on cubic polynomials $\phi \in W_3$. Let E be the vector subbundle of the trivial bundle $W_3 \times (T^3)_0$ on $(T^3)_0$ consisting of quadruples (ϕ, v_1, v_2, v_3) such that $\phi(v_1) = \phi(v_2) = \phi(v_3) = 0$. Finally, let E_0 be obtained from E by removing the zero section, i.e., by requiring that $\phi \neq 0$ in W_3. In summary, we have the following diagram of S_3-equivariant morphisms

$$\begin{tikzcd}
 & E_0 \arrow[r, hook, "i"] \arrow[ddl, "\alpha"] & E \arrow[d, "\beta"] & \\
 & & (T^3)_0 \arrow[r, hook, "j"] & T^3 \arrow[d, "\gamma"] \\
X & & & \mathbb{P}(V^*)
\end{tikzcd}$$

where $\alpha \colon (\phi, v_1, v_2, v_3) \mapsto ([\phi], [v_1], [v_2], [v_3])$ and β, γ are natural projections. Here the horizontal maps i and j are open immersions, β and γ are vector bundles, and α is a $(\mathbb{G}_m)^4$-torsor, where $(s, t_1, t_2, t_3) \in (\mathbb{G}_m)^4$ acts on E_0 by $(\phi, v_1, v_2, v_3) \mapsto (s\phi, t_1 v_1, t_2 v_2, t_3 v_3)$. By Lemma 2.1(d), ${}^P(T^3)_0$ is rational over ${}^P\mathbb{P}(V^*)$ and ${}^P E$ is rational over ${}^P(T^3)_0$.

On the other hand, the actions of $G := \mathrm{S}_3$ and $H := (\mathbb{G}_m)^4$ on E_0 skew-commute, relative to the S_3-action on H given by permuting the last three $\mathbb{G}_m$-factors (and leaving the first factor unchanged). Applying Proposition 2.6(b) to these skew-commuting actions of $G := \mathrm{S}_3$ and $H := (\mathbb{G}_m)^4$ on E_0, we see that ${}^P E_0$ is rational over ${}^P X$. Indeed, Proposition 2.6(b) applies in this situation, because ${}^P H = \mathbb{G}_m \times R_{A/F}(\mathbb{G}_m^3)$ is both special (see Section 2e) and rational.

Putting all of this together, and remembering that S_3 acts trivially on $\mathbb{P}(V^*)$ and thus ${}^P\mathbb{P}(V^*) = \mathbb{P}(V^*)$, we obtain the following diagram

$$\begin{tikzcd}
 & {}^P E_0 \arrow[r, hook] \arrow[ddl, "\text{rational}"] & {}^P E \arrow[d, "\text{rational}"] & \\
 & & {}^P(T^3)_0 \arrow[r, hook] & {}^P T^3 \arrow[d, "\text{rational}"] \\
{}^P X & & & \mathbb{P}(V^*) \arrow[d, "\text{rational}"] \\
 & & & \mathrm{Spec}(F).
\end{tikzcd}$$

By Lemma 2.1(b), the horizontal maps ${}^{P}i$ and ${}^{P}j$ are open immersions (hence, birational), and Claim 2 follows. This completes the proof of Theorem 4.1. □

5. Forms of $\overline{M}_{1,4}$

In this section we will prove the following.

Theorem 5.1. *Let F be a field of characteristic $\neq 2$. Then every F-form of $\overline{M}_{1,4}$ is stably rational.*

Let V be a 4-dimensional F-vector space and $W_2 := \mathrm{S}^2(V^*)$ be the 10-dimensional space of homogeneous quadratic polynomials on V. Consider the closed subvariety $X \subset \mathrm{Gr}(2, W_2) \times \mathbb{P}(V)^4$ given by

$$\begin{aligned} X := \ \{(Q, p_1, \dots, p_4) \,|\, & q(p_1) = \dots = q(p_4) = 0 \text{ for any } q \in Q, \\ & \text{and } p_1,\, p_2,\, p_3,\, p_4 \text{ are collinear in } \mathbb{P}(V)\,\}. \end{aligned}$$

The group S_4 acts on X by permuting the points p_1, p_2, p_3, p_4; this action commutes with the natural action of $\mathrm{PGL}(V)$.

Lemma 5.2. *(a) The rational quotient $X/\mathrm{PGL}(V)$ is S_4-equivariantly birationally isomorphic to $M_{1,4}$.*

(b) The $\mathrm{PGL}(V)$-action on X is generically free.

Proof. (a) Define a rational map $f\colon X \dashrightarrow M_{1,4}$ as follows. Given $(Q, p_1, \dots, p_4) \in X$, choose a basis q_1, q_2 of the 2-dimensional subspace $Q \subset W_2$ and set $f(Q, p_1, \dots, p_4) \in \overline{M}_{1,4}$ to be the isomorphism class of $(C, p_1, \dots, p_4)$, where $C \subset \mathbb{P}^3$ is the curve given by $q_1 = q_2 = 0$. Clearly $(C, p_1, \dots, p_4)$ does not depend on the choice of the basis q_1, q_2 of Q. For $(Q, p_1, \dots, p_4)$ in general position in X, C is a smooth curve of genus 1; see, e.g., [Ha77, Exercise I.7.2]. Hence, f is a well-defined rational map. By construction f is $\mathrm{S}_4 \times \mathrm{PGL}(V)$ equivariant, where $\mathrm{PGL}(V)$ acts trivially on $\overline{M}_{1,4}$. Thus f factors through the rational quotient $X/\mathrm{PGL}(V)$. We claim that the induced S_3-equivariant rational map $\overline{f}\colon X/\mathrm{PGL}(V) \dashrightarrow M_{1,4}$ is a birational isomorphism.

To prove this claim, we will construct the inverse $\overline{h}\colon M_{1,4} \to X/\mathrm{PGL}(V)$ to $\overline{f}$ as follows. Given $(C, p_1, \dots, p_4) \in M_{1,4}$, consider the invertible sheaf $\mathcal{O}_C(p_1 + \dots + p_4)$. By the Riemann-Roch theorem, the space of global sections $H^0(C, \mathcal{O}_C(p_1 + \dots + p_4))$ is 4-dimensional. Identifying this space with V^*, we obtain an embedding $h\colon C \hookrightarrow \mathbb{P}(V)$. The element of V^* corresponding to $1 \in H^0(C, \mathcal{O}_C(p_1 + \dots + p_4))$ cuts out a plane $L \subset \mathbb{P}(V)$ passing through $h(p_1), \dots, h(p_4)$. Moreover, by [Ha77, Exercise 3.6(b)], the space $H^0(\mathbb{P}(V), \mathcal{I}_{h(C)}(2))$ of global sections of the ideal sheaf $\mathcal{I}_{h(C)}(2)$ is 2-dimensional. Set $Q := H^0(\mathbb{P}(V), \mathcal{I}_{h(C)}(2))$. Then $(Q, h(p_1), \dots, h(p_4)) \in X$. Note that while $(Q, h(p_1), \dots, h(p_4)) \in X$ depends on how we identified $H^0(C, \mathcal{O}_C(p_1 + \ldots + p_4))$ with V^*, the image of $(Q, h(p_1), \dots, h(p_4))$ in $X/\mathrm{PGL}(V)$ does not. By the universal property of $M_{1,4}$ there is a rational map $\overline{h}\colon M_{1,4} \dashrightarrow X/\mathrm{PGL}(V)$ which sends $(C, p_1, \dots, p_4)$ to the class of $(Q, h(p_1), \dots, h(p_4)) \in X$ in $X/\mathrm{PGL}(V)$.

It remains to show that $\overline{f}\colon X/\mathrm{PGL}(V) \dashrightarrow M_{1,4}$ and $\overline{h}\colon M_{1,4} \dashrightarrow X/\mathrm{PGL}(V)$ are mutually inverse rational maps. To see that the composition $\overline{f}\,\overline{h}\colon M_{1,4} \dashrightarrow M_{1,4}$ is the identity map, note that $\overline{h}$ presents an abstract marked curve $(C, p_1, \dots, p_4)$ as an intersection of a pencil Q of quadrics in $\mathbb{P}(V) = \mathbb{P}^3$ (up to an element of $\mathrm{PGL}(V)$), and $\overline{f}$ "forgets" this presentation and returns $(C, p_1, \dots, p_4)$ as an abstract curve with four marked points.

To verify that $\overline{h}\,\overline{f}\colon X/\mathrm{PGL}(V) \dashrightarrow X/\mathrm{PGL}(V)$ is the identity map, recall that for $(Q, p_1, \dots, p_4) \in X$ in general position, $f(Q, p_1, \dots, , p_4)$ is the isomorphism class of $(C, p_1, \dots, p_4)$, where C is the curve of genus 1 cut out by a basis q_1, q_2 of Q. Since $\mathcal{O}_C(p_1 + \ldots + p_4)$ is $\mathcal{O}_C(1)$, up to an element of $\mathrm{PGL}(V)$, $(h(C), h(p_1), \dots, h(p_4))$ recovers $(C, p_1, \dots, p_4)$, and $H^0(\mathbb{P}(V), \mathcal{I}_{h(C)}(2))$ recovers Q. This completes the proof of part (a).

(b) follows from Proposition 2.8 by the same argument as in the proof of Lemma 4.2(b). $\square$

Proof of Theorem 5.1. By Corollary 2.10, every F-form of $\overline{M}_{1,4}$ is isomorphic to ${}^P\overline{M}_{1,4}$ for some S_4-torsor $P \to \mathrm{Spec}(F)$. By Lemma 4.2(a), ${}^P\overline{M}_{1,4}$ is birationally isomorphic to ${}^PX/\mathrm{PGL}(V)$. Thus it remains to show that ${}^PX/\mathrm{PGL}(V)$ is stably rational over F. We will fix the S_3-torsor P for the rest of the proof.

Claim 1: ${}^PX$ is rational over ${}^PX/\mathrm{PGL}(V)$.

The proof, based on Proposition 3.6(b), with $G = \mathrm{PGL}(V)$ and $Y = \mathbb{P}(V^*)$, follows the same pattern as the proof of Claim 1 in the previous section. Once again, in order to apply Proposition 3.6(b), we need to check items (i) - (iv). The proof of (i), (ii) and (iii) is exactly the same. To define a $\mathrm{PGL}(V)$-equivariant rational map ${}^PX \dashrightarrow \mathbb{P}(V^*)$, start with the $\mathrm{PGL}(V)$-equivariant rational map $X \dashrightarrow \mathbb{P}(V^*)$ sending a quadruple $(Q, p_1, \dots, p_4)$ in general position to the unique hyperplane in $\mathbb{P}^3$ passing through $p_1, \dots, p_4$, then twist this map by P. This completes the proof of Claim 1.

Claim 2: ${}^PX$ is stably rational over F.

To prove Claim 2, let

$T \to \mathbb{P}(V^*)$ be the tautological line bundle, whose fiber over the plane $\{l = 0\}$ in $\mathbb{P}(V)$ consists of vectors $v \in V$ such that $l(v) = 0$,

$(T^4)_0$ be the dense open subset of $T^4 := T \times_{\mathbb{P}(V^*)} \cdots \times_{\mathbb{P}(V^*)} T$ (4 times) consisting of quadruples $(v_1, \dots, v_4)$ which impose independent conditions on quadrics in $\mathbb{P}(V)$,

E be the vector subbundle of the trivial bundle $W_2 \times (T^4)_0$ on $(T^4)_0$ consisting of tuples $(\phi_1, v_1, \dots, v_4)$ such that $\phi(v_1) = \cdots = \phi(v_4) = 0$,

E^2_0 be the dense open subset of $E^2 = E \times_{(T^4)_0} E$ consisting of tuples $(\phi_1, \phi_2, v_1, \dots, v_4)$ such that ϕ_1 and ϕ_2 are linearly independent, and

X_0 be the dense open subset of X consisting of tuples $(Q, p_1, \dots, p_4)$ such that $p_1, \dots, p_4$ span a 2-dimensional projective plane in $\mathbb{P}(V)$.

Then we have the following diagram of S_4-equivariant morphisms

$$\begin{tikzcd}
 & (E^2)_0 \arrow[r, hook, "j"] \arrow[ddl, "\alpha"'] & E^2 \arrow[d, "\beta"] & \\
 & & (T^4)_0 \arrow[r, hook, "k"] & T^4 \arrow[d, "\gamma"] \\
X_0 \arrow[r, hook, "i"] & X & & \mathbb{P}(V^*)
\end{tikzcd}$$

where $\alpha\colon (\phi_1, \phi_2, v_1, \dots, v_4) \mapsto (\mathrm{Span}(\phi_1, \phi_2), [v_1], [v_2], [v_3], [v_4])$ and β, γ are natural projections. Note that the horizontal maps i, j and k are open immersions, β and γ are vector bundles, and α is a $\mathrm{GL}_2 \times (\mathbb{G}_m)^4$-torsor. Twisting by P, we obtain the following diagram of F-morphisms:

$$\begin{tikzcd}
 & {}^P(E^2)_0 \arrow[r, hook] \arrow[ddl, "\text{rational}"'] & {}^P E^2 \arrow[d, "\text{rational}"] & \\
 & & {}^P(T^4)_0 \arrow[r, hook] & {}^P(T^4) \arrow[d, "\text{rational}"] \\
{}^P X_0 \arrow[r, hook] & {}^P X & & \mathbb{P}(V^*) \arrow[d, "\text{rational}"] \\
 & & & \mathrm{Spec}(F).
\end{tikzcd}$$

Here ${}^P\mathbb{P}(V^*) = \mathbb{P}(V^*)$ because S_4 acts trivially on $\mathbb{P}(V^*)$; ${}^P(E^2)$ is rational over ${}^P(T^4)_0$ and ${}^P(T^4)$ is rational over $\mathbb{P}(V^*)$ by Lemma 2.1(d). Finally, to show that ${}^P(E^2)_0$ is rational over ${}^P X$, we apply Proposition 2.6(b) to the skew-commuting actions of S_4 and $(\mathbb{G}_m)^4$ on $(E^2)_0$. The twisted group

$$^P(\mathrm{GL}_2 \times \mathbb{G}_m^4) = \mathrm{GL}_2 \times R_{A/F}(\mathbb{G}_m)$$

(where A/F is the étale algebra of degree 4 associated to P) is both special (see Section 2e) and rational, and Proposition 2.6(b) applies.

Finally, the horizontal maps ${}^P i$, ${}^P j$ and ${}^P k$ are open immersions (and hence, birational) by Lemma 2.1(b). Thus the diagram shows that ${}^P X$ is stably rational over F. This completes the proof of Claim 2 and thus of Theorem 4.1. $\square$

6. Forms of $\overline{M}_{2,2}$

In this section we will prove the following.

Theorem 6.1. *Let F be a field of characteristic $\neq 2$. Then every F-form of $\overline{M}_{2,2}$ is stably rational.*

For the rest of this section V will denote a 3-dimensional F-vector space, and $W_4 := \mathrm{S}^4(V^*)$ will denote the 15-dimensional space of degree 4 homogeneous polynomials on V. Let $(\mathbb{P}(V^*) \times \mathbb{P}(V^*))_0$ be the dense open subset of $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ consisting of pairs (L_1, L_2) of distinct lines in $\mathbb{P}(V)$. Consider the subbundle E of the trivial vector bundle $W_4 \times (\mathbb{P}(V^*) \times \mathbb{P}(V^*))_0$ over $(\mathbb{P}(V^*) \times \mathbb{P}(V^*))_0$ consisting of triples (ϕ, L_1, L_2) such that the restriction of $\phi \in W_4$ to L_i ($i = 1, 2$) vanishes to second order at the intersection point $p = L_1 \cap L_2$. In other words, the plane curve $C \subset \mathbb{P}(V)$ given by $\{\phi = 0\}$ is singular at p and L_1, L_2 are tangent lines to C at p.

Lemma 6.2. *(a) The rational quotient $\mathbb{P}(E)/\mathrm{PGL}(V)$ is S_2-equivariantly birationally isomorphic to $M_{2,2}$.*

(b) The $\mathrm{PGL}(V)$-action on $\mathbb{P}(E)$ is generically free.

Proof. (a) We begin by defining a $\mathrm{S}_2 \times \mathrm{PGL}(V)$-equivariant rational map

$$f \colon \mathbb{P}(E) \dashrightarrow M_{2,2}$$

as follows. Given a triple $([\phi], L_1, L_2) \in \mathbb{P}(E)$, in general position, let $C \subset \mathbb{P}(V) = \mathbb{P}^2$ be the quartic curve cut out by ϕ, and p be the point of intersections of L_1 and L_2. Then C has a node at p and is smooth elsewhere, and thus is a curve of genus 1. Blowing up C at p, we obtain a smooth curve C', with two marked points, p_1 and p_2, corresponding to the tangent lines L_1 and L_2 to C at p. We define $f([\phi], L_1, L_2)$ to be the isomorphism class of (C', p_1, p_2). Clearly any $\mathrm{PGL}(V)$-translate of $([\phi], L_1, L_2)$ will have the same image under f. Thus f descends to an S_2-equivariant rational map

$$\overline{f} \colon \mathbb{P}(E)/\mathrm{PGL}(V) \dashrightarrow \mathrm{M}_{2,2} \; .$$

To prove that $\overline{f}$ is a birational isomorphism, we will construct the inverse $\overline{h} \colon M_{2,2} \to \mathbb{P}(E)/\mathrm{PGL}(V)$ to $\overline{f}$ as follows. Given a point $(C', p_1, p_2) \in M_{2,2}$, consider the invertible sheaf $\mathcal{O}_C(K + p_1 + p_2)$, where K is the canonical divisor on C'. By the Riemann-Roch theorem, the space of global sections $H^0(C, \mathcal{O}_C(K+p_1+p_2))$ is 3-dimensional. Identifying this space with V^*, we obtain a map

$$h \colon C \to \mathbb{P}(V)$$

of C into $\mathbb{P}(V) = \mathbb{P}^2$ as a curve of degree 4. Assuming that p_1 and p_2 are in general position in C, the image $C = h(C')$ of this map is a quartic curve with one node, and $h \colon C' \to C$ is the normalization map; see [H11, Example 5.15]. Moreover, C has two tangent lines at p, L_1 and L_2, which correspond to p_1 and p_2 under h. Thus we send $(C', p_1, p_2) \in \mathrm{M}_{2,2}$ to $([\phi], L_1, L_2) \in \mathbb{P}(E)$, where $\phi \in W_4$ is a defining equation for C. Once again, $([\phi], L_1, L_2)$ depends on the isomorphism we have chosen between V^* with $H^0(C, \mathcal{O}_C(K + p_1 + p_2))$, but the image $([\phi], L_1, L_2)$ in $\mathbb{P}(E)/\mathrm{PGL}(V)$ does not depend on this choice, giving rise to a rational map

$$\overline{h} \colon M_{2,2} \dashrightarrow \mathbb{P}(E)/\mathrm{PGL}(V) \, .$$

Both $\overline{f}$ and $\overline{h}$ are S_2-equivariant; it remains to show that they are mutually inverse. The composition $\bar{f}\,\bar{h} \colon M_{2,2} \dashrightarrow M_{2,2}$ is clearly the identity map. To prove that the composition $\bar{h}\,\bar{f}$ is the identity map on $\mathbb{P}(E)/\mathrm{PGL}(V)$, it suffices to establish the following claim. Suppose $C \subset \mathbb{P}(V)$ is a plane quartic curve with a single node at p, $\pi \colon C' \to C$ is the normalization map, and $p_1, p_2 \in C'$ are the preimages of p. Then $\pi^* \mathcal{O}_C(1) = \mathcal{O}_{C'}(K + p_1 + p_2)$, where K is the canonical divisor on C'.

To prove this claim, note that if L is a general line through p in $\mathbb{P}(V)$, cut out by a linear form $l \in V^*$, and q_1, q_2 are the two other intersection points of L with C (other than p), then clearly $H^0(C', \pi^{-1}(q_1) + \pi^{-1}(q_2))$ contains a non-constant function l'/l, where l and $l' \in V^*$ are linearly independent. By the Riemann-Roch Theorem, this implies that $\pi^{-1}(q_1) + \pi^{-1}(q_2) = K$ is the canonical divisor on C', and the claim follows.

(b) is a consequence of Proposition 2.8 with $g = n = 2$. The proof is the same as in Lemma 4.2(a). □

Proof of Theorem 6.1. By Corollary 2.10, every F-form of $\overline{M}_{2,2}$ is isomorphic to ${}^P\overline{M}_{2,2}$ for some S_2-torsor $P \to \mathrm{Spec}(F)$. By Lemma 6.2(a), ${}^P\overline{M}_{2,2}$ is birationally isomorphic to

${}^P\mathbb{P}(E)/\mathrm{PGL}(V)$. It thus remains to show that ${}^P\mathbb{P}(E)/\mathrm{PGL}(V)$ is stably rational over F. Let us fix the S_2-torsor $P \to \mathrm{Spec}(F)$ for the rest of the proof.

Claim 1: ${}^P\mathbb{P}(E)$ is rational over ${}^P\mathbb{P}(E)/\mathrm{PGL}(V)$.

We will deduce Claim 1 from Proposition 3.6(b), with $\mathbb{P}(E) = {}^P\mathbb{P}(E)$, $G = \mathrm{PGL}(V)$ and $Y = \mathbb{P}(W_4)$. The group $\mathrm{PGL}(V) \simeq \mathrm{PGL}_2$ is clearly rational. It follows from Lemma 6.2(b) that the $\mathrm{PGL}(V)$-action on ${}^P\,\mathbb{P}(E)$ is generically free. The $\mathrm{PGL}(V)$-action on Y is antiversal by Lemma 3.7 with $W = W_4$, $m = 1$, $n = \dim(V) = 3$ and $d = 4$. It thus remains to construct a $\mathrm{PGL}(V)$-equivariant rational map ${}^P\mathbb{P}(E) \dashrightarrow Y = \mathbb{P}(W_4)$. We begin with the $\mathrm{S}_2 \times \mathrm{PGL}(V)$-equivariant rational map $\mathbb{P}(E) \to \mathbb{P}(W_4)$ sending $([\phi], L_1, L_2) \in \mathbb{P}(E)$ to $[\phi] \in \mathbb{P}(W_4)$. Here S_2 acts trivially on $\mathbb{P}(W_4)$. Twisting by P, we obtain a $\mathrm{PGL}(V)$-equivariant rational map ${}^P\mathbb{P}(E) \dashrightarrow {}^P\mathbb{P}(W_4) = \mathbb{P}(W_4)$, as desired. Claim 1 now follows from Proposition 3.6(b).

Claim 2: ${}^P\mathbb{P}(E)$ is stably rational over F.

To prove Claim 2, recall that at the beginning of this section we defined $(\mathbb{P}(V^*)\times\mathbb{P}(V^*))_0$ as the dense open subset of $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ consisting of pairs of distinct lines (L_1, L_2) and E as a vector bundle on $(\mathbb{P}(V^*)\times\mathbb{P}(V^*))_0$. Let E_0 be the dense open subset obtained from E by removing the zero section. That is, points of E_0 are triples $(\phi, L_1, L_2) \in E$ such that $0 \neq \phi \in W_4$. In summary, we have a diagram of S_2-equivariant maps

$$\begin{tikzcd}
E_0 \arrow[r, hook, "i"] \arrow[d, "\alpha"'] & E \arrow[dr, "\beta"] & & \\
\mathbb{P}(E) & & (\mathbb{P}(V^*) \times \mathbb{P}(V^*))_0 \arrow[r, hook, "j"] & \mathbb{P}(V^*) \times \mathbb{P}(V^*)
\end{tikzcd}$$

given by $\alpha(\phi, L_1, L_2) = ([\phi], L_1, L_2)$ and $\beta(\phi, L_1, L_2) = (L_1, L_2)$. Then α is a $\mathbb{G}_m$-torsor, the horizontal maps i and j are open immersions, and β is a vector bundle. Twisting by P, we obtain the following diagram

$$\begin{tikzcd}
{}^P E_0 \arrow[r, hook] \arrow[d, "\text{rational}"'] & {}^P E \arrow[dr, "\text{rational}"] & & \\
{}^P\mathbb{P}(E) & & {}^P(\mathbb{P}(V^*) \times \mathbb{P}(V^*))_0 \arrow[r, hook] & {}^P(\mathbb{P}(V^*) \times \mathbb{P}(V^*)) \arrow[d, "\text{rational}"'] \\
& & & \mathrm{Spec}(F).
\end{tikzcd}$$

Here ${}^P E_0$ is rational over ${}^P(\mathbb{P}(V^*) \times \mathbb{P}(V^*))_0$ by Lemma 2.1(d).

To show that ${}^P E_0$ is rational over ${}^P\mathbb{P}(E)$, we appeal to Proposition 2.6(b). The actions of $G = \mathrm{S}_2$ and $H = \mathbb{G}_m$ on E_0 commute, i.e., skew-commute relative to the trivial action of S_2 on $\mathbb{G}_m$. Since S_2 acts trivially on $\mathbb{G}_m$, ${}^P\mathbb{G}_m \simeq \mathbb{G}_m$ is special and rational, and Proposition 2.6(b) tells us that ${}^P E_0$ is rational over ${}^P\mathbb{P}(E)$, as desired.

To see that ${}^P(\mathbb{P}(V^*) \times \mathbb{P}(V^*))$ is rational over F, note that ${}^P(\mathbb{P}(V^*) \times \mathbb{P}(V^*)) = R_{A/F}(\mathbb{P}(V_A^*))$, where A/F is the 2-dimensional étale algebra associated to the class of P in $H^1(F, \mathrm{S}_2)$; see the last paragraph of Section 2d. On the other hand, $R_{A/F}(\mathbb{P}(V_A^*))$ is rational over F by Lemma 2.3(d).

Finally, the horizontal maps ${}^P i$ and ${}^P j$ are immersions by Lemma 2.1(b). We conclude that ${}^P\mathbb{P}(E)$ is stably rational over F, thus completing the proof of Claim 2 and of Theorem 6.1. $\square$

7. Forms of $\overline{M}_{2,3}$

In this section we will prove the following.

Theorem 7.1. *Let F be a field of characteristic $\neq 2$. Then every F-form of $\overline{M}_{2,3}$ is stably rational.*

For the rest of this section V will denote a 2-dimensional F-vector space, and $W_{2,3}$ will denote the 12-dimensional space of degree bihomogeneous polynomials on $V \times V$ of bidegree $(2,3)$. The group

$$G := \mathrm{PGL}(V) \times \mathrm{PGL}(V) \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$$

acts on $\mathbb{P}(V)^3 \times \mathbb{P}(V)$ by $(g,h)\colon (a_1,a_2,a_3,b) \mapsto (ga_1,ga_2,ga_3,hb)$ and the symmetric group S_3 by permuting a_1,a_2,a_3. Let $(\mathbb{P}(V)^3 \times \mathbb{P}(V))_0$ be the $\mathrm{S}_3 \times G$-invariant dense open subset of $\mathbb{P}(V)^3 \times \mathbb{P}(V)$ consisting of quadruples (a_1,a_2,a_3,b) such that (a_1,b), (a_2,b) and $(a_3,b) \in \mathbb{P}(V) \times \mathbb{P}(V)$ impose three independent conditions on bihomogeneous polynomials of bidegree $(2,3)$. Let E be the rank 9 vector subbundle of the trivial bundle $W_{2,3} \times (\mathbb{P}(V)^3 \times \mathbb{P}(V))_0$ over $(\mathbb{P}(V)^3 \times \mathbb{P}(V))_0$ consisting of tuples (ϕ, a_1,a_2,a_3,b) such that

$$\phi(a_1,b) = \phi(a_2,b) = \phi(a_3,b) = 0\,.$$

Denote the associated projective bundle over $(\mathbb{P}(V)^3 \times \mathbb{P}(V))_0$ by $\mathbb{P}(E)$. The symmetric group S_3 acts on $\mathbb{P}(E)$ by permuting a_1,a_2,a_3, and $G = \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ acts on $\mathbb{P}(E)$ by

$$(g,h)\colon ([\phi], a_1,a_2,a_3,b) \mapsto ([\phi^{g,h}], ga_1,ga_2,ga_3,hb)$$

for any $(g,h) \in \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ and $a_1,a_2,a_3,b \in \mathbb{P}(V)$. Here $\phi^{g,h}(x,y) := \phi(gx,hy)$.

Lemma 7.2. *(a) The rational quotient $\mathbb{P}(E)/G$ is S_3-equivariantly birationally isomorphic to $M_{2,3}$.*

(b) The G-action on $\mathbb{P}(E)$ is generically free.

Proof. (a) A bihomogeneous polynomial $\phi \in W_{2,3}$ in general position cuts out a smooth curve $C \subset \mathbb{P}(V) \times \mathbb{P}(V) = \mathbb{P}^1 \times \mathbb{P}^1$ of genus 2; see [Ha77, Exercise 5.6(c)]. Sending $([\phi], a_1,a_2,a_3,b)$ to the isomorphism class of the marked curve (C,p_1,p_2,p_3), where $p_i = (a_i,b) \in \mathbb{P}(V) \times \mathbb{P}(V)$, gives rise to a $G \times \mathrm{S}_3$-equivariant rational map

$$f\colon \mathbb{P}(E) \dashrightarrow M_{2,3}\,.$$

Here G acts trivially on $M_{2,3}$. We claim that the induced S_3-equivariant rational map

$$\overline{f}\colon \mathbb{P}(E)/G \dashrightarrow \mathrm{M}_{2,3}$$

is a birational isomorphism. To prove this claim we will define the inverse

$$\overline{h}\colon M_{2,3} \dashrightarrow \mathbb{P}(E)/G$$

to f as follows. Let K be the canonical divisor on C. By the Riemann-Roch theorem, the spaces of global sections $H^0(C, \mathcal{O}_C(K))$ and $H^0(C, \mathcal{O}_C(p_1 + p_2 + p_3))$ are both 2-dimensional. Identifying them with V^*, we obtain ramified covers

$$\pi_1 \text{ and } \pi_2 \colon C \to \mathbb{P}(V)$$

of degrees 2 and 3 respectively; see [H11, Examples 5.11 and 5.13]. (Note that π_2 is not well defined for every choice of p_1, p_2 and p_3 but is well defined for p_1, p_2 and p_3 in general position.) For p_1, p_2, p_3 in general position, $C' := (\pi_1 \times \pi_2)(C)$ is a curve of bidegree $(2, 3)$ in $\mathbb{P}(V) \times \mathbb{P}(V)$, $\pi_1 \times \pi_2$ is an isomorphism between C and C', and $\pi_2(p_1) = \pi_2(p_2) = \pi_2(p_3)$ in $\mathbb{P}(V)$. Denote a defining polynomial for C' in $\mathbb{P}(V) \times \mathbb{P}(V)$ by $\phi \in W_{2,3}$ and set $a_i = \pi_1(p_i)$ for $i = 1, 2, 3$ and $b := \pi_2(p_1) = \pi_2(p_2) = \pi_2(p_3)$. The resulting point $([\phi], a_1, a_2, a_3, b) \in \mathbb{P}(E)$ depends on how we identified $H^0(C, \mathcal{O}_C(K))$ and $H^0(C, \mathcal{O}_C(p_1 + p_2 + p_3))$ with V^*, but its image in $\mathbb{P}(E)/G$ does not. This gives rise to a rational map $\overline{h} \colon M_{2,3} \dashrightarrow \mathbb{P}(E)/G$ given by

$$\overline{h}(C, p_1, p_2, p_3) := ([\phi], a_1, a_2, a_3, b)\,.$$

Now one readily checks that $\overline{f}$ and $\overline{h}$ are mutually inverse S_3-equivariant birational isomorphisms between $\mathbb{P}(E)/G$ and $M_{2,3}$.

(b) is, once again, a consequence of Proposition 2.8. The proof is the same as in Lemma 4.2(b). □

Proof of Theorem 7.1. Once again, in view of Corollary 2.10 and Lemma 7.2(a), it suffices to show that ${}^P\mathbb{P}(E)/G$ is stably rational over F for every S_3-torsor $P \to \mathrm{Spec}(F)$.

Claim 1: ${}^P\mathbb{P}(E)$ is rational over ${}^P\mathbb{P}(E)/G$.

We will prove Claim 1 by applying Proposition 3.6(b) to the $G = \mathrm{PGL}(V) \times \mathrm{PGL}(V)$-action on ${}^P\,\mathbb{P}(E)$, with

$$Y = {}^P(\mathbb{P}(V)^3 \times \mathbb{P}(V)) = R_{A/F}(\mathbb{P}(V)) \times \mathbb{P}(V),$$

where A/F is the étale algebra of degree 3 representing the class of P in $H^1(F, \mathrm{S}_3)$. Here the $G = \mathrm{PGL}(V) \times \mathrm{PGL}(V)$-action is the product of the natural action of the first factor of $\mathrm{PGL}(V)$ on $R_{A/F}(\mathbb{P}(V))$ and the second factor of $\mathrm{PGL}(V)$ on $\mathbb{P}(V)$. To apply Proposition 3.6, we need to check that

(i) G is rational,

(ii) the G-action on ${}^P\mathbb{P}(E)$ is generically free,

(iii) the G-action on Y is anti-versal, and

(iv) there exists a G-equivariant rational map ${}^P\mathbb{P}(E) \dashrightarrow Y$.

(i) is obvious and (ii) follows from Lemma 7.2(b). To prove (iii), note that the $\mathrm{PGL}(V)$-action on $\mathbb{P}(V)$ is anti-versal by Example 3.3 (or as a special case of Lemma 3.7). Since $[A : F] = 3$ is prime to $\dim(V) = 2$, the $\mathrm{PGL}(V)$-action on $R_{A/F}(\mathbb{P}(V))$ is also anti-versal by Lemma 3.4(c). Hence, by Lemma 3.4(b), the action of $G = \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ on $Y = R_{A/F}(\mathbb{P}(V)) \times \mathbb{P}(V)$ is also anti-versal. The proof of (iii) is now complete. To proof (iv), twist the natural $\mathrm{S}_3 \times G$-equivariant projection $\mathbb{P}(E) \to \mathbb{P}(V)^3 \times \mathbb{P}(V)$ taking $([\phi], a_1, a_2, a_3, b)$ to (a_1, a_2, a_3, b), by P to obtain a desired G-equivariant morphism ${}^P\mathbb{P}(E) \to R_{A/F}(\mathbb{P}(V)) \times \mathbb{P}(V) = Y$. This finishes the proof of Claim 1.

Claim 2: ${}^P\mathbb{P}(E)$ is stably rational over F.

To prove Claim 2, let E_0 be the dense open subset obtained from E by removing the zero section. That is, E_0 consists of tuples $(\phi, a_1, a_2, a_3, b) \in E$ such that $0 \neq \phi \in W_{2,3}$. Then we have a diagram of S_2-equivariant maps

$$\begin{tikzcd} E_0 \arrow[r, hook, "i"] \arrow[d, "\alpha"'] & E \arrow[drr, "\beta"] & & \\ \mathbb{P}(E) & & (\mathbb{P}(V^*)^3 \times \mathbb{P}(V^*))_0 \arrow[r, hook, "j"] & \mathbb{P}(V^*)^3 \times \mathbb{P}(V^*) \end{tikzcd}$$

The horizontal maps i and j are open immersions, $\alpha\colon (\phi, a_1, a_2, a_3, b) \mapsto ([\phi], a_1, a_2, a_3, b)$ is a $\mathbb{G}_m$-torsor, and $\beta\colon (\phi, a_1, a_2, a_3, b) \mapsto (a_1, a_2, a_3, b)$ is a vector bundle. Twisting by P, we obtain the following diagram

$$\begin{tikzcd} {}^P E_0 \arrow[r, hook] \arrow[d, "\text{rational}"'] & {}^P E \arrow[drr, "\text{rational}"] & & \\ \mathbb{P}(E) & & {}^P(\mathbb{P}(V)^3 \times \mathbb{P}(V))_0 \arrow[r, hook] & {}^P(\mathbb{P}(V)^3 \times \mathbb{P}(V)) \arrow[d, "\text{rational}"'] \\ & & & \mathrm{Spec}(F). \end{tikzcd}$$

We will now justify the rationality assertions in this diagram. First note that since β is an S_3-equivariant vector bundle, ${}^P E_0$ is rational over ${}^P(\mathbb{P}(V)^3 \times \mathbb{P}(V))_0$ by Lemma 2.1(d). The rationality of ${}^P E_0$ over $\mathbb{P}(E)$ follows from Proposition 2.6(b), applied to the commuting actions of S_3 and $\mathbb{G}_m$ on E_0 in the same way as in the previous section. The rationality of ${}^P(\mathbb{P}(V)^3 \times \mathbb{P}(V)) = R_{A/F}(\mathbb{P}(V_A)) \times \mathbb{P}(V)$ over F follows from Lemma 2.3(d). (Here A/F is the degree 3 étale algebra associated to the S_3-torsor $P \to \mathrm{Spec}(F)$, as above.)

Finally, the horizontal maps in the above diagram are open immersions by Lemma 2.1(b). Claim 2 now follows from this diagram. This completes the proof of Theorem 7.1. $\square$

8. Forms of $\overline{M}_{3,n}$

In this section we will prove the following.

Theorem 8.1. *Let F be a field of characteristic $\neq 2$ and $1 \leqslant n \leqslant 14$ be an integer. Then every F-form of $\overline{M}_{3,n}$ is stably rational.*

Recall that every non-hyperelliptic curve of genus 3 can is embedded, via its canonical linear system, as a smooth curve of degree 4 in $\mathbb{P}^2$. Conversely, every smooth curve of degree 4 in $\mathbb{P}^2$ is a canonically embedded curve of genus 3; see [Ha77, Example IV.5.2.1].

Let V be a 3-dimensional F-vector space and $W_4 := \mathrm{S}^4(V^*)$ be the 15-dimensional space of homogeneous polynomials of degree 4 on V. For $1 \leqslant n \leqslant 14$, let $\mathbb{P}(V)^n_0$ be a S_n-invariant dense open subset of $\mathbb{P}(V)^n$ consisting of n-tuples $(p_1, \dots, p_n)$ which impose n independent conditions on quartic polynomials. Now consider the rank $15 - n$ subbundle $E \to \mathbb{P}(V)^n_0$ of the rank 15 trivial bundle $W_4 \times \mathbb{P}(V)^n_0 \to \mathbb{P}(V)^n_0$ consisting of tuples $(\phi, p_1, \dots, p_n)$ such that $\phi(p_i) = 0$ for $i = 1, \dots, n$. Denote the associated projective bundle over $\mathbb{P}(V)^n_0$ by $\mathbb{P}(E)$.

Lemma 8.2. *(a) $M_{3,n}$ is S_n-equivariantly birationally isomorphic to $\mathbb{P}(E)/\mathrm{PGL}(V)$.*
(b) The $\mathrm{PGL}(V)$-action on $\mathbb{P}(E)$ is generically free.

Proof. (a) Let $f\colon \mathbb{P}(E) \dashrightarrow M_{3,n}$ be the S_n-equivariant rational map which associates to a point $([\phi], p_1, \ldots, p_n)$ the quartic curve cut out by $\phi \in W_4$ with n marked points $p_1, \ldots, p_n$. This map factors through $\mathbb{P}(E)/\mathrm{PGL}(V)$. Note that the actions of $\mathrm{PGL}(V)$ and S_n commute. Denote the resulting S_n-equivariant rational map by $\overline{\phi}\colon \mathbb{P}(E)/\mathrm{PGL}(V) \dashrightarrow M_{3,n}$.

To show that $\overline{\phi}$ is a birational isomorphism, we will construct the inverse map as follows. Let C be a smooth non-hyperelliptic curve of genus 3, with n marked points $x_1, \ldots, x_n$. The canonical linear system $H^0(C, \mathcal{O}_C(K))$ and V are both 3-dimensional F-vector spaces. Identifying $H^0(C, \mathcal{O}_C(K))$ with V^*, we obtain a canonical embedding $h\colon C \to \mathbb{P}(V)$ as a plane quartic. Set $p_i := h(x_i)$. If $0 \neq \phi \in W_4$ is a quartic polynomial vanishing on $h(C)$, then $([\phi], p_1, \ldots, p_n) \in \mathbb{P}(E)$. Note that $([\phi], p_1, \ldots, p_n) \in \mathbb{P}(E)$ depends on the isomorphism of F-vector spaces we chose between $H^0(C, \mathcal{O}_C(K))$ with V^*. On the other hand, the image of $([\phi], p_1, \ldots, p_n)$ in $\mathbb{P}(E)/\mathrm{PGL}(V)$ does not. This gives rise to a rational map $\overline{h}\colon M_{3,n} \dashrightarrow \mathbb{P}(E)/\mathrm{PGL}(V)$ which takes $(C, x_1, \ldots, x_n)$ to $([\phi], p_1, \ldots, p_n)$ as above.

It remains to check that $\overline{f}\colon \mathbb{P}(E)/\mathrm{PGL}(V) \dashrightarrow M_{3,n}$ and $\overline{h}\colon M_{3,n} \dashrightarrow \mathbb{P}(E)/\mathrm{PGL}(V)$ are mutually inverse rational maps. To see that

$$\overline{f}\,\overline{h}\colon M_{3,n} \dashrightarrow M_{3,n}$$

is the identity map, note that after $\overline{h}$ embeds $(C, x_1, \ldots, x_n)$ into $\mathbb{P}(V) = \mathbb{P}^2$ as a canonical curve and $\overline{h}$ "forgets" this embedding and returns $(C, x_1, \ldots, x_n)$ as an abstract curve with n marked points. On the other hand, $\overline{h}\,\overline{f}\colon \mathbb{P}(E)/\mathrm{PGL}(V) \dashrightarrow \mathbb{P}(E)/\mathrm{PGL}(V)$ is the identity map, because the canonical linear system on C is $H^0(C, \mathcal{O}_C(1))$; see [Ha77, Example IV.5.2.1]. This completes the proof of Lemma 8.2.

(b) is, once again, a consequence of Proposition 2.8, as in the proof of Lemma 4.2(a). □

Proof of Theorem 8.1. By Corollary 2.10, every F-form of $\overline{M}_{3,n}$ is isomorphic to ${}^P\overline{M}_{3,n}$ for some S_n-torsor $P \to \mathrm{Spec}(F)$. By Lemma 8.2(a), ${}^P\overline{M}_{3,n}$ is birationally isomorphic to ${}^P\mathbb{P}(E)/\mathrm{PGL}(V)$. Thus it remains to show that ${}^P\mathbb{P}(E)/\mathrm{PGL}(V)$ is stably rational over F. We will fix the S_n-torsor $P \to \mathrm{Spec}(F)$ for the rest of the proof.

Claim 1: ${}^P\mathbb{P}(E)$ is rational over ${}^P\mathbb{P}(E)/\mathrm{PGL}(V)$.

We will deduce Claim 1 from Proposition 3.6(b), with $X = {}^P\mathbb{P}(E)$, $G = \mathrm{PGL}(V)$ and $Y = \mathbb{P}(W_4)$. To see that Proposition 3.6 applies in this situation, note that (i) the group $\mathrm{PGL}(V)$ is clearly rational, (ii) the $\mathrm{PGL}(V)$-action on ${}^P\mathbb{P}(E)$ is generically free by Lemma 8.2(b), (iii) the PGL_n-action on Y is anti-versal by Lemma 3.7 with $W = W_4$, $m = 1$, $n = \dim(V) = 3$ and $d = 4$. Moreover, (iv) there exists a $\mathrm{PGL}(V)$-equivariant rational map ${}^P\mathbb{P}(E) \dashrightarrow Y = \mathbb{P}(W_4)$. To construct it, twist the natural projection $\mathbb{P}(E) \to \mathbb{P}(W_2)$ by P. The proof of Claim 1 is now complete.

Claim 2: ${}^P\mathbb{P}(E)$ is stably rational over F.

To prove Claim 2, let E_0 be the dense open subset obtained from E by removing the zero section. That is, E_0 consists of tuples $(\phi, p_1, \ldots, p_n) \in E$ such that $0 \neq \phi \in W_3$.

Then we have a diagram of S_n-equivariant maps

$$\begin{tikzcd} E_0 \arrow[r, hook, "i"] \arrow[d, "\alpha"'] & E \arrow[dr, "\beta"] & & \\ \mathbb{P}(E) & & \mathbb{P}(V)^n_0 \arrow[r, hook, "j"] & \mathbb{P}(V)^n \end{tikzcd}$$

The horizontal maps i and j are open immersions, $\alpha\colon (\phi, p_1, \dots, p_n) \mapsto ([\phi], p_1, \dots, p_n)$ is a $\mathbb{G}_m$-torsor, and $\beta\colon (\phi, p_1, \dots, p_n) \mapsto (p_1, \dots, p_n)$ is a vector bundle. Twisting by P, we obtain the following diagram

$$\begin{tikzcd} {}^P E_0 \arrow[r, hook] \arrow[d, "\text{rational}"'] & {}^P E \arrow[dr, "\text{rational}"] & & \\ {}^P\mathbb{P}(E) & & {}^P(\mathbb{P}(V)^n_0) \arrow[r, hook] & {}^P(\mathbb{P}(V)^n) \arrow[d, "\text{rational}"'] \\ & & & \mathrm{Spec}(F). \end{tikzcd}$$

Once again, the horizontal maps are open immersions by Lemma 2.1(b) and the rationality of ${}^P E$ over ${}^P\mathbb{P}(V)^n_0$ follows from Lemma 2.1(d). To show that ${}^P E_0$ is rational over ${}^P\mathbb{P}(E)$, we apply Proposition 2.6(b) to the commuting actions of $G = \mathrm{S}_n$ and $\mathbb{G}_m$ on E_0, as in the previous section. The rationality of ${}^P(\mathbb{P}(V)^n)$ over F by Lemma 2.3(d), since ${}^P(\mathbb{P}(V)^n) = R_{A/F}(\mathbb{P}(V))$, where A/F is the étale algebra associated to the S_n-torsor $P \to \mathrm{Spec}(F)$.

Claim 2 now follows from the above diagram. This completes the proof of Theorem 7.1. □

9. Forms of $\overline{M}_{4,n}$

In this section we will prove the following theorem.

Theorem 9.1. *Let F be a field of characteristic $\neq 2$ and $1 \leqslant n \leqslant 9$ be an integer. Then every F-form of $\overline{M}_{4,n}$ is stably rational.*

Our proof will be based on the fact that every non-hyperelliptic curve of genus 4 can is embedded, via its canonical linear system, as a degree 6 curve in $\mathbb{P}^3$. This embedded canonical curve is a complete intersection of a unique irreducible quadric surface Q and an irreducible cubic surface S in $\mathbb{P}^3$. Moreover, the cubic polynomial s which cuts out S, is uniquely determined up to replacing s by $s' = \alpha s + lq$, where $\alpha \in F^*$ is a non-zero constant, and l is a linear form. Conversely, any irreducible non-singular curve in $\mathbb{P}^3$, which is a complete intersection of a quadric surface and a cubic surface, is a canonically embedded curve of genus 4. For proofs of these assertions, see [Ha77, Example IV.5.2.2].

Let V be a 4-dimensional vector space, $W_i := \mathrm{Sym}^i(V^*)$ be the $\binom{3+i}{3}$-dimensional vector space of homogeneous polynomials of degree i on V.

Lemma 9.2. *Assume $n \leqslant 9$. There exists a $\mathrm{S}_n \times \mathrm{PGL}(V)$-invariant dense open subset $\mathbb{P}(V)^n_0$ of $\mathbb{P}(V)^n$ such that for every $(p_1, \dots, p_n) \in \mathbb{P}(V)^n_0$,* *(i) $p_1, \dots, p_n$ impose independent conditions on quadric hypersurfaces in $\mathbb{P}(V)$,*

(ii) $p_1, \dots, p_n$ impose independent conditions on cubic hypersurfaces in $\mathbb{P}(V)$, and

(iii) there is a smooth quadric hypersurface $Q \subset \mathbb{P}(V)$ and a smooth cubic hypersurface $S \subset \mathbb{P}(V)$ such that $C := Q \cap S$ is a complete intersection smooth irreducible curve passing through $p_1, \dots, p_n$.

Proof. (i) and (ii) are clearly open conditions in $\mathbb{P}(V)^n$, invariant under the natural $\mathrm{PGL}(V) \times \mathrm{S}_n$-action. Since $\dim(W_2) = 10$, $\dim(W_3) = 20$ and we are assuming that $n \leqslant 9$, there exists a $\mathrm{S}_n \times \mathrm{PGL}(V)$-invariant dense open subset $E_n \subset \mathbb{P}(V)^n$ such that (i) and (ii) hold for every $(p_1, \dots, p_n)$ in U_n.

Now consider the closed subvariety I of $W_2 \times W_3 \times U_n$ consisting of tuples $(q, s, p_1, \dots, p_n)$ such that $q(p_i) = s(p_i) = 0$ for every $i = 1, \dots, n$, along with the natural projections

$$\begin{tikzcd} & I \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ W_2 \times W_3 & & U_n \arrow[r, hook] & \mathbb{P}(V)^n \end{tikzcd}$$

given by $\pi_1(q, s, p_1, \dots, p_n) \mapsto (q, s)$ and $\pi_1(q, s, p_1, \dots, p_n) \mapsto (p_1, \dots, p_n)$. Note that π_2 is a vector bundle over U_n; in particular, I is irreducible. Note also that both π_1 and π_2 are invariant under the commuting natural $\mathrm{GL}(V)$- and S_n-actions. (Here S_n acts trivially on W_2 and W_3.) By Bertini's theorem, there is a $\mathrm{GL}(V)$-invariant dense open subset $(W_2 \times W_3)_0$ consisting of pairs (q, s) such that the quadric surface Q given by $q = 0$ and the cubic surface S given by $s = 0$ intersect transversely, in a smooth irreducible curve in $\mathbb{P}(V) = \mathbb{P}^3$. The preimage $I_0 := \pi_1^{-1}(W_2 \times W_3)$ is a dense open $\mathrm{S}_n \times \mathrm{GL}(V)$-invariant subset of I. The image $\pi_2(I_0)$ contains a dense subset $\mathrm{S}_n \times \mathrm{PGL}(V)$-invariant subset of $\mathbb{P}(V)^n$. This is our desired open subset $(\mathbb{P}(V)^n)_0$. $\square$

From now on, let us assume that $n \leqslant 9$, so that we can apply Lemma 9.2. Set

$$E_2 := \{(\phi, p_1, \dots, p_n) \mid \phi(p_1) = \dots = \phi(p_n) = 0\} \subset W_2 \times \mathbb{P}(V)_0^n .$$

We will view E_2 as a rank $10 - n$ subbundle of the trivial bundle over $\mathbb{P}(V)_0^n$ and will denote the associated projective bundle over $\mathbb{P}(V)_0^n$ by $\mathbb{P}(E_2)$. Let $\Lambda_3 \to \mathbb{P}(E_2)$ be the rank $20 - n$ vector bundle whose fiber over $([q], p_1, \dots, p_n)$ consist of cubic forms $s \in W_3$ such that $s(p_1) = \dots = s(p_n) = 0$. Set $\Lambda_{1.2}$ to be the rank 4 vector subbundle of Λ_3 whose fiber over $([q], p_1, \dots, p_n)$ consist of cubic forms $s \in W_3$ of the form $s = l \cdot q$ as l ranges over $W_1 = V^*$. All of these vector bundles are equivariant with respect to the natural commuting actions of $\mathrm{GL}(V)$ (by coordinate changes in V) and S_n (by permuting the points $p_1, \dots, p_n$).

Lemma 9.3. *Assume $1 \leq n \leqslant 9$. Let $\Lambda_3/\Lambda_{1,2}$ denote the quotient vector bundle of rank $16 - n$ over $\mathbb{P}(E_2)$, and $\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ denote the associated projective bundle. Then*

(a) $M_{4,n}$ is S_n-equivariantly birationally isomorphic to $\mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V)$.

(b) The $\mathrm{PGL}(V)$-action on $\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ is generically free.

Proof. (a) Consider the natural rational S_n-equivariant map

$$\Lambda_3 \dashrightarrow M_{4,n}$$

which takes a point $(q, s, p_1, \dots, p_n) \in \Lambda_3$ to the smooth irreducible curve $C \subset \mathbb{P}(V)$ of degree 6 obtained by intersecting the quadric surface $\{q = 0\}$ and the cubic surface $\{s = 0\}$. Here $(p_1, \dots, p_n)$ denotes a point of $(\mathbb{P}(V)^n)_0$, q is a quadratic polynomial vanishing at $p_1, \dots, p_n$, and s is a cubic polynomial vanishing at $p_1, \dots, p_n$. For a point of Λ_3 in general position C is a curve of genus 4 in $\mathbb{P}(V) = \mathbb{P}^3$, and $p_1, \dots, p_n \in C$. This gives rise to a rational map $\Lambda_3 \dashrightarrow M_{4,n}$ taking $(q, s, p_1, \dots, p_n)$ to the isomorphism class of $(C, p_1, \dots, p_n)$. This map descends to a rational map $f\colon \mathbb{P}(\Lambda_3/\Lambda_{1,2}) \dashrightarrow M_{4,n}$. Moreover, f is $\mathrm{S}_n \times \mathrm{PGL}(V)$-equivariant, and factors through the rational quotient $\mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V)$, yielding an S_n-equivariant rational map

$$\overline{f}\colon \mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V) \dashrightarrow M_{4,n}. \tag{9.4}$$

In order to complete the proof of the lemma, it remains to show that $\overline{f}$ is a birational isomorphism. We will do so by constructing the inverse

$$\overline{h}\colon M_{4,n} \dashrightarrow \mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V)\,. \tag{9.5}$$

Let C be a non-hyperelliptic curve of genus 4 with n marked points $x_1, \dots, x_n$. The canonical linear system $H^0(C, \mathcal{O}_C(K))$ and V are both 3-dimensional F-vector spaces. Identifying $H^0(C, \mathcal{O}_C(K))$ with V^*, we obtain a canonical embedding $h\colon C \to \mathbb{P}(V)$. Let $p_i := h(x_i)$, $q \in W_2$ a homogeneous quadratic form on V vanishing on $h(C)$, and $0 \neq s \in W_3$ be a cubic form vanishing on $h(C)$. The discussion at the beginning of this section shows that h uniquely determines the point $([s], q, p_1, \dots, p_n)$ in $\mathbb{P}(\Lambda_3/\Lambda_{1,2})$. Moreover, the projection of this point to $\mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V)$ depends only on $(C, x_1, \dots, x_n)$ and not on the isomorphism of F-vector spaces we chose between $H^0(C, \mathcal{O}_C(K))$ and V^*. This gives rise to a rational map (9.5) which takes $(C, x_1, \dots, x_n)$ to $([s], q, p_1, \dots, p_n)$.

It remains to show that the S_n-equivariant rational maps (9.4) and (9.5) are mutually inverse. To see that the composition

$$\bar{\phi}\,\overline{h}\colon M_{4,n} \dashrightarrow M_{4,n}$$

is the identity map, note that $\overline{h}$ embeds $(C, x_1, \dots, x_n)$ into $\mathbb{P}(V) = \mathbb{P}^3$ as a canonical curve and $\overline{\phi}$ "forgets" this embedding and returns $(C, x_1, \dots, x_n)$ as an abstract curve with n marked points. On the other hand,

$$\overline{h}\,\overline{f}\colon \mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V) \dashrightarrow \mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V)$$

is the identity map, because the canonical linear system on a complete intersection of a smooth quadric surface and a smooth cubic surface in $\mathbb{P}^3$ is $H^0(C, \mathcal{O}_C(1))$; see [Ha77, Example IV.5.2.2]. This completes the proof of Lemma 9.3.

(b) We argue as in the proof of Lemma 4.2(b). The stabilizer H of a point of $\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ represented by $(s, q, p_1, \dots, p_n)$ naturally embeds in $\mathrm{Aut}(C, p_1, \dots, p_n)$, where C is the curve in $\mathbb{P}(V) = \mathbb{P}^3$ given by $q = s = 0$. For $(s, q, p_1, \dots, p_n)$ in general position in Λ_3, C is a smooth irreducible curve of genus 4. Moreover, since the rational map f constructed in part (a) is dominant, we may assume that $\mathrm{Aut}(C, p_1, \dots, p_n) = \{1\}$; see Proposition 2.8. Thus $H = \{1\}$, as desired. □

Proof of Theorem 9.1. By Corollary 2.10, every F-form of $\overline{M}_{4,n}$ is isomorphic to ${}^{P}\overline{M}_{4,n}$ for some S_n-torsor $P \to \mathrm{Spec}(F)$. On the other hand, in view of Lemma 9.3(a), it suffices to show that the quotient variety ${}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V)$ is stably rational. We will fix the S_n-torsor $P \to \mathrm{Spec}(F)$ for the rest of the proof.

Claim 1: ${}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ is rational over ${}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2})/\mathrm{PGL}(V)$.

We will deduce Claim 1 from Proposition 3.6(b), with $X = {}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ and $G = \mathrm{PGL}(V)$ and $Y = \mathrm{Gr}(5, W_3)$. To prove that Proposition 3.6(b) applies in this situation, note that (i) the group $\mathrm{PGL}(V) \simeq \mathrm{PGL}_4$ is rational, (ii) The $\mathrm{PGL}(V)$-action on ${}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ is generically free by Lemma 9.3(b), and (iii) the $\mathrm{PGL}(V)$-action on Y is anti-versal by Lemma 3.7 with $W = W_3$, $n = \dim(V) = 4$, $d = 3$ and $m = 5$. Finally, (iv) there exists $\mathrm{PGL}(V)$-equivariant rational map ${}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2}) \to Y = \mathrm{Gr}(5, W_3)$. To construct this rational map, start with $\alpha\colon \Lambda_3 \dashrightarrow \mathrm{Gr}(5, W_3)$ taking $(q, s, p_1, \ldots, p_n)$ to the subspace of W_3 spanned by the cubic forms s and $l \cdot q$, as l ranges over V^*. For $(q, s, p_1, \ldots, p_n) \in \Lambda_3$ in general position this subspace of W_3 will be 5-dimensional. One readily checks that α descends to a $\mathrm{PGL}(V) \times \mathrm{S}_n$-equivariant rational map

$$\overline{\alpha}\colon \mathbb{P}(\Lambda_3/\Lambda_{1,2}) \to \mathrm{Gr}(5, W_3)\,,$$

where S_n acts trivially on $\mathrm{Gr}(5, W_3)$. Twisting by P, we obtain a desired $\mathrm{PGL}(V)$-equivariant rational map

$${}^P\overline{\alpha}\colon\ {}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2}) \dashrightarrow \mathrm{Gr}(5, W_3)\,,$$

This completes the proof of Claim 1.

Claim 2: ${}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ is stably rational over F.

To prove Claim 2, let $(\Lambda_3/\Lambda_{1,2})_0$ be the dense open subset obtained from $\Lambda_3/\Lambda_{1,2}$ by removing the zero section. Then we have a diagram of S_n-equivariant maps

$$\begin{tikzcd}
(\Lambda_3/\Lambda_{12})_0 \arrow[r, hook] \arrow[ddd, "\alpha"] & \Lambda_3/\Lambda_{1,2} \arrow[dd, "\beta"] & & & \\
& & (E_2)_0 \arrow[r, hook] \arrow[dl, "\gamma"] & E_2 \arrow[dd, "\delta"] & \\
& \mathbb{P}(E_2) & & & \\
\mathbb{P}(\Lambda_3/\Lambda_{1,2}) & & & \mathbb{P}(V)^n_0 \arrow[r, hook] & \mathbb{P}(V)^n
\end{tikzcd}$$

Here the horizontal maps are open immersions, α and γ are $\mathbb{G}_m$-torsors, and β and δ are vector bundles. Twisting by P and applying Lemma 2.1(d) and Proposition 2.6(b), as we did in the previous sections, we obtain the following diagram

$$\begin{tikzcd}
{}^P(\Lambda_3/\Lambda_{12})_0 \arrow[r, hook] \arrow[dddd, "\text{rational}"'] & {}^P(\Lambda_3/\Lambda_{1,2}) \arrow[dd, "\text{rational}"'] & & & \\
& & {}^P(E_2)_0 \arrow[r, hook] \arrow[dl, "\text{rational}"] & {}^P E_2 \arrow[ddd, "\text{rational}"] & \\
& {}^P\mathbb{P}(E_2) & & & \\
& & & & \\
{}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2}) & & & {}^P(\mathbb{P}(V)^n_0) \arrow[r, hook] & {}^P(\mathbb{P}(V)^n) \arrow[d, "\text{rational}"] \\
& & & & \mathrm{Spec}(F).
\end{tikzcd}$$

Here the horizontal maps are open immersions by Lemma 2.1(b), and the rationality assertions in this diagram are established in the same way as in the proof of Claim 2 in the previous section. In particular, ${}^P(\Lambda_3/\Lambda_{1,2})$ is rational over ${}^P\mathbb{P}(E_2)$ and ${}^P E_2$ is rational over ${}^P(\mathbb{P}(V)^n)$ by Lemma 2.1(d). The rationality of ${}^P(\Lambda_3/\Lambda_{1,2})_0$ over ${}^P\mathbb{P}(\Lambda_3/\Lambda_{1,2})$ and of ${}^P(E_2)_0$ over ${}^P\mathbb{P}(E_2)$ follows from applying Proposition 2.6(b) to the commuting actions of $G = \mathrm{S}_n$ and $H = \mathbb{G}_m$. Finally, the rationality of ${}^P(\mathbb{P}(V)^n)$ over F follows from Lemma 2.3(d), since ${}^P(\mathbb{P}(V)^n) = R_{A/F}(\mathbb{P}(V)_A)$, where A/F is the étale algebra associated to the S_n-torsor $P \to \mathrm{Spec}(F)$. Claim 2 now follows from the above diagram, and the proof of Theorem 9.1 is complete. $\square$

10. Forms of $\overline{M}_{5,n}$

In this section we will prove the following.

Theorem 10.1. *Let F be a field of characteristic $\neq 2$ and $1 \leqslant n \leqslant 12$ be an integer. Then every F-form of $\overline{M}_{5,n}$ is stably rational.*

Our proof will be based on the fact that a general curve of genus 5 (and more precisely, any smooth curve of genus 5 with no g^1_3) can is embedded, via its canonical linear system, as a complete intersection of three quadric hypersurfaces in $\mathbb{P}^4$. Conversely, any irreducible non-singular curve, which is a complete intersection of three quadrics in $\mathbb{P}^4$, is a canonically embedded curve of genus 5. For proofs of these assertions, see [Ha77, Example IV.5.5.3].

Let V be a 5-dimensional F-vector space and $W_2 := \mathrm{S}^2(V^*)$ be the 15-dimensional space of homogeneous polynomials of degree 2 on V. For $1 \leqslant n \leqslant 14$, let $(\mathbb{P}(V)^n)_0$ be a S_n-invariant dense open subset of $\mathbb{P}(V)^n$ consisting of n-tuples $(p_1, \ldots, p_n)$ which impose n independent conditions on qudratic polynomials. Now consider the vector bundle $E \to (\mathbb{P}^4)^n_0$ of rank $15 - n$ whose fiber of E over $(p_1, \ldots, p_n)$ consists of homogeneous quadratic polynomials vanishing at $p_1, \ldots, p_n$. Denote the Grassmannian bundle of 3-dimensional subspaces associated to E by $\mathrm{Gr}(3, E)$. The natural action of $\mathrm{PGL}(V)$ on $\mathbb{P}(V)$ induces $\mathrm{PGL}(V)$-actions on $\mathbb{P}(V)^n$, $\mathbb{P}(V)^n_0$, E and $\mathrm{Gr}(3, E)$.

Lemma 10.2. *(a) $M_{5,n}$ is S_n-equivariantly birationally isomorphic to* $\mathrm{Gr}(3,E)/\mathrm{PGL}(V)$.

(b) The $\mathrm{PGL}(V)$*-action on* $\mathrm{Gr}(3,E)$ *is generically free.*

Proof. (a) Let $f\colon \mathrm{Gr}(3,E) \dashrightarrow M_{5,n}$ be the S_n-equivariant rational map which associates to a point $(L, p_1, \ldots, p_n)$ the curve $\{\phi = 0 \mid \phi \in L\}$ with n marked points $p_1, \ldots, p_n$. Here L denotes a 3-dimensional subspace of $E_{(p_1,\ldots,p_n)}$. Since $f(L, p_1, \ldots, p_n)$ remains unchanged when we translate $(L, p_1, \ldots, p_n)$ by an element of $\mathrm{PGL}(V)$, f factors through $\mathrm{Gr}(3,E)/\mathrm{PGL}(V)$. Denote the resulting rational map by $\overline{f}\colon \mathrm{Gr}(3,E)/\mathrm{PGL}(V) \dashrightarrow M_{3,n}$. Note that the actions of $\mathrm{PGL}(V)$ and S_n on $\mathrm{Gr}(3,E)$ commute. Hence, the rational map $\overline{\phi}$ is S_n-equivariant.

To show that $\overline{f}$ is a birational isomorphism, we will construct the inverse map as follows. Let C be a smooth curve of genus 5 with no g^1_3, equipped with n marked points $x_1, \ldots, x_n$. Identifying V^* with the canonical linear system on C (both are 5-dimensional F-vector spaces) we obtain a canonical embedding $h\colon C \to \mathbb{P}(V)$. Set $p_i := h(x_i)$. The space $L := H^0(\mathbb{P}(V), \mathcal{I}_{h(C)}(2))$ of global sections of the ideal sheaf $\mathcal{I}_{h(C)}(2)$ is 3-dimensional. The resulting point $(L, p_1, \ldots, p_n)$ of $\mathrm{Gr}(3,E)$ depends on the choice of isomorphism between V^* and $H^0(C, O_C(K))$ but the projection of $(L, p_1, \ldots, p_n)$ to $\mathrm{Gr}(3,E)/\mathrm{PGL}(V)$ does not. This gives rise to a rational map $\overline{h}\colon M_{5,n} \dashrightarrow \mathrm{Gr}(3,E)/\mathrm{PGL}(V)$ which takes $(C, x_1, \ldots, x_n)$ to $(L, p_1, \ldots, p_n)$, as above.

Now observe that the composition $\overline{f}\,\overline{h}\colon M_{5,n} \dashrightarrow M_{5,n}$ is the identity map. Indeed, $\overline{h}$ embeds $(X, x_1, \ldots, x_n)$ into $\mathbb{P}(V)$ as a canonical curve, and $\overline{h}$ "forgets" this embedding and returns $(X, x_1, \ldots, x_3)$ as an abstract curve with n marked points. On the other hand, $\overline{h}\,\overline{f}\colon \mathrm{Gr}(3,E)/\mathrm{PGL}(V) \dashrightarrow \mathrm{Gr}(3,E)/\mathrm{PGL}(V)$ is the identity map, because the canonical linear system on a complete intersection X of three smooth quadric hypersurfaces in $\mathbb{P}^4$ is $H^0(X, \mathcal{O}_X(1))$; see [Ha77, Example IV.5.5.3]. This completes the proof of Lemma 10.2.

(b) is deduced from Proposition 2.8 (with $g = 5$) in the same way as in the proof of Lemma 4.2(b). □

Proof of Theorem 10.1. By Corollary 2.10, every F-form of $\overline{M}_{5,n}$ is isomorphic to ${}^P\overline{M}_{5,n}$ for some S_n-torsor P over $\mathrm{Spec}(F)$. By Lemma 10.2(b), ${}^P\overline{M}_{5,n}$ is birationally isomorphic to ${}^P\,\mathrm{Gr}(3,E)/\mathrm{PGL}(V)$. Thus it remains to show that ${}^P\,\mathrm{Gr}(3,E)/\mathrm{PGL}(V)$ is stably rational over F. We fix the $\mathrm{PGL}(V)$-torsor $P \to \mathrm{Spec}(F)$ for the rest of the proof.

Claim 1: ${}^P\,\mathrm{Gr}(3,E)$ is rational over ${}^P\,\mathrm{Gr}(3,E)/\mathrm{PGL}(V)$.

We will deduce Claim 1 from Proposition 3.6(b), with $X = {}^P\,\mathrm{Gr}(3,E)$, $G = \mathrm{PGL}(V)$ and $Y = \mathrm{Gr}(3,W_2)$. To check that Proposition 3.6 applies in this situation, note that (i) the group $\mathrm{PGL}(V) \simeq \mathrm{PGL}_5$ is rational, (ii) the $\mathrm{PGL}(V)$-action on $\mathrm{Gr}(3,E)$ is generically free by Lemma 10.2(a), (iii) the $\mathrm{PGL}(V)$-action on Y is anti-versal by Lemma 3.7 with $W = W_2$, $n = \dim(V) = 5$, $d = 2$ and $m = 3$, and (iv) there exists a $\mathrm{PGL}(V)$-equivariant map ${}^P\,\mathrm{Gr}(3,E) \to Y = \mathbb{P}(3,W_2)$. To construct this map, recall that E is, by definition, a subbundle of the trivial bundle over $\mathbb{P}(V)^n_0$ with fiber W_2. Twisting the tautological $\mathrm{PGL}(V) \times \mathrm{S}_n$-equivariant morphism

$$\mathrm{Gr}(3,E) \to \mathrm{Gr}(3,W_2)$$

by P, we obtain a desired $\mathrm{PGL}(V)$-equivariant morphism

$${}^P\,\mathrm{Gr}(3,E) \to {}^P\,\mathrm{Gr}(3,W_2) = \mathrm{Gr}(3,W_2)\,.$$

This completes the proof of Claim 1.

Claim 2: ${}^P\operatorname{Gr}(3,E)$ is stably rational over F.

To prove Claim 2, let $(E^3)_0$ be a dense open subset of $E^3 := E \times_{\mathbb{P}(V)_0^n} E \times_{\mathbb{P}(V)_0^n} E$ whose elements are tuples $(q_1, q_2, q_3, p_1, \dots, p_n)$ such that q_1, q_2 and $q_3 \in W_2(V)$ are linearly independent over F. Consider the following diagram of S_n-equivariant morphisms

$$\begin{tikzcd}
(E^3)_0 \arrow[r, hook, "i"] \arrow[d, "\alpha"] & E^3 \arrow[dr, "\beta"] & & \\
\operatorname{Gr}(3,E) & & \mathbb{P}(V)_0^n \arrow[r, hook, "j"] & \mathbb{P}(V)^n
\end{tikzcd}$$

The horizontal maps i and j are open immersions,

$$\alpha\colon (q_1, q_2, q_3, p_1, \dots, p_n) \mapsto (\operatorname{Span}(q_1, q_2, q_3), p_1, \dots, p_n)$$

is a GL_3-torsor, and $\beta\colon (q_1, q_2, q_3, p_1, \dots, p_n) \mapsto (p_1, \dots, p_n)$ is a vector bundle. Twisting by P, and applying Lemma 2.1, Lemma 2.3 and Proposition 2.6(b), as in the previous sections, we obtain the following diagram

$$\begin{tikzcd}
{}^P(E^3)_0 \arrow[r, hook] \arrow[d, "\text{rational}"] & {}^P E^3 \arrow[dr, "\text{rational}"] & & \\
\operatorname{Gr}(3,E) & & {}^P(\mathbb{P}(V)_0^n) \arrow[r, hook] & {}^P(\mathbb{P}(V)^n) \arrow[d, "\text{rational}"] \\
& & & \operatorname{Spec}(F).
\end{tikzcd}$$

This completes the proof of Claim 2 and thus of Theorem 10.1. □

Acknowledgements

We are grateful to A. Massarenti, M. Mella, M. Talpo and A. Vistoli for helpful comments.

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