CROSS–DISJOINT PAIRS OF CLOUDS IN THE INTERVAL LATTICE

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Abstract

Let $\mathcal{I}_n$ be the lattice of intervals in the Boolean lattice $\mathcal{L}_n$. For $\mathcal{A}, \mathcal{B} \subset \mathcal{I}_n$ the pair of clouds $(\mathcal{A}, \mathcal{B})$ is cross–disjoint, if $I \cap J = \phi$ for $I \in \mathcal{A}$, $J \in \mathcal{B}$. We prove that for such pairs $|\mathcal{A}||\mathcal{B}| \leq 3^{2n-2}$ and that this bound is best possible.

Optimal pairs are up to obvious isomorphisms unique. The proof is based on a new bound on cross intersecting families in $\mathcal{L}_n$ with a weight distribution. It implies also an Intersection Theorem for multisets of Erdös and Schönheim [9].

Consider the set $[n]=\{1,2,\dots,n\}$, the set of all its subsets $\mathcal{L}_n$, and the lattice of intervals $\mathcal{I}_n=\big\{I=[A,B]:A,B\in\mathcal{L}_n\big\}$, where $[A,B]=\{C\in\mathcal{L}_n:A\subset C\subset B\}$, if $A\subset B$, and $[A,B]=I_\phi$ (the empty interval), if $A\not\subset B$. The lattice operations $\wedge$ and $\vee$ are defined by

$$[A,B]\wedge[A',B']=[A,B]\cap[A',B'], \tag{1.1}$$

$$[A,B]\vee[A',B']=[A\cap A',B\cup B']. \tag{1.2}$$

Here the empty interval I_ϕ is represented by $\big[[n],\phi\big]$. The pair $(\mathcal{A},\mathcal{B})$ with $\mathcal{A},\mathcal{B}\subset\mathcal{I}_n\smallsetminus\{I_\phi\}$ is cross–disjoint, if

$$I\wedge J=I_\phi \ \text{ for } \ I\in\mathcal{A},J\in\mathcal{B}. \tag{1.3}$$

Let us denote the set of those pairs by $\mathcal{D}_n$.

Theorem 1. *For* $n=1,2,\dots$

$$\max\big\{|\mathcal{A}||\mathcal{B}|:(\mathcal{A},\mathcal{B})\in\mathcal{D}_n\big\}=3^{2n-2}.$$

Equality is assumed for

$$\mathcal{A}^*=\big\{I\in\mathcal{I}_n:I=[A,B],1\notin B\big\},\mathcal{B}^*=\big\{I\in\mathcal{I}_n:I=[A,B],1\in A\big\}.$$

All optimal pairs are obtained by replacing 1 in the definition of $\mathcal{A}^*$ *and* $\mathcal{B}^*$ *by any element* m *of* $[n]$, *and by exchanging the roles of these two sets.*

We shall relate *cross–disjoint* pairs of clouds from $\mathcal{I}_n$ to *cross–intersecting* pairs of clouds from $\mathcal{L}_n$ with a suitable weight.

Recall from [1] that $(\mathcal{U},\mathcal{V})$ with $\mathcal{U},\mathcal{V}\subset\mathcal{L}_n$ is cross–intersecting, if

$$U\cap V\neq\phi \ \text{ for } \ U\in\mathcal{U} \ \text{ and } \ V\in\mathcal{V}. \tag{1.4}$$

We denote the set of these pairs by $\mathcal{P}_n$. Furthermore we introduce the weight $w:\mathcal{L}_n\to\mathbb{N}$ by

$$w(A)=2^{n-|A|} \ \text{ for } \ A\in\mathcal{L}_n. \tag{1.5}$$

Theorem 2. *For* $(\mathcal{U},\mathcal{V})\in\mathcal{P}_n$

$$W(\mathcal{U})W(\mathcal{V})\triangleq\sum_{U\in\mathcal{U}}w(U)\cdot\sum_{V\in\mathcal{V}}w(V)\leq 3^{2(n-1)}$$

and the bound is best possible. Moreover, for any optimal pair $(\mathcal{U},\mathcal{V})$ *there exists a* $t\in[n]$ *such that* $\mathcal{U}=\mathcal{V}=\{A\in\mathcal{L}_n:t\in A\}$.

2. Another description for $\mathcal{I}_n \smallsetminus \{I_\phi\}$

We associate $[A,B] \in \mathcal{I}_n \smallsetminus \{I_\phi\}$ with a ternary sequence $\Psi\big([A,B]\big) = (x_1, x_2, \dots, x_n)$, where

$$x_t = \begin{cases} 0 & \text{if } \ t \notin B \\ 1 & \text{if } \ t \in A \\ 2 & \text{if } \ t \in B \smallsetminus A \ . \end{cases}$$

$\Psi : \mathcal{I}_n \smallsetminus \{I_\phi\} \to \{0,1,2\}^n$ is bijective.

If $\Psi\big([A,B]\big) = x^n$ and $\Psi\big([A',B']\big) = y^n$, then

$$[A,B] \wedge [A',B'] = I_\phi \Leftrightarrow \exists t \in [n] : \{x_t, y_t\} = \{0,1\}. \tag{2.1}$$

For $(\mathcal{A},\mathcal{B}) \in \mathcal{D}_n$ the associated pair $(\mathcal{X},\mathcal{Y}) = \big(\Psi(\mathcal{A}), \Psi(\mathcal{B})\big)$ has the property:

$$\text{For } \ x^n \in \mathcal{X}, y^n \in \mathcal{Y} \qquad \{x_t, y_t\} = \{0,1\} \ \text{ for some } \ t \in [n]. \tag{2.2}$$

We can view $(\mathcal{X},\mathcal{Y})$ as families of cross–disjoint subcubes of the n–dimensional unit cube or as families of cross–disjoint cylindersets in $\{0,1\}^n$ in the sense of measure or probability theory. In this interpretation 2 stands for the set $\{0,1\}$.

Henceforth we consider pairs $(\mathcal{X},\mathcal{Y})$; $\mathcal{X},\mathcal{Y} \subset \{0,1,2\}^n$; satisfying (2.2). The set of these pairs is denoted by $\mathcal{D}_n^*$. Our first goal in proving Theorem 1 is to show that for $(\mathcal{X},\mathcal{Y}) \in \mathcal{D}_n^*$

$$|\mathcal{X}||\mathcal{Y}| \leq 3^{2(n-1)}. \tag{2.3}$$

3. Down–up–shifts

The proof of Theorem 1 goes in several steps. At first we show here that any $(\mathcal{X},\mathcal{Y}) \in \mathcal{D}_n^*$ can be transformed into another pair in $\mathcal{D}_n^*$ with the same cardinalities and with invariance under down–up–shifts. They are defined as follows.

For any $\mathcal{Z} \subset \{0,1,2,\}^n$ and any $t \in [n]$ set

$$\begin{aligned} d_t(\mathcal{Z}) = \Big\{&(z_1, \dots, z_{t-1}, i, z_{t+1}, \dots, z_n) : i = 0, \dots, j-1; j \geq 1 \ \text{ and} \\ &\big|\big\{z : (z_1, \dots, z_{t-1}, z, z_{t+1}, \dots, z_n) \in \mathcal{Z}\big\}\big| = j\Big\}. \end{aligned} \tag{3.1}$$

This is the down–shift of $\mathcal{Z}$ in the t–th component. Similarly, $u_t(\mathcal{Z})$, the up–shift of $\mathcal{Z}$ in the t–th component is obtained by exchanging 0 and 1 in the t–th component of the sequences in $d_t(\mathcal{Z})$. We formulate an immediate consequence of our definitions.

Lemma 1. *For any* $(\mathcal{X},\mathcal{Y}) \in \mathcal{D}_n^*$ *and* $t \in [n]$ *also* $\big(d_t(\mathcal{X}), u_t(\mathcal{Y})\big) \in \mathcal{D}_n^*$.

We say that $(\mathcal{X},\mathcal{Y})$ with $\mathcal{X},\mathcal{Y} \subset \{0,1,2\}^n$ is down–up–extremal, if

$$\big(d_t(\mathcal{X}), u_t(\mathcal{Y})\big) = (\mathcal{X},\mathcal{Y}) \ \text{ for all } \ t \in [n]. \tag{3.2}$$

4. Relation to cross–intersection in $\mathcal{L}_n$

Next we introduce the mappings $\sigma_i : \{0,1,2\}^n \to \mathcal{L}^n$ by

$$\sigma_i(z^n) = \{t : z_t = i, 1 \leq t \leq n\} \text{ for } i = 0,1. \tag{3.3}$$

We also put for $\mathcal{Z} \subset \{0,1,2\}^n$

$$\sigma_i(\mathcal{Z}) = \{\sigma_i(z^n) : z^n \in \mathcal{Z}\}. \tag{3.4}$$

These mappings make it possible to convert cross–disjoint pairs of clouds from the interval lattice $\mathcal{I}_n$ into cross–intersecting pairs of clouds from the Boolean lattice $\mathcal{L}_n$.

More precisely we have the following result.

Lemma 2. *Suppose that $(\mathcal{X},\mathcal{Y})$ with $\mathcal{X},\mathcal{Y} \subset \{0,1,2\}^n$ is down–up–extremal. Then $(\mathcal{X},\mathcal{Y})$ is cross–disjoint exactly if $\big(\sigma_0(\mathcal{X}),\sigma_1(\mathcal{Y})\big)$ is cross–intersecting, that is, $X \cap Y \neq \phi$ for $X \in \sigma_0(\mathcal{X})$ and $Y \in \sigma_1(\mathcal{Y})$.*

Proof: Suppose that $(\mathcal{X},\mathcal{Y})$ is cross–disjoint, but that $\big(\sigma_0(\mathcal{X}),\sigma_1(\mathcal{Y})\big)$ is not cross–intersecting. Then there exist $x^n \in \mathcal{X}$, $y^n \in \mathcal{Y}$, and a non–empty set $E \subset [n]$ such that $(x_t,y_t) = (1,0)$ for $t \in E$ and $\{x_t,y_t\} \neq \{1,0\}$ for $t \notin E$. However, since $(\mathcal{X},\mathcal{Y})$ is down–up–extremal, the sequence x'^n obtained from x^n by replacing for $t \in E$ $x_t = 1$ by $x'_t = 0$ must be in $\mathcal{X}$ and this sequence is not disjoint with y^n . This contradiction proves that $\big(\sigma_0(\mathcal{X}),\sigma_1(\mathcal{Y})\big)$ is cross–intersecting. The reverse implication is obvious.

5. Theorem 1 from Theorem 2

Notice that for $A \subset [n]$

$$|\sigma_0^{-1}(A)| = |\sigma_1^{-1}(A)| = 2^{n-|A|}. \tag{5.1}$$

Therefore also for any $A,B \subset [n]$ and $\mathcal{X},\mathcal{Y} \subset \{0,1,2\}^n$

$$|\sigma_0^{-1}(A) \cap \mathcal{X}| \leq 2^{n-|A|},\ |\sigma_1^{-1}(B) \cap \mathcal{Y}| \leq 2^{n-|B|}. \tag{5.2}$$

Now in upper bounding $|\mathcal{X}||\mathcal{Y}|$ for $(\mathcal{X},\mathcal{Y}) \in \mathcal{D}_n^*$ we can assume by Lemma 1 that $(\mathcal{X},\mathcal{Y})$ is down–up–extremal and by Lemma 2 that $\big(\sigma_0(\mathcal{X}),\sigma_1(\mathcal{Y})\big) \in \mathcal{P}_n$. Hence Theorem 2 implies that for $\mathcal{U} = \sigma_0(\mathcal{X})$ and $\mathcal{V} = \sigma_1(\mathcal{Y})$

$$W(\mathcal{U})W(\mathcal{V}) \leq 3^{2(n-1)}$$

and thus by (5.2) and (5.1)

$$
\begin{aligned}
|\mathcal{X}||\mathcal{Y}| &= \sum_{U\in\mathcal{U}} |\sigma_0^{-1}(U)\cap\mathcal{X}|\cdot\sum_{V\in\mathcal{V}}|\sigma_1^{-1}(V)\cap\mathcal{Y}| \\
&\leq \sum_{U\in\mathcal{U}} 2^{n-|U|}\cdot\sum_{V\in\mathcal{V}} 2^{n-|V|} = W(\mathcal{U})\cdot W(\mathcal{V})\leq 3^{2(n-1)}.
\end{aligned} \tag{5.3}
$$

The characterisation of the optimal pairs follows from the one in Theorem 2. We use right away the sequence terminology. If $(\mathcal{X},\mathcal{Y})\in\mathcal{D}_n^*$ is optimal, then applications of operations (d_t,u_t) and $\sigma_0,\sigma_1)$ lead to an optimal $(\mathcal{U},\mathcal{V})\in\mathcal{P}_n$ by (5.3).

By the uniqueness part of Theorem 2 for some $t\in[n]$, w.l.o.g. say $t=n$, we have $\mathcal{U}=\mathcal{V}=\{A\in\mathcal{L}_n : t\in A\}$. Furthermore $\big(\sigma_0^{-1}(\mathcal{U}),\sigma_1^{-1}(\mathcal{V})\big)=\big(\{0,1,2\}^{n-1}\times\{0\},\{0,1,2\}^{n-1}\times\{1\}\big)$.

It remains to be seen that (d_n^{-1},u_n^{-1}) leads to no non–isomorphic pairs. We have $d_n^{-1}\big(\{0,1,2\}^{n-1}\times\{0\}\big)=\mathcal{X}(0)\times\{0\}\,\dot\cup\,\mathcal{X}(1)\times\{1\}\,\dot\cup\,\mathcal{X}(2)\times\{2\}$, $u_n^{-1}\big(\{0,1,2\}^{n-1}\times\{1\}\big)=\mathcal{Y}(0)\times\{0\}\dot\cup\mathcal{Y}(1)\times\{1\}\dot\cup\mathcal{Y}(2)\times\{2\}$, where by the optimality the $\mathcal{X}(i)$'s and also the $\mathcal{Y}(i)$'s partition $\{0,1,2\}^{n-1}$. Therefore for some i and some j $(2,2,\dots,2)\in\mathcal{X}(i)\cap\mathcal{Y}(j)$. But now by (2.2) necessarily $\{i,j\}=\{0,1\}$ and $\mathcal{X}(i')=\phi$ for $i\neq i'$, $\mathcal{Y}(i')=\phi$ for $j\neq j'$. We have arrived at the desired form.

6. Auxiliary results for proving Theorem 2

Obviously in deriving an upper bound on $W(\mathcal{U})W(\mathcal{V})$ for $(\mathcal{U},\mathcal{V})\in\mathcal{P}_n$ we can always assume that $\mathcal{U}$ and $\mathcal{V}$ are upsets.

Moreover we can replace $(\mathcal{U},\mathcal{V})$ by the pair of images $\big(S_{ij}(\mathcal{U}),S_{ij}(\mathcal{V})\big)$ under the familiar left–shifting S_{ij} :

For any $\mathcal{E}\subset\mathcal{L}_n$

$$
S_{ij}(E)=\begin{cases} E\triangle\{i,j\}, & \text{if } i\notin E, j\in E \text{ and } E\triangle\{i,j\}\notin\mathcal{E} \\ E & \text{otherwise.}\end{cases} \tag{6.1}
$$

for $E\in\mathcal{E}$ and

$$
S_{ij}(\mathcal{E})=\big\{S_{ij}(E) : E\in\mathcal{E}\big\}.
$$

Just verify that $(\mathcal{U},\mathcal{V})\in\mathcal{P}_n$ implies $\big(S_{ij}(\mathcal{U}),S_{ij}(\mathcal{V})\big)\in\mathcal{P}_n$, that $|S_{ij}(\mathcal{U})|=|\mathcal{U}|, |S_{ij}(\mathcal{V})|=|\mathcal{V}|$, and that

$$
W\big(S_{ij}(\mathcal{U})\big)=W(\mathcal{U}), W\big(S_{ij}(\mathcal{V})\big)=W(\mathcal{V}). \tag{6.2}
$$

Clearly, finitely many applications of left–shifting operators results in a pair, which is invariant under further such operations. We call such a pair left–shifted.

Let now $(\mathcal{U},\mathcal{V})\in\mathcal{P}_n$ be a pair of left–shifted upsets.

For the analysis of such pairs we introduce the following sets and families of sets.

For $A\subset[n]$ its projection on $[n-1]$ is

$$ {}_pA = A\cap[n-1] \tag{6.3}$$

and for $\mathcal{A}\subset\mathcal{L}_n$ we define

$$ {}_p\mathcal{A}=\{{}_pA : A\in\mathcal{A}\}. \tag{6.4}$$

Furthermore we partition $\mathcal{A}$ into

$$ \mathcal{A}_0=\{A\in\mathcal{A}: n\notin A\},\ \mathcal{A}_1=\{A\in\mathcal{A}: n\in A\}. \tag{6.5}$$

Thus also $\mathcal{U}_0,\mathcal{U}_1,\mathcal{V}_0,\mathcal{V}_1$, ${}_p\mathcal{U}_i$ and ${}_p\mathcal{V}_i$ $(i=0,1)$ are well–defined.

Since $\mathcal{U}$ and $\mathcal{V}$ are upsets

$$ {}_p\mathcal{U}_0\subset{}_p\mathcal{U}_1 \text{ and } {}_p\mathcal{V}_0\subset{}_p\mathcal{V}_1. \tag{6.6}$$

Lemma 3. *If $(\mathcal{U},\mathcal{V})\in\mathcal{P}_n$ cannot be enlarged without violating the cross–intersection property and $\mathcal{U},\mathcal{V}$ are left–shifted upsets, then for all $U\in\mathcal{U}_1$ with ${}_pU\in{}_p\mathcal{U}_1\smallsetminus{}_p\mathcal{U}_0$ there exists a $V\in\mathcal{V}_1$ with ${}_pV\in{}_p\mathcal{V}_1\smallsetminus{}_p\mathcal{V}_0$ such that*

(i) ${}_pU\cap{}_pV=\phi$, ${}_pU\cup{}_pV=[n-1]$

and

(ii) for all V' with ${}_pV'\in{}_p\mathcal{V}_1\smallsetminus{}_p\mathcal{V}_0$ and $V'\neq V$ necessarily ${}_pU\cap{}_pV'\neq\phi$.

Exchanging the roles of $\mathcal{U}$ and $\mathcal{V}$ gives analogous statements.

Furthermore

(iii) ${}_p\mathcal{U}_0={}_p\mathcal{U}_1\Leftrightarrow{}_p\mathcal{V}_0={}_p\mathcal{V}_1$.

Proof: (i) For every $U\in\mathcal{U}_1$ with ${}_pU\in{}_p\mathcal{U}_1\smallsetminus{}_p\mathcal{U}_0$ there must exist a $V\in\mathcal{V}_1$ with ${}_pV\in{}_p\mathcal{V}_1\smallsetminus{}_p\mathcal{V}_0$ with ${}_pU\cap{}_pV=\phi$, because otherwise U is intersecting on $[n-1]$ with all $V^*\in\mathcal{V}_1$ and by (6.6) with all $V^*\in\mathcal{V}$, and thus one can enlarge $\mathcal{U}$ by $U\smallsetminus\{n\}$ in contradiction to our assumptions.

Furthermore for this V ${}_pU\cup{}_pV=[n-1]$, because otherwise for some $i\in[n-1]\smallsetminus{}_pU\cup{}_pV$ $U^*=S_{in}(U)\in\mathcal{U}$ by assumption and $U^*\cap V=\phi$ in contradiction to the fact that $(\mathcal{U},\mathcal{V})\in\mathcal{P}_n$.

(ii) The forgoing argument shows that ${}_pV'\not\subset{}_pV$. and that necessarily ${}_pU\cap{}_pV'\neq\phi$, because ${}_pU\cup{}_pV=[n-1]$.

(iii) This follows from the fact that $(\mathcal{U},\mathcal{V})$ cannot be enlarged.

7. Proof of Theorem 2

We proceed by induction on n. The case $n=1$ is verified by inspection. For $n \geq 2$ we can consider a $(\mathcal{U}, \mathcal{V})$ satisfying the assumptions of Lemma 3.

Case: ${}_p\mathcal{U}_0 = {}_p\mathcal{U}_1$.

By (iii) in Lemma 3 we have also ${}_p\mathcal{V}_0 = {}_p\mathcal{V}_1$. However $({}_p\mathcal{U}_0, {}_p\mathcal{V}_0) \in \mathcal{P}_{n-1}$ and by induction hypothesis

$$W({}_p\mathcal{U}_0)W({}_p\mathcal{V}_0) \leq 3^{2(n-2)}. \tag{7.1}$$

Now just calculate that in the present case

$$\begin{aligned} W(\mathcal{U})W(\mathcal{V}) &= \left[W({}_p\mathcal{U}_0)\cdot 2 + W({}_p\mathcal{U}_1)\right]\cdot\left[W({}_p\mathcal{V}_0)\cdot 2 + W({}_p\mathcal{V}_1)\right] \\ &= 3W({}_p\mathcal{U}_0)\cdot 3W({}_p\mathcal{V}_0) \leq 3^{2(n-1)}. \end{aligned}$$

Case: ${}_p\mathcal{U}_0 \neq {}_p\mathcal{U}_1$.

By Lemma 3 ${}_p\mathcal{V}_0 \neq {}_p\mathcal{V}_1$ and there are subsets $U \in \mathcal{U}_1$, $V \in \mathcal{V}_1$ satisfying (i). For all $V' \in \mathcal{V}_0$ necessarily ${}_pU \cap {}_pV' \neq \phi$ and again by Lemma 3 also for all $V' \in \mathcal{V}_1 \setminus \{V\}$, $V' \neq V$, ${}_pU \cap {}_pV' \neq \phi$. This means that $\big(\mathcal{U} \cup \{U \setminus \{n\}\}, \mathcal{V} \setminus \{V\}\big) \in \mathcal{P}_n$ and symmetrically $\big(\mathcal{U} \setminus \{U\}, \mathcal{V} \cup \{V \setminus \{n\}\}\big) \in \mathcal{P}_n$.

Moreover we see that

$$W\big(\mathcal{U} \cup \{U \setminus \{n\}\}\big) = W(\mathcal{U}) + 2^{n-|U|+1}, \tag{7.2}$$

$$W\big(\mathcal{V} \setminus \{V\}\big) = W(\mathcal{V}) - 2^{n-|V|} = W(\mathcal{V}) - 2^{|U|-1} \text{ (by (i))}, \tag{7.3}$$

$$W\big(\mathcal{U} \setminus \{U\}\big) = W(\mathcal{U}) - 2^{n-|U|}, \tag{7.4}$$

and

$$W\big(\mathcal{V} \cup \{V - \{n\}\}\big) = W(\mathcal{V}) + 2^{n-|V|+1} = W(\mathcal{V}) + 2^{|U|} \text{ (by (i))}. \tag{7.5}$$

By the optimality of $(\mathcal{U}, \mathcal{V})$ we conclude with (7.2) and (7.3) that

$$\begin{aligned} W(\mathcal{U})W(\mathcal{V}) &\geq W\big(\mathcal{U} \cup \{U \setminus \{n\}\}\big)W\big(\mathcal{V} \setminus \{V\}\big) \\ &= \big(W(\mathcal{U}) + 2^{n-|U|+1}\big)\big(W(\mathcal{V}) - 2^{|U|-1}\big) \\ &= W(\mathcal{U})W(\mathcal{V}) - 2^{|U|-1}W(\mathcal{U}) + 2^{n-|U|+1}W(\mathcal{V}) - 2^n \end{aligned} \tag{7.6}$$

and with (7.4) and (7.5) that

$$\begin{aligned} W(\mathcal{U})W(\mathcal{V}) &\geq W\big(\mathcal{U} \setminus \{U\}\big)W\big(V \cup \{V \setminus \{n\}\}\big) \\ &= \big(W(\mathcal{U}) - 2^{n-|U|}\big)\big(W(\mathcal{V}) + 2^{|U|}\big) \\ &= W(\mathcal{U})W(\mathcal{V}) + 2^{|U|}W(\mathcal{U}) - 2^{n-|U|}W(\mathcal{V}) - 2^n. \end{aligned} \tag{7.7}$$

Now (7.6) and (7.7) yield

$$-2^{|U|-1}W(\mathcal{U}) + 2^{n-|U|+1}W(\mathcal{V}) \le 2^n \tag{7.8}$$

and

$$2^{|U|}W(\mathcal{U}) - 2^{n-|U|}W(\mathcal{V}) \le 2^n. \tag{7.9}$$

The double of the left hand side in (7.8) plus the left hand side of (7.9) equals $3 \cdot 2^{n-|U|}W(\mathcal{V})$ and satisfies

$$3 \cdot 2^{n-|U|}W(\mathcal{V}) \le 3 \cdot 2^n.$$

This is equivalent to

$$W(\mathcal{V}) \le 2^{|U|}. \tag{7.10}$$

Similarly, by doubling the left hand side of (4.9) and adding to it the left hand side of (4.9) leads to the inequality

$$W(\mathcal{U}) \le 2^{n-|U|+1}. \tag{7.11}$$

The two inequalities imply

$$W(\mathcal{U})W(\mathcal{V}) \le 2^{n+1} < 3^{2(n-1)} \quad \text{for} \quad n \ge 2. \tag{7.12}$$

We calculate that for $\mathcal{U} = \mathcal{V} = \{A \subset [n] : 1 \in A\}$

$$W(\mathcal{U})W(\mathcal{V}) = 3^{2(n-1)}.$$

Finally we prove uniqueness. We have learnt already that for optimal left–shifted pairs $(\mathcal{U}, \mathcal{V})$ necessarily ${}_p\mathcal{U}_0 = {}_p\mathcal{U}_1$, ${}_p\mathcal{V}_0 = {}_p\mathcal{V}_1$ and that by induction hypothesis $W({}_p\mathcal{U}_0)W({}_p\mathcal{V}_0) = 3^{2(n-2)}$. Thus $\mathcal{U} = \mathcal{V} = \{A \subset [n] : 1 \in A\}$. In general, every optimal pair $(\mathcal{U}^*, \mathcal{V}^*)$ can be left–shifted to $(\mathcal{U}, \mathcal{V})$. Since the left–shifting operators don't change cardinalities of subsets, there must be a singleton $\{t\}$ in both, $\mathcal{U}^*$ and $\mathcal{V}^*$. Consequently we have $\mathcal{U}^* = \mathcal{V}^* = \{A \subset [n] : t \in A\}$.

8. A common generalization of Theorem 2 and a Theorem of Erdös/Schönheim [9]

In deriving their Intersection Theorem for multisets Erdös and Schönheim established first an Intersection Theorem with weights for $\mathcal{L}_n$. Those weights $w(A), A \in \mathcal{L}_n$, are *increasing* in $|A|$, whereas our weights $w(A) = 2^{n-|A|}$ used in Theorem 2 are *decreasing* in $|A|$. The latter does not allow to just choose the "heavier" one of A and $A^c = [n] \setminus A$ in order to construct an optimal configuration. This difference makes things more difficult in our case. Nevertheless we can give a unified approach.

Let $\mathcal{W} = \{w_i : 1 \leq i \leq n\}$ be positive reals which give rise to the weight w on $\mathcal{L}_n$:

$$w(A) = \prod_{t \in A} w_t \quad \text{for} \quad A \subset [n] \tag{8.1}$$

and

$$W(\mathcal{A}) = \prod_{A \in \mathcal{A}} w(A) \quad \text{for} \quad \mathcal{A} \subset \mathcal{L}_n. \tag{8.2}$$

Define

$$\alpha(n, w) = \max\big\{W(\mathcal{A}) : \mathcal{A} \subset \mathcal{L}_n \text{ is intersecting}\big\} \tag{8.3}$$

(i.e. $A \cap B \neq \phi$ for $A, B \in \mathcal{A}$).

We recall first a result of [9].

Theorem ES.

$$\alpha(n, w) \leq \frac{1}{2} \sum_{A \subset [n]} \max\big(w(A), w(A^c)\big) \tag{8.4}$$

and the bound is best possible when $w_i \geq 1$ *for* $i \in [n]$.

Proof: Clearly an intersecting $\mathcal{A}$ can have at most one of the sets A and A^c as member.

One can construct an optimal intersecting family $\mathcal{A}(n, w)$ in case

$$w_i \geq 1 \quad \text{for} \quad i \in [n] \tag{8.5}$$

as follows:

a) If $w(A) > w(A^c)$, then $A \in \mathcal{A}(n, w)$.

b) If $w(A) = w(A^c)$ and $|A| > |A^c|$, then $A \in \mathcal{A}(n, w)$.

c) If $w(A) = w(A^c)$ and $|A| = |A^c|$, then take anyone of A, A^c into $\mathcal{A}(n, w)$ and keep the other out of $\mathcal{A}(n, w)$. Clearly, $W\big(\mathcal{A}(n, w)\big) = \frac{1}{2} \sum_{A \subset [n]} \max\big(w(A), w(A^c)\big)$.

By (8.5) and a), b), and c) $\mathcal{A}(n,w)$ is an upset and also intersecting, because for $A,B \in \mathcal{A}(n,w)$ $A \cap B = \phi$ implies $A^c \supset B$ and thus $A^c \in \mathcal{A}(n,w)$ in contradiction to a) – c).

However, without condition (8.5) the $\mathcal{A}(n,w)$ described above need not be an upset or intersecting.

For example when $w_i < 1$ for all $i \in [n]$, then the biggest weight is assigned to the empty set, which cannot occur in an intersecting family. Therefore (8.4) is not tight.

Fortunately an analysis of the proof of our Theorem 2 leads us to the right generalisation.

First of all by relabelling we can always assume that

$$w_1 \geq w_2 \geq \cdots \geq w_n. \tag{8.6}$$

Let now m be the largest index with $w_m \geq 1$, if it exists, and otherwise set $m = 0$. Set $\mathcal{W}' = \{w_i : i \in [m]\}$, $w'(B) = \prod\limits_{i \in B} w_i$ for $B \subset [m]$.

Next define

$$\mathcal{A}^*(n,w) = \begin{cases} \{A \subset [n] : A \cap [m] \in \mathcal{A}(m,w')\}, & \text{if } m \geq 1 \\ \{A \subset [n] : 1 \in A\}, & \text{if } m = 0. \end{cases} \tag{8.7}$$

Clearly,

$$\mathcal{A}^*(n,w) = \mathcal{A}(n,w), \quad \text{if (8.5) holds.} \tag{8.8}$$

Theorem 3.

$$\alpha(n,w) = W\big(\mathcal{A}^*(n,w)\big). \tag{8.9}$$

Proof: We use induction on $n-m$.

The case $n-m=0$ or $n=m$ is the case covered by Theorem ES.

Case: $n-m>0$

Suppose that $\mathcal{A}$ is an optimal intersecting family, that is,

$$W(\mathcal{A}) = \alpha(n,w). \tag{8.10}$$

Since (8.6) holds, the left–pushing operator S_{ij} can be applied, because it does not decrease the total weight. We can therefore assume that $\mathcal{A}$ is invariant under such operations. Also we can assume that $\mathcal{A}$ is an upset, because adding an $A' \subset [n]$ to $\mathcal{A}$ with $A' \supset A$ for some $A \in \mathcal{A}$ does not affect the intersection property and could only increase the total weight.

We use again the projection p on $[n-1]$ and our earlier definitions ${}_pA$, ${}_p\mathcal{A}$, $\mathcal{A}_i$, and ${}_p\mathcal{A}_i$ $(i=0,1)$. Since $\mathcal{A}$ is an upset

$${}_p\mathcal{A}_0 \subset {}_p\mathcal{A}_1. \tag{8.11}$$

Case: ${}_p\mathcal{A}_0 = {}_p\mathcal{A}_1$.

Since ${}_p\mathcal{A}_0$ is intersecting, by induction hypothesis in this case

$$\begin{aligned} W(\mathcal{A}) &= W({}_p\mathcal{A}_0) + w_n W({}_p\mathcal{A}_1) = (1+w_n)W({}_p\mathcal{A}_1) \\ &\leq (1+w_n)W\big(\mathcal{A}^*(n-1,w'')\big) = W\big(\mathcal{A}^*(n,w)\big), \end{aligned}$$

where $w'' = (w_i)_{i=1}^{n-1}$ and the lase identity follows with definition (8.7).

Case: ${}_p\mathcal{A} \neq {}_p\mathcal{A}_1$

Here there is an $A \in \mathcal{A}_1$ with $A \setminus \{n\} \notin \mathcal{A}_0$ and there must be a $B \in \mathcal{A}_1$ with

$$B \cap A = \{n\}, \tag{8.12}$$

because otherwise one can enlarge $\mathcal{A}$ by $A \setminus \{n\}$. Now the same ideas as used in the proof of Lemma 3 apply and give

$${}_pA \cup {}_pB = [n-1] \tag{8.13}$$

and consequently that the B with these properties is unique, because otherwise there is an $i \in {}_pA \cup {}_pB$ and, since $S_{in}(A) \in \mathcal{A}$, by (8.12)

$$S_{in}(A) \cap B = \big({}_pA \cup \{i\}\big) \cap \big(B_p \cup \{n\}\big) = \phi.$$

This is a contradiction.

Since A and B can be exchanged, we can assume that

$$w(A) \geq w(B) \tag{8.14}$$

and consequently that

$$w\big(A \setminus \{n\}\big) = \frac{w(A)}{w_n} > w(A) \geq w(B), \tag{8.15}$$

because $w_n < 1$, if $n - m > 0$.

Finally, since B is the unique member of $\mathcal{A}$ satisfying (8.12) $\big(\mathcal{A} \setminus \{B\}\big) \cup \big\{A \setminus \{n\}\big\}$ is intersecting and by (8.15) has bigger weight than $\mathcal{A}$. This contradicts the optimality of $\mathcal{A}$. The case ${}_p\mathcal{A}_o \neq {}_p\mathcal{A}_1$ cannot arise.

9. Maximal families of disjoint intervals

One might wonder what can be said about families $\mathcal{A} \subset \mathcal{I}_n$ with

$$A \wedge B = I_\phi \quad \text{for} \quad A, B \in \mathcal{A}. \tag{9.1}$$

The family $\mathcal{A}$ corresponds to a set $\mathcal{A}^* \subset \{0,1,2\}^n$ by the mapping Ψ of Section 2. $\mathcal{A}^*$ has the property:

$$\text{for all } \; x^n, y^n \in \{0,1,2\}^n \; \text{ for some } \; t \in [n] \; \{x_t, y_t\} = \{0,1\}. \tag{9.2}$$

One readily verifies that $|\mathcal{A}^*| \leq 2^n$ and equality occurs for $\mathcal{A}^* = \{0,1\}^n$. In fact the problem is equivalent to Shannon's zero error capacity problem for the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. As Shannon noticed in [11], it equals $\log_2 2 = 1$.

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