COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **15** (4) (1995) 475–480

A SPLITTING PROPERTY OF MAXIMAL ANTICHAINS

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Received August 1, 1993
Revised December 13, 1994

In any *dense* poset $\mathcal{P}$ (and in any Boolean lattice in particular) every maximal antichain S may be partitioned into disjoint subsets S_1 and S_2, such that the union of the downset of S_1 with the upset of S_2 yields the entire poset: $\mathcal{D}(S_1)\cup\mathcal{U}(S_2)=\mathcal{P}$. To find a similar splitting of maximal antichains in posets is NP-hard in general.

1. Introduction

Let $\mathcal{P}=(P,<_P)$ be a poset and let H be a subset of P. The *downset* $\mathcal{D}(H)$ of the subset H is:

$$\mathcal{D}(H) = \{x \in P : \exists s \in H\big(x \leq s\big)\}.$$

The *upset* of H is:

$$\mathcal{U}(H) = \{x \in P : \exists s \in H\big(s \leq x\big)\}.$$

We introduce also the sets

$$\mathcal{D}^*(H) = \{x \in P : \exists s \in H(x < s)\}$$

and

$$\mathcal{U}^*(H) = \{x \in P : \exists s \in H(s < x)\}.$$

A subset $S \subset P$ is called an *antichain* or a *Sperner system* if no two elements of S are comparable. An antichain $\mathcal{S}$ is *maximal* if for every antichain $S' \subset P$, $S \subset S'$ implies $S=S'$. It is easy to see that antichain S is maximal iff

$$\mathcal{D}(S) \cup \mathcal{U}(S) = P. \tag{1}$$

Mathematics Subject Classification (1991): 05 D 05, 06 A 07

[1] Research was partially carried out when the second author visited the first author in Bielefeld and it was partially supported by OTKA grant T016358

0209–9683/95/$6.00

Let X be an n-element set and let S be the set of all its k-element subsets. Furthermore let $S_1 \subset S$ be the family of all k-element subsets containing a fixed element of X, and let $S_2 = S \setminus S_1$. Then it is easy to check that $\mathcal{D}(S_1) \cup \mathcal{U}(S_2) = 2^X$. In fact a similar *splitting* can be achieved for every maximal antichain of 2^X. The purpose of this note is to prove this claim and to generalize it to a wide class of posets.

The structure of this paper is as follows: Section 2 describes maximal antichains in general posets which satisfy the splitting property. Section 3 determines a large class of posets where every maximal antichain possesses such splittings. Section 4 gives further examples for posets which satisfy the splitting property, and which do not. The last part, Section 5 shows that to find a suitable splitting of a maximal antichain in posets is NP-hard in general.

Finally, for the convenience, let us introduce the following notation: For $H, G \subset P$ we write $H > | < G$ iff for all $h \in H$ and all $s \in G$ elements h and s are incomparable. For $s, s' \in P$ and $G \subset P$ we also write $s > | < s'$ instead of $\{s\} > | < \{s'\}$ and $s > | < G$ instead of $\{s\} > | < G$.

2. Dense sets in posets

In this section we describe a class of maximal antichains which satisfies the property described above.

Definition. Let $\mathcal{P} = (P, <_P)$ be a finite partially ordered set (poset). A subset $H \subset P$ is called *dense* in the poset $\mathcal{P}$ if any non-empty *open interval* $\langle x, y\rangle = \{z \in P : x <_p z <_p y\})$ which intersects H contains at least two elements of H. We also say that a maximal antichain $S \subset P$ satisfies the *splitting property* if there exists a partition of S into disjoint subsets S_1 and S_2 such that

$$\mathcal{U}(S_1) \cup \mathcal{D}(S_2) = P \tag{2}$$

holds. The poset $\mathcal{P}$ *has the splitting property* if every maximal antichain $\mathcal{S}$ in $\mathcal{P}$ satisfies the splitting condition expressed in (2).

It is easy to find a poset $\mathcal{P}$ and a maximal antichain S in $\mathcal{P}$ such that S has no splitting. Indeed if $\mathcal{P}$ has a (non-maximal and non-minimal) element s which is comparable to any other element of P then $S = \{s\}$ is a maximal antichain and does not satisfy the splitting property. On the other hand we prove the following statement:

Theorem 2.1. *Let S be a maximal, dense antichain in the poset $\mathcal{P}$. Then S satisfies the splitting property.*

Proof. Let S be a dense maximal antichain in the poset $\mathcal{P}$ and let $<_{\text{ord}}$ be an arbitrary linear ordering of S. For every element $x \in \mathcal{D}^*(S)$ let $f(x)$ be the greatest element $s \in S$ (with respect to $<_{\text{ord}}$) such that $x <_p s$. Set

$$S_1 = \{f(x) \ : \ x \in \mathcal{D}^*(S)\}$$
$$S_2 = S \setminus S_1.$$

We claim that S_1 and S_2 satisfy the splitting condition (2). Assume the contrary. Then there exists $y \in \mathcal{U}^*(S)$ such that $y \notin \mathcal{U}(S_2)$. Let $f(y)$ be the smallest element $s \in S$ (with respect to $<_{\mathrm{ord}}$) such that $s <_p y$. We know that $f(y) \notin S_2$ therefore there exists an element $x \in \mathcal{D}^*(S)$ such that $f(x) = f(y) = s$. Then the open interval $\langle x, y \rangle$ contains the element s, therefore there exists an $s' \in \langle x, y \rangle \cap (S \setminus \{s\})$. The linear ordering gives us an order between s and s'. If, say, $s <_{\mathrm{ord}} s'$ then, due to the definitions, $f(x)$ cannot be s, a contradiction. The other case, $s' <_{\mathrm{ord}} s$, leads to a contradiction with $f(y) = s$. ∎

We remark here that the splitting of a maximal antichain is not unique in general.

3. Dense posets

In this section a wide class of posets will be described where every maximal antichain possesses the splitting property.

Definition. The poset $\mathcal{P}$ is called *(weakly) dense* if every non-empty open interval $\langle x, y \rangle$ contains at least two elements. The poset $\mathcal{P}$ is called *(strongly) dense* if for every non-empty interval $\langle x, y \rangle$ and for every element of this interval $z \in \langle x, y \rangle$ there exists another element z' of the interval such that z and z' are incomparable (that is $z > | < z'$).

It is easy to show that for finite posets the two notions are equivalent, that is any weakly dense finite poset $\mathcal{P}$ is strongly dense as well. But for infinite posets the two notions do not necessarily coincide. For example the totally ordered chain of the rational numbers is weekly dense but not strongly dense. Thus in the case of finite posets we simply consider *dense* posets.

Theorem 3.1. *Let $\mathcal{P}$ be a dense poset. Then every maximal antichain S satisfies the splitting property.*

Here we give two proofs. We remark that the first argument can be applied to the proof of Theorem 2.1, as well. The second proof reduces the statement to an application of Theorem 2.1.

First proof. We apply induction on the cardinality of S.

Base case. The unique element of the maximal antichain $S = \{s\}$ is comparable to any other element of P. It is impossible that $x < s < y$ for some $x, y \in P$ because otherwise by (strong) denseness there is a $z \in \langle x, y \rangle$ with $z > | < s$ and this contradicts the maximality of S. Therefore we have either a maximal element 1 or a minimal element 0 equal to s and the splitting property holds.

General case. Let $\mathcal{P} = (P, <_P)$ be a finite dense poset and let S be a maximal antichain in $\mathcal{P}$. Assume that for every finite poset and for every maximal antichain with less elements than S the statement is true.

Assume first that for some $s \in S$ neither

$$\mathcal{D}^*(s) \subset \mathcal{D}^*\big(S \setminus \{s\}\big) \tag{3}$$

nor

$$\mathcal{U}^*(s) \subset \mathcal{U}^*(S \setminus \{s\}) \tag{4}$$

holds, that is, for some $x \in \mathcal{D}^*(s)$ and some $y \in \mathcal{U}^*(s)$

$$\{x, y\} >|< S \setminus \{s\} \tag{5}$$

(because S is antichain). Since $s \in \langle x,y \rangle$, by the (strong) denseness of $\mathcal{P}$ there is a $z \in \langle x,y \rangle$ with $z >|< s$. However, $z >|< S$ also holds, because $z > s'$ (or $z < s'$) for some $s' \in S$ would imply $s' < y$ (or $x < s'$) in contradiction to (5). But this fact contradicts the maximality of S.

We know therefore that for every $s \in S$ condition (3) or condition (4) holds. By symmetry we can assume that condition (3) holds for an arbitrary, but fixed, s. Then

Claim 3.2. *The poset $\mathcal{P}' = (P \setminus \mathcal{U}(s), <_P)$ is dense and the antichain $S' = S \setminus \{s\}$ is maximal in it.*

Proof. Due to condition (3) and the maximality of S any element $x \in \mathcal{P}'$ is comparable to some elements of S'. So S' is maximal in $\mathcal{P}'$. Furthermore, for any x and $y \in P'$ in obvious notation

$$\langle x, y \rangle_{\mathcal{P}'} = \langle x, y \rangle_{\mathcal{P}} \tag{6}$$

holds. Indeed, otherwise there would be an element $z \in \langle x, y \rangle_{\mathcal{P}}$ such that $z \notin P'$. But in that case $s < z$ and therefore $s < y$, so $y \in \mathcal{U}(s)$, a contradiction. By relation (6) the poset $\mathcal{P}'$ is dense. This proves Claim 3.2.

By our inductive hypothesis we have a partition (S_1', S_2') of S' with $\mathcal{D}(S_1') \cup \mathcal{U}(S_2') = P'$ and hence

$$D(S_1') \cup \mathcal{U}(S_2' \cup \{s\}) = P. \qquad \blacksquare$$

Second proof. We proceed by constructing a subposet $\mathcal{P}'$ in $\mathcal{P}$ such that the maximal antichain S has the partitioning property in $\mathcal{P}'$ if and only if it has the property in $\mathcal{P}$. Let D be the set of maximal elements of $\mathcal{D}^*(S)$ in $\mathcal{P}$ and let U be the set of the minimal elements of $\mathcal{U}^*(S)$ in $\mathcal{P}$. Take the subposet $\mathcal{P}'$ where $P' = D \cup S \cup U$ and $<_{\mathcal{P}'}$ is the induced order. Our definition of $\mathcal{P}'$ necessitates that a good splitting of $\mathcal{P}'$ supplies a good splitting of $\mathcal{P}$. Obviously, S is maximal in $\mathcal{P}'$, since $\mathcal{P}'$ is an induced subposet of $\mathcal{P}$. Furthermore the antichain S is dense in $\mathcal{P}'$ otherwise either $\mathcal{P}$ was not dense or S was not maximal. The application of Theorem 2.1 to $\mathcal{P}'$ completes the proof. $\blacksquare$

4. Examples

In this section we furnish some posets which satisfy the splitting property and some which do not.

At first we remark that Theorem 2.1 and Theorem 3.1 are incomparable. This follows from the fact that there exists a dense poset $\mathcal{P}$, and a maximal antichain S in $\mathcal{P}$ which is not dense there. To show that, let $P=\{a,b,c,d,e,f,g\}$ and let the covering relations be $a<c<f$, $a<d<f$, $b<c<g$, $b<d<g$, finally $b<e<g$. This poset is dense but the maximal antichain $\{a,e\}$ is not dense in $\mathcal{P}$. (Observe that $S_1=\{e\}$ and $S_2=\{a\}$ is a good splitting.)

Many well-known posets satisfy the splitting property. For example:

Corollary 4.1. *Every finite geometric lattice (and every Boolean lattice in particular) satisfies the splitting property.*

Proof. Since every geometric lattice is relatively complemented, it is dense as well. Upon application of Theorem 3.1 the result follows. ∎

Furthermore we remark that the notion of 'denseness' is not necessary to ensure the splitting property. The following lattice is not dense, no maximal antichain is dense in the poset either, yet it satisfies the splitting property. The symbol $<$ means (again) 'covering' in the poset:

Example 4.2. (Faigle, [1]) The poset

$$\{a<b<c<d<e; a<f<d; b<g<e\}$$

is not dense, and no (non-trivial) maximal antichain is dense in the poset, but the poset satisfies the splitting property. ∎

On the other hand, there are even distributive lattices which do not satisfy the splitting property. We remark, that the splitting property of an antichain bears no relationship to the LYM property or even to the Sperner property of the poset, since it is easy to find posets satisfying one property and violating the other. We also remark that the existence of the splitting property in Boolean lattices reveals that there are (many) monotonic Boolean functions with the property that the minimal elements of the Boolean function are incomparable with the maximal elements of the complement.

We close this section with a conjecture. As mentioned earlier the notions of weak and strong denseness are different for infinite posets. The totally ordered chain of the rational numbers is weakly dense but it clearly does not have the splitting property. However we conjecture:

Conjecture 4.3. *Any countable infinite strongly dense poset has the splitting property.*

5. Splitting in general is NP-hard

It would be interesting to determine exactly which posets have the splitting property. The following NP-completeness result is a cause for some pessimism in this regard. We follow the terminology of Garey and Johnson [2].

INSTANCE: A finite poset $\mathcal{P}$.

QUESTION: Is there a partition of each maximal antichain $\mathcal{S}$ in $\mathcal{P}$ into two subsets S_1, S_2 such that $\mathcal{D}(S_1)\cup\mathcal{U}(S_2)=\mathcal{P}$?

Define MONOSAT as those instances of SATISFIABILITY where each clause contains only complemented or uncomplemented literals. It is straightforward to reduce instances of SATISFIABILITY to instances of MONOSAT. Indeed, it is mentioned in [2] that Gold showed that 'MONOTONE 3SAT' is NP-complete.

We now construct a rank 3 poset $\mathcal{P}=(P,<_P)$ from an instance of MONOSAT with variables $X=\{x_1,\cdots,x_n\}$ and clauses $\mathcal{C}=\{C_1,\cdots,C_m\}$. We partition the clauses $\mathcal{C}$ into two sets $\mathcal{C}^-$ and $\mathcal{C}^+$ respectively containing the complemented and non-complemented clauses of $\mathcal{C}$. Let $P=\mathcal{C}^-\cup X\cup\mathcal{C}^+$. For each x_i, C_j such that $x_i\in C_j\in\mathcal{C}^+$ let $x_i<_P C_j$. For each x_i, C_j such that $\overline{x}_i\in C_j\in\mathcal{C}^-$ let $C_j<_P x_i$. Clearly, the maximal antichain X has the splitting property in $\mathcal{P}$ if and only if the collection of clauses $\mathcal{C}=\mathcal{C}^-\cup\mathcal{C}^+$ is satisfiable. Let $C_i<C_j$ for each incomparable pair $C_i\in\mathcal{C}^-$, $C_j\in\mathcal{C}^+$. Now each element in $\mathcal{C}^-$ is comparable with every element of $\mathcal{C}^+$. Consequently, every maximal antichain in $\mathcal{P}$ other than X satisfies the splitting property. Thus, the poset $\mathcal{P}$ has the splitting property iff the MONOSAT instance is satisfiable.

References

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