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Density Inequalities for Sets of Multiples

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For finite sets $A, B \subset \mathbb{N}$, the set of positive integers, consider the set of least common multiples $[A, B] = \{[a, b]: a \in A, b \in B\}$, the set of largest common divisors $(A, B) = \{(a, b): a \in A, b \in B\}$, the set of products $A \times B = \{a \cdot b: a \in A, b \in B\}$, and the sets of their multiples $M(A) = A \times \mathbb{N}$, $M(B)$, $M[A, B]$, $M(A, B)$, and $M(A \times B)$, resp. Our discoveries are the inequalities

$$\mathbf{d}M(A, B)\,\mathbf{d}M[A, B] \geqslant \mathbf{d}M(A) \cdot \mathbf{d}M(B) \geqslant \mathbf{d}M(A \times B),$$

where $\mathbf{d}$ denotes the asymptotic density. The first inequality is by the factor $\mathbf{d}M(A, B)$ sharper than Behrend's well-known inequality. This in turn is a generalisation of an earlier inequality of Rohrbach and Heilbronn, which settled a conjecture of Hasse concerning an identity due to Direchlet. Our second inequality does not seem to have predecessors.

1. INTRODUCTION

In addition to the notions presented in the abstract we need the following. For the numbers $u, v \in \mathbb{N}$ we write $u \mid v$ if u divides v. In case $(u, v) = 1$ u and v are said to be relatively prime. If $u \leqslant v$, then we set $\langle u, v \rangle = \{u, u+1, ..., v\}$, and for any $A \in \mathbb{N}$ we set $A_n = A \cap \langle 1, n \rangle$ and $|A|$ for the cardinality of A.

The asymptotic density $\mathbf{d}A$ of A is defined by

$$\mathbf{d}A = \lim_{n \to \infty} \frac{|A_n|}{n}, \tag{1.1}$$

if the limit exists.

We associate with A the *set of multiples*

$$M(A) = \{m \in \mathbb{N}: a \mid m \text{ for some } a \in A\} \tag{1.2}$$

and the *set of non-multiples*

$$N(A) = \mathbb{N} \backslash M(A).$$

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This naturally leads to the concept of *primitive sets* (or sequences). $B \subset \mathbb{N}$ is said to be primitive, if

$$b \nmid b' \quad \text{for} \quad b, b' \in B; b \neq b'. \tag{1.3}$$

Clearly, every $A \subset \mathbb{N}$ contains a unique subset $P(A)$, which is primitive and satisfies

$$M(P(A)) = M(A). \tag{1.4}$$

Actually,

$$P(A) = \{a \in A: \nexists a' \in A, a' \neq a, \text{and } a' \mid a\}.$$

For a finite set $A \subset \mathbb{N}$ $\mathbf{d}M(A)$ exists and with the principle of inclusion-exclusion can be written in the form

$$\mathbf{d}M(A) = \sum_{a \in A} \frac{1}{a} - \sum_{a < a'} \frac{1}{[a, a']} + \sum_{a < a' < a''} \frac{1}{[a, a', a'']} - \cdots. \tag{1.5}$$

Similarly,

$$\mathbf{d}N(A) = 1 - \sum_{a \in A} \frac{1}{a} + \sum_{a < a'} \frac{1}{[a, a']} \cdots. \tag{1.6}$$

In this terminology Behrend's inequality ([4]; see also [5], [6]) takes the form

$$\mathbf{d}N(A \cup B) \geqslant \mathbf{d}N(A)\,\mathbf{d}N(B) \tag{1.7}$$

for any finite sets A, $B \subset \mathbb{N}$.

The equality holds exactly if for the primitive sets $P(A)$ and $P(B)$

$$(a, b) = 1 \quad \text{for all} \quad a \in P(A) \quad \text{and} \quad b \in P(B). \tag{1.8}$$

In conclusion we reformulate Behrend's inequality. Clearly, (1.7) is equivalent to

$$1 - \mathbf{d}M(A \cup B) \geqslant (1 - \mathbf{d}M(A))(1 - \mathbf{d}M(B)). \tag{1.9}$$

Using the identity

$$\mathbf{d}M(A \cup B) = \mathbf{d}M(A) + \mathbf{d}M(B) - \mathbf{d}(M(A) \cap M(B)) \tag{1.10}$$

we get another equivalent form

$$\mathbf{d}(M(A) \cap M(B)) \geqslant \mathbf{d}M(A) \cdot \mathbf{d}M(B). \tag{1.11}$$

Since

$$\mathbf{d}(M(A) \cap M(B)) = \mathbf{d}M[A, B], \tag{1.12}$$

we get the desired form

$$\mathbf{d}M[A, B] \geqslant \mathbf{d}M(A) \cdot \mathbf{d}M(B). \tag{1.13}$$

It was this form, which inspired us also to look for an anology to the AD-inequality [7] (and its generalisation [8]) and which led also to our first inequality stated in the abstract. In terms of Behrend's original formulation (1.7) the path to our sharper inequality is more hidden. It then takes the form

$$\mathbf{d}N(A \cup B) \geqslant \mathbf{d}N(A)\, \mathbf{d}N(B) + \mathbf{d}N(A, B)(1 - \mathbf{d}N[A, B]). \tag{1.14}$$

2. Auxiliary Results

The two lemmas of this section are used in the proofs of the inequalities. Equivalent formulations are presented, because they may be used elsewhere. We adapt the notation

$$mA = \{ma \colon a \in A\}. \tag{2.1}$$

Lemma 1. *For any finite sets A, $B \subset \mathbb{N}$ and any number $m \in \mathbb{N}$*

(i) $\mathbf{d}(M(A) \cap M(B)) \leqslant m\mathbf{d}(M(mA) \cap M(B))$

(ii) $\mathbf{d}M(mA \cup B) \leqslant (1/m)\, \mathbf{d}M(A \cup B) + ((m-1)/m)\, \mathbf{d}M(B)$

(iii) $\mathbf{d}(N(mA) \cap N(B)) \geqslant \mathbf{d}(N(A) \cap N(B)) + (m-1)\, N(B)$

(iv) $\mathbf{d}N(mA \cup B) \geqslant (1/m)\, \mathbf{d}N(A \cup B) + ((m-1)/m)\, \mathbf{d}N(B)$.

Equalities hold iff $M(mA) \cap M(B) \subset M(mB)$.

A sufficient condition for equality is $(\{m\}, B) = 1$.

Proof. (i) Obviously $(1/m)(M(mA) \cap M(B)) \supset (1/m)(M(mA) \cap M(mB)) = (M(A) \cap M(B))$ and therefore $\mathbf{d}((1/m)(M(mA) \cap M(B)) = m\mathbf{d}(M(mA) \cap M(B))) \geqslant \mathbf{d}(M(A) \cap M(B))$. Furthermore, equality holds exactly under the specified condition. Also, this condition holds, if $(\{m\}, B) = 1$, because then $[mA, B] = [mA, mB]$ and $M(mA) \cap M(B) = M[mA, B] = M[mA, mB] = M(mA) \cap M(mB)$.

(ii) Since

$$\mathbf{d}M(mA \cup B) = \mathbf{d}M(mA) + \mathbf{d}M(B) - \mathbf{d}(M(mA) \cap M(B)), \tag{2.2}$$

$$\mathbf{d}M(A \cup B) = \mathbf{d}M(A) + \mathbf{d}M(B) - \mathbf{d}(M(A) \cap M(B)) \tag{2.3}$$

we can write (ii) in the form $\mathbf{d}M(mA)+\mathbf{d}M(B)-\mathbf{d}(M(mA)\cap M(B))\leqslant (1/m)\,\mathbf{d}M(A)+(1/m)\,\mathbf{d}M(B)-(1/m)\,\mathbf{d}(M(A)\cap M(B))+((m-1)/m)\,\mathbf{d}M(B)$ or in the form $\mathbf{d}(M(A)\cap M(B))\leqslant m\mathbf{d}(M(mA)\cap M(B))+\mathbf{d}M(A)-m\,\mathbf{d}M(mA)=m\mathbf{d}(M(mA)\cap M(B))$, which is (i).

(iv) We use the equivalent formulation $1-\mathbf{d}M(mA\cup B)\geqslant (1/m)(1-\mathbf{d}M(A\cup B))+((m-1)/m)(1-\mathbf{d}M(B))$. This is obviously equivalent with (ii).

(iii) $\mathbf{d}(N(A)\cap N(B))=\mathbf{d}N(A\cup B)=1-\mathbf{d}M(A\cup B)$, and by (ii) $1-\mathbf{d}M(A\cup B)\leqslant 1-m\mathbf{d}M(mA\cup B)+(m-1)\,\mathbf{d}M(B)$.

Therefore $\mathbf{d}(N(A)\cap N(B))\leqslant 1-m(1-\mathbf{d}(N(mA)\cap N(B))+(m-1)(1-\mathbf{d}N(B))=\mathbf{d}(N(mA)\cap N(B))-(m-1)\,\mathbf{d}N(B)$, and thus (iii).

LEMMA 2. *For finite sets $C\subset A\subset \mathbb{N}$, $D\subset B\subset \mathbb{N}$ we have*

$$\mathbf{d}M[A,B]+\mathbf{d}M[C,D]\geqslant \mathbf{d}M[A,D]+\mathbf{d}M[C,B] \tag{2.4}$$

with equality if and only if $M[A,B]\subset M(C)\cup M(D)$.

Proof. The inequality (2.4) is equivalent to

$$\mathbf{d}(M(A)\cap M(B))+\mathbf{d}(M(C)\cap M(D))$$
$$\geqslant \mathbf{d}(M(A)\cap M(D))+\mathbf{d}(M(C)\cap M(B))$$

or

$$\mathbf{d}(M(A)\cap M(B))-\mathbf{d}(M(A)\cap M(D))$$
$$\geqslant \mathbf{d}(M(C)\cap M(B))-\mathbf{d}(M(C)\cap M(D)).$$

Since $(M(A)\cap M(B))\supset M(A)\cap M(D)$ and $(M(A)\cap M(B))\backslash(M(A)\cap M(D))=M(A)\cap(M(B)\backslash M(D))$ and similarly $(M(C)\cap M(B))\backslash(M(C)\cap M(D))=M(C)\cap(M(B)\backslash M(D))$, (2.4) is equivalent to $\mathbf{d}(M(A)\cap(M(B)\backslash M(D)))\geqslant \mathbf{d}(M(C)\cap(M(B)\backslash M(D)))$, and this holds, because $M(A)\supset M(C)$ and $\mathbf{d}$ is monotonic.

3. PROOF OF THE INEQUALITY $\mathbf{d}M(A,B)\,\mathbf{d}M[A,B]\geqslant \mathbf{d}M(A)\cdot\mathbf{d}M(B)$

We proceed by induction on $T=\sum_{a\in A}a+\sum_{b\in B}b$. For $T=2$ the result holds. Now, if

$$(a,b)=1 \quad \text{for all} \quad a\in A \quad \text{and all} \quad b\in B, \tag{3.1}$$

then $\mathbf{d}M(A, B) = 1$ and also by the formula (1.5)

$$\mathbf{d}M[A, B] = \mathbf{d}M(A) \cdot \mathbf{d}M(B).$$

We can assume therefore that for some prime p $A_1 = \{a \in A : p \mid a\} \neq \phi$ and $B_1 = \{b \in B : p \mid b\} \neq \phi$. Set $A_2 = A \backslash A_1$ and $B_2 = B \backslash B_1$ and observe that $A = pA'_1 \dot{\cup} A_2$, $B = pB'_1 \dot{\cup} B_2$, $(p, A_2) = 1$, and $(p, B_2) = 1$, where

$$A'_1 = \left\{\frac{a}{p} : a \in A_1\right\} \quad \text{and} \quad B'_1 = \left\{\frac{b}{p} : b \in B_1\right\}. \tag{3.2}$$

We consider

$$A' = A'_1 \cup A_2 \quad \text{and} \quad B' = B'_1 \cup B_2. \tag{3.3}$$

By Lemma 1(ii)

$$\mathbf{d}M(A) = \mathbf{d}M(pA'_1 \cup A_2) = \frac{1}{p}\mathbf{d}M(A') + \left(1 - \frac{1}{p}\right)\mathbf{d}M(A_2) \tag{3.4}$$

and also

$$\mathbf{d}M(B) = \frac{1}{p}\mathbf{d}M(B') + \left(1 - \frac{1}{p}\right)\mathbf{d}M(B_2). \tag{3.5}$$

Next we calculate $\mathbf{d}M[A, B]$ and $\mathbf{d}M(A, B)$. Now,

$$\begin{aligned}[A, B] &= [A_1, B_1] \cup [A_1, B_2] \cup [A_2, B_1] \cup [A_2, B_2] \\ &= p[A'_1, B'_1] \cup p[A'_1, B_2] \cup p[A_2, B'_1] \cup [A_2, B_2].\end{aligned}$$

Again by Lemma 1(ii)

$$\mathbf{d}M([A, B]) = \frac{1}{p}\mathbf{d}M([A', B']) + \frac{p-1}{p}\mathbf{d}M([A_2, B_2]). \tag{3.6}$$

Also,

$$\begin{aligned}(A, B) &= (A_1, B_1) \cup (A_1, B_2) \cup (A_2, B_1) \cup (A_2, B_2) \\ &= p(A'_1, B'_1) \cup (A'_1, B_2) \cup (A_2, B'_1) \cup (A_2, B_2)\end{aligned}$$ and by Lemma 1(ii)

$$\mathbf{d}M(A, B) = \frac{1}{p}\mathbf{d}M(A', B') + \frac{p-1}{p}\mathbf{d}M((A'_1, B_2) \cup (A_2, B'_1) \cup (A_2, B_2)). \tag{3.7}$$

We use the abbreviation

$$L = (A'_1, B_2) \cup (A_2, B'_1) \cup (A_2, B_2). \tag{3.8}$$

In the light of (3.4), (3.5), (3.6), (3.7), and (3.8) we have to prove that

$$\left(\frac{1}{p}\,\mathbf{d}M(A') + \frac{p-1}{p}\,\mathbf{d}M(A_2)\right)\left(\frac{1}{p}\,\mathbf{d}M(B') + \frac{p-1}{p}\,\mathbf{d}M(B_2)\right)$$
$$\leqslant \left(\frac{1}{p}\,\mathbf{d}M[A', B'] + \frac{p-1}{p}\,\mathbf{d}M[A_2, B_2]\right)$$
$$\cdot\left(\frac{1}{p}\,\mathbf{d}M(A', B') + \frac{p-1}{p}\,\mathbf{d}M(L)\right). \tag{3.9}$$

By induction hypothesis we have

$$\frac{1}{p}\,\mathbf{d}M(A')\cdot\frac{1}{p}\,\mathbf{d}M(B') \leqslant \frac{1}{p}\,\mathbf{d}M[A', B']\cdot\frac{1}{p}\,\mathbf{d}M(A', B')$$

and also

$$\frac{p-1}{p}\,\mathbf{d}M(A_2)\frac{p-1}{p}\,\mathbf{d}M(B_2) \leqslant \frac{p-1}{p}\,\mathbf{d}M[A_2, B_2]\cdot\frac{p-1}{p}\,\mathbf{d}M(A_2, B_2)$$
$$\leqslant \frac{p-1}{p}\,\mathbf{d}M[A_2, B_2]\cdot\frac{p-1}{p}\,\mathbf{d}M(L).$$

Therefore, sufficient for (3.9) is

$$\mathbf{d}M(A')\cdot\mathbf{d}M(B_2) + \mathbf{d}M(A_2)\,\mathbf{d}M(B')$$
$$\leqslant \mathbf{d}M[A', B']\,\mathbf{d}M(L) + \mathbf{d}M[A_2, B_2]\cdot\mathbf{d}M(A', B'). \tag{3.10}$$

Since by induction hypothesis

$$\mathbf{d}M(A')\,\mathbf{d}M(B_2) \leqslant \mathbf{d}M[A', B_2]\cdot\mathbf{d}M(A', B_2)$$

and also

$$\mathbf{d}M(A_2)\,\mathbf{d}M(B') \leqslant \mathbf{d}M[A_2, B']\cdot\mathbf{d}M(A_2, B')$$

sufficient for (3.10) is in turn

$$\mathbf{d}M[A', B_2]\,\mathbf{d}M(A', B_2) + \mathbf{d}M[A_2, B']\,\mathbf{d}M(A_2, B')$$
$$\leqslant \mathbf{d}M[A', B']\,\mathbf{d}M(L) + \mathbf{d}M[A_2, B_2]\cdot\mathbf{d}M(A', B'). \tag{3.11}$$

Since $\mathbf{d}M(A', B') \geqslant \mathbf{d}M(L) \geqslant \max(\mathbf{d}M(A', B_2), \mathbf{d}M(A_2, B'))$, of course, sufficient is also

$$(\mathbf{d}M[A', B_2] + \mathbf{d}M[A_2, B']) \cdot \mathbf{d}M(L) \leqslant (\mathbf{d}[A', B'] + \mathbf{d}M[A_2, B_2])\, \mathbf{d}M(L).$$

After cancellation of the factor $\mathbf{d}M(L)$ this last inequality holds by Lemma 2.

4. Proof of the Inequality $\mathbf{d}M(A)\,\mathbf{d}M(B) \geqslant \mathbf{d}M(A \times B)$

We proceed again by induction on $T = \sum_{a \in A} a + \sum_{b \in B} b$. In the case (3.1) clearly $A \times B = [A, B]$ and since here also $\mathbf{d}M[A, B] = \mathbf{d}M(A)\,\mathbf{d}M(B)$ we have even the equality $\mathbf{d}M(A)\,\mathbf{d}M(B) = \mathbf{d}M(A \times B)$.

Otherwise we define sets $A_1, B_1, A_2, B_2, A'_1, B'_1, A'$, and B' again as in Section 3. In addition to (3.4) and (3.5) we calculate now

$$A \times B = p^2 \cdot A'_1 \times B'_1 \cup pA'_1 \times B_2 \cup pA_2 \times B'_1 \cup A_2 \times B_2$$

and with Lemma 1(ii)

$$\mathbf{d}M(A \times B) = \frac{1}{p}\,\mathbf{d}M(C) + \frac{p-1}{p}\,\mathbf{d}M(A_2 \times B_2), \tag{4.1}$$

where

$$C = p \cdot A'_1 \times B'_1 \cup A'_1 \times B_2 \cup A_2 \times B'_1 \cup A_2 \times B_2. \tag{4.2}$$

We derive first an upper bound for $\mathbf{d}M(A \times B)$. Again by Lemma 1(ii)

$$\mathbf{d}M(C) \leqslant \frac{1}{p}\,\mathbf{d}M(A' \times B') + \frac{p-1}{p}\,\mathbf{d}M(D),$$

where

$$D = (A'_1 \times B_2) \cup (A_2 \times B'_1) \cup (A_2 \times B_2).$$

Consequently

$$\begin{aligned}\mathbf{d}M(A \times B) &\leqslant \frac{1}{p}\left(\frac{1}{p}\,\mathbf{d}M(A' \times B') + \frac{p-1}{p}\,\mathbf{d}M(D)\right) + \frac{p-1}{p}\,\mathbf{d}M(A_2 \times B_2)\\ &= \frac{1}{p^2}\,\mathbf{d}M(A' \times B') + \frac{p-1}{p^2}\,(\mathbf{d}M(D) + \mathbf{d}M(A_2 \times B_2))\\ &\quad + \frac{(p-1)^2}{p^2}\,\mathbf{d}M(A_2 \times B_2).\end{aligned} \tag{4.3}$$

We compare this quantity with

$$\mathbf{d}M(A)\,\mathbf{d}M(B) \leqslant \left(\frac{1}{p}\,\mathbf{d}M(A') + \frac{p-1}{p}\,\mathbf{d}M(A_2)\right) \cdot \left(\frac{1}{p}\,\mathbf{d}M(B') + \frac{p-1}{p}\,\mathbf{d}M(B_2)\right)$$

$$= \frac{1}{p^2}\,\mathbf{d}M(A')\,\mathbf{d}M(B') + \frac{(p-1)^2}{p^2}\,\mathbf{d}M(A_2)\,\mathbf{d}M(B_2)$$

$$+ \frac{p-1}{p^2}\,(\mathbf{d}M(A')\,\mathbf{d}M(B_2) + \mathbf{d}M(A_2)\,\mathbf{d}M(B')). \tag{4.4}$$

By induction hypothesis we have

$$\mathbf{d}M(A' \times B') \leqslant \mathbf{d}M(A')\,\mathbf{d}M(B') \tag{4.5}$$

and

$$\mathbf{d}M(A_2 \times B_2) \leqslant \mathbf{d}M(A_2)\,\mathbf{d}M(B_2). \tag{4.6}$$

It is now sufficient for us to show that

$$\mathbf{d}M(D) + \mathbf{d}M(A_2 \times B_2) \leqslant \mathbf{d}M(A')\,\mathbf{d}M(B_2) + \mathbf{d}M(A_2)\,\mathbf{d}M(B'). \tag{4.7}$$

Again by induction hypothesis for this it is sufficient to show that

$$\mathbf{d}M(D) + \mathbf{d}M(A_2 \times B_2) \leqslant \mathbf{d}M(A' \times B_2) + \mathbf{d}M(A_2 \times B')$$

or (equivalently) that

$$\mathbf{d}(M(A'_1 \times B_2) \cup M(A_2 \times B'_1) \cup M(A_2 \times B_2))$$
$$- \mathbf{d}M(A' \times B_2) \leqslant \mathbf{d}M(A_2 \times B') - \mathbf{d}M(A_2 \times B_2). \tag{4.8}$$

Since $M(A' \times B_2) \supset M(A'_1 \times B_2) \cup M(A_2 \times B_2)$ and $M(A_2 \times B') \supset M(A_2 \times B_2)$, this in turn is a consequence of

$$\mathbf{d}(M(A_2 \times B'_1) \backslash (M(A'_1 \times B_2) \cup M(A_2 \times B_2)))$$
$$\leqslant \mathbf{d}(M(A_2 \times B'_1) \backslash M(A_2 \times B_2)), \tag{4.9}$$

which obviously holds, because **d** is monotonically increasing in sets.

5. On the Characterization of Equality in $\mathbf{d}M(A)\,\mathbf{d}M(B) \leqslant \mathbf{d}M[A, B] \cdot \mathbf{d}M(A, B)$.

For many of the basic inequalities in mathematics conditions are known, which completely specify the cases of equality. We have mentioned that this is also the case for the predessor of the inequality stated above. However, for itself it turns out that the *characterization of equality* constitutes a formidable task. Comparable instances are discussed in [9].

In the present case, however, we do have a *necessary condition* or in short *a criterion* for equality. It is stated as Lemma 3 below.

We denote by $\mathscr{E}$ the set of pairs for which equality holds. It is clear that instead of pairs $\{A, B\}$ it suffices to consider pairs $\{P(A), P(B)\}$ of primitive sets (see (1.4)). Let $\mathscr{E}_0$ be the set of those for which equality holds. Consider now any $\{A, B\} \in \mathscr{E}_0$. Clearly, $1 \in (A, B)$ implies $\mathbf{d}M(A, B) = 1$. By Behrend's result discussed in the Introduction we know therefore that *in this case*

$$\{A, B\} \in \mathscr{E}_0 \quad \text{iff} \quad (A, B) = \{1\}. \tag{5.1}$$

The case $1 \notin (A, B)$ is much more complicated. It is convenient to use for finite sets $C = \{c_1, ..., c_l\} \subset \mathbb{N}$ the notation

$$(C) = (c_1, c_2, ..., c_k) = \text{g.c.d.} \quad \text{of} \quad c_1, c_2, ..., c_k. \tag{5.2}$$

LEMMA 3. *For $\{A, B\} \in \mathscr{E}_0$ with $1 \notin (A, B)$ we always have*

$$(A)(B) > 1.$$

Proof. Assume to the contrary that $(A) = (B) = 1$ and that $\{A, B\}$ minimizes $T = \sum_{a \in A} a + \sum_{b \in B} b$.

Under our assumptions there is a prime p such that we have the representation $A = pA_1 \cup A_2$, $B = pB_1 \cup B_2$; A_1, $B_1 \neq \phi$; $A_2, B_2 \neq \phi$, and

$$(p, A_2) = (p, B_2) = \{1\}. \tag{5.3}$$

(Here and elsewhere (c, C) means $(\{c\}, C)$).

An inspection of the proof of our inequality in Section 3 shows that $\{A, B\} \in \mathscr{E}$ occurs exactly if the following six relations hold.

(1) $\{A_1 \cup A_2, B_1 \cup B_2\} \in \mathscr{E}$

(2) $\{A_1 \cup A_2, B_2\} \in \mathscr{E}$

(3) $\{A_2, B_1 \cup B_2\} \in \mathscr{E}$

(4) $\{A_2, B_2\} \in \mathscr{E}$

(5) $\mathbf{d}M[A_1 \cup A_2, B_1 \cup B_2] + \mathbf{d}M[A_2, B_2] = \mathbf{d}M[A_1 \cup A_2, B_2] + \mathbf{d}M[A_2, B_1 \cup B_2]$

(6) $\mathbf{d}M(A_2, B_2) = \mathbf{d}(M(A_1, B_2) \cup M(A_2, B_1) \cup M(A_2, B_2))$.

We start with the pair $\{A_1 \cup A_2, B_1 \cup B_2\} \in \mathscr{E}$ and look at the associated "primitive pair" $\{A^*, B^*\} = \{P(A_1 \cup A_2), P(B_1 \cup B_2)\} \in \mathscr{E}_0$.

Since the original sets $A = pA_1 \cup A_2$ and $B = pB_1 \cup B_2$ are primitive, we can write

$$A^* = A_1 \cup A_2^*, B^* = B_1 \cup B_2^*, \tag{5.4}$$

where $A_2^* \subset A_2$ and $B_2^* \subset B_2$.

Since $\sum_{a \in A^*} a + \sum_{b \in B^*} b < T$, the minimality of $\{A, B\}$ implies that either

(a) $\mathbf{d}M(A^*, B^*) = 1$ or (b) $\mathbf{d}M(A^*, B^*) < 1$ and $(A^*)(B^*) > 1$.

In the case $(A^*) > 1$ necessarily $(A_1 \cup A_2) > 1$ and therefore $(A) = (pA_1 \cup A_2) > 1$, which contradicts our assumption $(A) = (B) = 1$. This excludes the case (b).

Suppose now that (a) holds. Since A^*, B^* are primitive, Behrend's equality characterisation implies

$$(A^*, B^*) = \{1\}. \tag{5.5}$$

Suppose now that $A_2^* \neq \phi$ (or $B_2^* \neq \phi$).

Then in particular $(A_2^*, B_1) = \{1\}$ and consequently $(A_2^*, pB_1) = \{1\}$, because $(A_2^*, p) = \{1\}$. Moreover, since $A_2^* \subset A_2 \subset A$ and $pB_1 \subset B$, we get $1 \in (A, B)$ in contradiction to our assumption $\mathbf{d}M(A, B) \neq 1$. Hence, $A_2^* = B_2^* \neq \phi$. Equivalently, since $\{A, B\}$ is primitive,

$$\forall a_2 \in A_2 (b_2 \in B_2)\ \exists a_1 \in A_1 (b_1 \in B_1)\colon a_1 \mid a_2 (b_1 \mid b_2). \tag{5.6}$$

Moreover, $a_1 < a_2$, $b_1 < b_2$.

Now we use (6) and observe that it is equivalent to the set equation

$$M(A_2, B_2) = M(A_1, B_2) \cup M(A_2, B_1) \cup M(A_2, B_2). \tag{5.7}$$

Let $c = (a_2, b_2) > 1$ be the smallest number in $M(A_2, B_2)$. By (5.6) we have

$$a_2 = l \cdot a_1;\ b_2 = rb_1;\ a_1 \in A_1,\ b_1 \in B_1;\ l, r > 1.$$

From (5.5) we know that $(a_1, b_1) = 1$. Therefore

$$f = (a_2, b_1) = (l \cdot a_1, b_1) = (l, b_1) \in M(A_2, B_1),$$

$$g = (a_1, b_2) = (a_1, rb_1) = (a_1, r) \in M(A_1, B_2).$$

We have $l = f \cdot l'$, $b_1 = f \cdot b_1'$, $a_1 = ga_1'$, $r = g \cdot r'$ and $f; g > 1$ (because otherwise $\mathbf{d}M(A, B) = 1$).

Now we conclude that $c = (a_2, b_2) = (l \cdot a_1, r \cdot b_1) = (f \cdot l' \cdot g \cdot a_1', g \cdot r' \cdot f \cdot b_1') \geqslant f \cdot g > \max\{f, g\}$ and that therefore $f \in M(A_2, B_1)$, $g \in M(A_1, B_2)$, and $f, g < c$.

Since c is minimal in $M(A_2, B_2)$ we have $f, g \notin M(A_2, B_2)$ in contradiction to the definitions of f, g and (5.7).

Remark. An algorithm for deciding on equality for a given pair $\{A, B\}$ can be found in the preprint [1]. It is based on the criterion in Lemma 3.

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