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Incomparability and Intersection Properties of Boolean Interval Lattices and Chain Posets

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In a canonical way, we establish an AZ-identity (see [2]) and its consequences, the LYM-inequality and the Sperner property, for the Boolean interval lattice. Furthermore, the Bollobas inequality for the Boolean interval lattice turns out to be just the LYM-inequality for the Boolean lattice. We also present an Intersection Theorem for this lattice.

Perhaps more surprising is that by our approach the conjecture of P. L. Erdös *et al.* [7] and Z. Füredi concerning an Erdös–Ko–Rado-type intersection property for the poset of Boolean chains could also be established. In fact, we give two seemingly elegant proofs.

1. THE BOOLEAN INTERNAL LATTICE $\mathscr{I}_n$

The main objects of our investigation are collections of intervals in the Boolean lattice $\mathscr{B}_n$; that is, the family of all subsets of $[n]=\{1, 2, \ldots, n\}$ endowed with union and intersection as lattice operations.

For $A, B \in \mathscr{B}_n$, we define the interval

$$[A, B]=\{C: A \subset C \subset B\} \tag{1.1}$$

in $\mathscr{B}_n$, and if $A \not\subset B$ here, then we speak of the empty interval $I_\varnothing$.

The Boolean interval lattice $\mathscr{I}_n$ is the set of all intervals in $\mathscr{B}_n$ endowed with the following 'meet' and 'join' operations denoted by '$\wedge$' and '$\vee$':

$$\begin{aligned}[A, B] \wedge [A', B'] &= \{C: C \in [A, B] \cap [A', B']\} \\ &= \begin{cases} [A \cup A', B \cap B'], & \text{if } A \cup A' \subset B \cap B', \\ I_\varnothing, & \text{otherwise}, \end{cases} \end{aligned} \tag{1.2}$$

$$[A, B] \vee [A', B'] = [A \cap A', B \cup B']. \tag{1.3}$$

The lattice properties are readily verified. Note that the meet can be viewed as a Boolean intersection. However, for the join we have

$$[A, B] \vee [A', B'] \supset [A, B] \cup [A', B']$$

and often there is no equality.

Clearly, we can define a partial order '$\leqslant$' by

$$[A, B] \leqslant [A', B'] \Leftrightarrow [A, B] \subset [A', B'] \qquad \text{or (equivalently)} \qquad A' \subset A \subset B \subset B'. \tag{1.4}$$

We define a rank function $\rho: \mathscr{I}_n \to \mathbb{N} \cup \{0\}$ by

$$\rho([A, B]) = \begin{cases} 0, & \text{if } [A, B] = I_\varnothing, \\ |B \setminus A| + 1, & \text{if } [A, B] \neq I_\varnothing. \end{cases}$$

One readily verifies that ρ is upper semimodular; that is,

$$\rho([A, B] \vee [A', B']) + \rho([A, B] \wedge [A', B']) \leqslant \rho([A, B]) + \rho([A', B']).$$

This is not used in this paper.

However, we frequently use an equivalent description of non-empty intervals.

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Instead of $[A, B]$ we write $\langle C, D\rangle$, where $C = A$ and $D = B \setminus A$. Note that $C \cap D = \emptyset$ and, given any disjoint subsets C and D of $[n]$, $\langle C, D\rangle$ is the interval corresponding to $[C, C \cup D]$.

Now $\langle C, D\rangle \leqslant \langle C', D'\rangle \Leftrightarrow C' \subset C$ and $(C \setminus C') \cup D \subset D' \Rightarrow C' \subset C$ and $D \subset D'$.

One also readily verifies that

$$\langle C, D\rangle \wedge \langle C', D'\rangle = \langle C \cup C', D \cap D'\rangle, \qquad \text{if } \langle C, D\rangle \wedge \langle C', D'\rangle \neq I_{\emptyset}. \tag{1.5}$$

Finally (with a little abuse of notation), we also use ρ for the second interval description:

$$\rho(\langle C, D\rangle) = |D| + 1. \tag{1.6}$$

2. The AZ-identity, the LYM-inequality, and the Spencer Property for $\mathscr{I}_n$

Let us introduce $\mathscr{B}_n^k = \binom{[n]}{k}$ and let us denote by $\mathscr{I}_n^k$ the set of intervals from $\mathscr{I}_n$ of rank k $(0 \leqslant k \leqslant n+1)$.

Observe first that, for all $I \in \mathscr{I}_n^k$,

$$|\{I' \in \mathscr{I}_n^{k+1} : I' \supset I\}| = n - k + 1 \tag{2.1}$$

and that

$$|\{I' \in \mathscr{I}_n^{k-1} : I' \subset I\}| = 2(k-1). \tag{2.2}$$

This regularity property of a lattice is sufficient for the LYM-inequality to hold. We move directly to the AZ-identity. For any $\mathscr{A} \subset \mathscr{I}_n$ and any $I = [A, B] \in \mathscr{I}_n$ with

$$\mathscr{A}_I = \{K \in \mathscr{A} : K \subset I\} \neq \emptyset \tag{2.3}$$

write

$$\mathscr{A}_I = \{[A_i, B_i] : 1 \leqslant i \leqslant \alpha\} \tag{2.4}$$

and define

$$W_{\mathscr{A}}(I) = \left(|B| - \left|\bigcup_{i=1}^{\alpha} A_i\right|\right) + \left(\left|\bigcap_{i=1}^{\alpha} B_i\right| - |A|\right). \tag{2.5}$$

If (2.3) does not hold, set $W_{\mathscr{A}}(I) = 0$.

Theorem 1 (AZ-identity). *For any $\mathscr{A} \subset \mathscr{I}_n$,*

$$\sum_{I \in \mathscr{I}_n} \frac{W_{\mathscr{A}}(I)}{2^{n-\rho(I)+2}(\rho(I)-1)\binom{n}{\rho(I)-1}} \equiv 1.$$

Proof. By (2.1) (or (2.2)), $\mathscr{I}_n$ has exactly

$$2^n \prod_{k=1}^{n+1} (n-k+1)\left(\text{or } 2^n \prod_{k=2}^{n+1} (k-1)\right) = 2^n n!$$

maximal chains. Also, exactly

$$(n - \rho(I) + 1)!\, W_{\mathscr{A}}(I) 2^{\rho(I)-2} (\rho(I) - 2)!$$

maximal chains leave the upset $\mathscr{U}(\mathscr{A}) = \{K \in \mathscr{I}_n : \exists I' \in \mathscr{A} \text{ with } K \geqslant I'\}$ in $I = [A, B]$. Since $\rho(I) = |B \setminus A| + 1$, we obtain

$$\sum_{I \in \mathscr{U}(\mathscr{A})} (n - \rho(I) + 1)!\, W_{\mathscr{A}}(I) 2^{\rho(I)-2} (\rho(I) - 2)! = 2^n n!$$

Since $W_{\mathscr{A}}(I) = 0$ for $I \notin \mathscr{U}(\mathscr{A})$, the identity follows. □

The Whitney numbers w_k of $\mathscr{I}_n$ are defined by

$$w_k = |\mathscr{I}_n^k| \qquad \text{for } 0 \leqslant k \leqslant n+1. \tag{2.6}$$

They can be evaluated.

LEMMA 1. *We have*:
(i) $w_k = \binom{n}{k-1} 2^{n-k+1}$ *for* $0 < k \leqslant n+1$ *and* $w_0 = 1$ *and, consequently,*
(ii) $|\mathscr{I}_n| = \sum_{k=0}^{n+1} w_k = 3^n + 1$.

PROOF. The set $\mathscr{I}_n^k$ is exactly the set of intervals $\langle C, D\rangle$ with $D \in \mathscr{B}_n^{k-1}$ and $C \subset D^c = [n] \setminus D$. Therefore (i) holds and (ii) follows, because $|\mathscr{I}_n| = \sum_{k=0}^{n+1} |\mathscr{I}_n^k|$. □

COROLLARY 1 (LYM-inequality). *For any antichain* $\mathscr{A} \subset \mathscr{I}_n$,

$$\sum_{k=0}^{n+1} \frac{|\mathscr{A} \cap \mathscr{I}_n^k|}{w_k} \leqslant 1.$$

PROOF. For $\mathscr{I} \in \mathscr{A}$, by the antichain property and by (2.5),

$$W_{\mathscr{A}}(I) = (|B| - |A|) + (|B| - |A|) = 2(\rho(I) - 1)$$

and, by Theorem 1,

$$\sum_{I \in \mathscr{A}} \frac{2(\rho(I)-1)}{2^{n-\rho(I)+2}(\rho(I)-1)\binom{n}{\rho(I)-1}} \leqslant 1.$$

Using Lemma 1(i), we obtain

$$\sum_{I \in \mathscr{A}} \frac{1}{w_{\rho(I)}} \leqslant 1$$

and thus the result. □

COROLLARY 2 (Sperner property). (i) *For every antichain* $\mathscr{A} \subset \mathscr{I}_n$,

$$|\mathscr{A}| \leqslant \max_{0<k\leqslant n+1} w_k = \binom{n}{\left\lceil \frac{n+1}{3} \right\rceil - 1} 2^{n - \lceil \frac{n+1}{3} \rceil + 1}.$$

(ii)

$$\max_{0<k\leqslant n+1} w_k = \begin{cases} w_{l+1}, & \textit{if } n+1 = 3l+1, \\ w_{l+1}, & \textit{if } n+1 = 3l+2, \\ w_{l+1} = w_l, & \textit{if } n+1 = 3l. \end{cases}$$

(iii) *The antichains* $\mathscr{A}$ *of maximal length are*

$$\mathscr{A} = \begin{cases} \mathscr{I}_n^{l+1}, & \textit{if } n+1 = 3l+m, \quad m = 0, 1, 2, \\ \mathscr{I}_n^l, & \textit{if } n+1 = 3l. \end{cases}$$

Thus, if $3 \mid n+1$, *then there are two optimal antichains.*

PROOF. Corollary 1 implies that $|\mathscr{A}| \leqslant \max_{0<k\leqslant n+1} w_k$. The condition $w_k \geqslant \max(w_{k-1}, w_{k+1})$ gives the necessary condition for w_k to be maximal: $\lceil \frac{n+1}{3} \rceil \leqslant k \leqslant \lfloor \frac{n+1}{3} \rfloor + 1$. It can also be verified to be sufficient. Thus (i) follows. Also (ii) is a consequence.

The antichains specified in (iii) are thus optimal. It remains to be seen that there are no others in the case $3 \mid n+1$.

Let $\mathscr{A}$ be a antichain of maximal length w_l. By Theorem 1 (or even Corollary 1), necessarily,

$$|\mathscr{A} \cap \mathscr{I}_n^l| + |\mathscr{A} \cap \mathscr{I}_n^{l+1}| = w_l$$

and, again by Theorem 1, $W_{\mathscr{A}}(I) = 0$ for $I \notin \mathscr{A}$. Now suppose that $\mathscr{A} \neq \mathscr{I}_n^l, \mathscr{I}_n^{l+1}$. For all $I = [A, B] \in \mathscr{I}_n^{l+1} \setminus \mathscr{A}$, by its definition $W_{\mathscr{A}}(I) = 0$ implies that, for all $x \in B \setminus A$, $A' = A \cup \{x\}$ satisfies $[A', B] \in \mathscr{A} \cap \mathscr{I}_n^l$ and $B' = B \setminus \{x\}$ satisfies $[A, B'] \in \mathscr{A} \cap \mathscr{I}_n^l$. This means that all sub-intervals of I, which have a rank l, are in $\mathscr{A} \cap \mathscr{I}_n^l$. However, no $I \in \mathscr{A} \cap \mathscr{I}_n^{l+1}$ has a sub-interval in $\mathscr{A} \cap \mathscr{I}_n^l$. This means that the bipartite graph $(\mathscr{I}_n^l, \mathscr{I}_n^{l+1}, \subset)$ has two connected components. This is impossible, because $I = [A, B] \in \mathscr{I}_n^{l+1}$ is connected to $J = [\varnothing, B'] \in \mathscr{I}_n^{l+1}$ by alternating deleting elements from A and B and each two vertices $[\phi, B']$ and $[\phi, B'']$ are connected in this graph. □

3. Intersecting Systems in $\mathscr{I}_n$ and $\mathscr{I}_n^k$, the Erdös–Ko–Rado Property and Uniqueness

The goal of our investigations is to understand how intersecting systems, which have been studied extensively in Boolean lattices [see [4, 6]) and also in other structures (see [4] and [5, 8–10, 14, 15]), behave in $\mathscr{I}_n$ and $\mathscr{I}_n^k$.

We call $S \subset \mathscr{I}_n$ an *intersecting system*, if for all $I, I' \in S$,

$$I \wedge I' = I \cap I' \neq \varnothing. \tag{3.1}$$

Also, we say that S is *saturated*, if it is not a proper subset of an intersecting system.

A simple and basic saturated intersecting system is, for $C \subset [n]$,

$$\mathscr{I}_n(C) = \{I \in \mathscr{I}_n : C \in I\}. \tag{3.2}$$

We show first that its cardinality is independent of C.

Lemma 2. *For all $C \subset [n]$,*

$$|\mathscr{I}_n(C)| = 2^n.$$

Proof. The intervals containing C are of the form $[A, B]$, $A \subset C \subset B$. Clearly, there are $2^{|C|} 2^{n-|C|}$ such intervals. □

Next we show that all saturated intersecting systems are of the form (3.2).

Lemma 3. *For every intersecting system $S \subset \mathscr{I}_n$ a $D \in \mathscr{I}_n$, $D \neq I_\varnothing$, exists with $D \leqslant I$ for all $I \in S$. Furthermore, if S is saturated, then $S = \mathscr{I}_n(C)$ for some non-empty $C \subset [n]$.*

Proof. Write $S = \{[A_t, B_t] : t \in T\}$ and note that $A_i \subseteq B_j$ $(i, j \in T)$ implies

$$\bigcup_{t \in T} A_t \subset \bigcap_{t \in T} B_t. \tag{3.3}$$

So, the interval $D = [\bigcup_{t \in T} A_t, \bigcap_{t \in T} B_t]$ satisfies $D \neq I_\varnothing$ and $D \leqslant I$ $(I \in S)$. Furthermore, when S is saturated, then $D = [C, C]$ for some $C \subset [n]$ and $D \in S$, $S = \mathscr{I}_n(C)$. □

Whereas in $\mathscr{I}_n$ the intersecting systems $\mathscr{I}_n(C)$ $(\varnothing \neq C \subset [n])$ are exactly the largest intersecting systems, in $\mathscr{B}_n$ the systems $\{X \subset [n] : x \in X\}$ $(x \in [n])$ are not the only intersecting systems of maximal cardinality.

However, connections between these lattices can be established via their Whitney numbers. For this, we define

$$\mathscr{I}_n^k(C) = \mathscr{I}_n(C) \cap \mathscr{I}_n^k. \tag{3.4}$$

Lemma 4. $|\mathscr{I}_n^k(C)| = \binom{n}{k-1}$.

Proof. Note that

$$\mathscr{I}_n^k(C) = \left\{\langle C \cap E^c, E\rangle : E \in \binom{[n]}{k-1}\right\} = \left\{[C \setminus E, C \cup E] : E \in \left\{\begin{matrix}[n]\\ k-1\end{matrix}\right)\right\}$$

and thus the claimed identity. □

We now consider intersecting systems of intervals of rank k. This is analogous to the case of k elements sets, considered originally in [6]. It is remarkable that in the new situation we have uniqueness in the sense that only the $\mathscr{I}_n^k(C)$'s appear as optimal systems.

Theorem 2. *For every intersecting system $S \subset \mathscr{I}_n^k$,*

$$S \leqslant \binom{n}{k-1}$$

and the $\mathscr{I}_n^k(C)$ $(C \subset [n],\ 1 \leqslant |C| \leqslant n-k+1)$ *are exactly the intersecting systems achieving equality.*

The analysis proceeds in terms of a useful concept of parallelism.

We say that the interval $\langle C, D\rangle \in \mathscr{I}_n \setminus \{I_\varnothing\}$ has *direction* $d(\langle C, D\rangle) = D$. The empty interval $I_\varnothing$ has no direction.

Intervals with the same direction are called parallel. We write $I \parallel I'$, if I and I' are parallel. Obviously,

$$\rho(I) = \rho(I'), \qquad \text{if } I \parallel I'. \tag{3.5}$$

The next property is familiar from geometry.

Lemma 5. *Parallel intervals are disjoint or, formally,*

$$I \parallel I', I \neq I' \Rightarrow I \cap I' = I \wedge I' = I_\varnothing.$$

Proof. For $I = \langle C, D\rangle = [C, C \cup D]$, $I' = \langle C', D\rangle = [C', C' \cup D]$, $C \neq C'$ and $C \cap D = C' \cap D = \varnothing$ we have $I \cap I' = [C \cup C', (C \cup D) \cap (C' \cup D)] = [C \cup C', (C \cap C') \cup D] = I_\varnothing$, because $(C \cup C') \cap D = \varnothing$ and $C \cup C' \not\subset C \cap C'$ for $C \neq C'$. Consequently, $C \cup C' \nsubseteq (C \cap C') \cup D$. □

Using this result one readily verifies the next statements, which shed new light on Lemmas 1 and 4.

Lemma 6. (i) *For every direction $D \subset [n]$, the intervals $\langle C, D\rangle$ and $C \subset D^c$ partition $\mathscr{I}_n$.*

(ii) *$\mathscr{I}_n$ can be partitioned into 2^n families of parallel intervals.*

(iii) *$\mathscr{I}_n^k$ can be partitioned into $\binom{n}{k-1}$ families, each having 2^{n-k+1} parallel intervals.*

PROOF OF THEOREM 2. Clearly, Lemma 6(iii) and Lemma 5 imply that $|S| \leqslant \binom{n}{k-1}$.

Furthermore, by Lemma 3, $S \subset \mathscr{I}_n(C)$, and since by assumption $S \subset \mathscr{I}_n^k$, we conclude that $S \subset \mathscr{I}_n(C) \cap \mathscr{I}_n^k$. If S is optimal, then necessarily $S = \mathscr{I}_n(C) \cap \mathscr{I}_n^k = \mathscr{I}_n^k(C)$, by Lemma 3. □

REMARK 1. It is natural also to consider intersecting systems with a qualified constraint. We call $S \subset \mathscr{I}_n$ *d-intersecting* if, for all $I, I' \in S$,

$$\rho(I \wedge I') \geqslant d. \tag{3.6}$$

Our previous definition is included in the case $d = 1$.

There is a simple reduction for the cases $d \geqslant 2$.

For $I = \langle C, D \rangle$, $I' = \langle C', D' \rangle$, (3.6) is equivalent to $\langle C, D \rangle \wedge \langle C', D' \rangle = \langle C \cup C', D \cap D' \rangle$ with $|D \cap D'| \geqslant d - 1$ and $D \neq D'$.

Therefore for every d-intersecting system $S \subset \mathscr{I}_n$ there corresponds a $(d-1)$-intersecting system of $\mathscr{B}_n$ $\mathscr{D} = \{D : \exists \langle C, D \rangle \in S\}$ of the *same cardinality*. Conversely, to every $(d-1)$-intersecting system of $\mathscr{B}_n$ $\mathscr{D}$ there corresponds a d-intersecting system of $\mathscr{I}_n$ of the same cardinality; namely, for any $E \in \mathscr{B}_n$,

$$S = \{\langle E \setminus D, D \rangle : D \in \mathscr{D}\}.$$

Similarly, there is such a correspondence between intersecting systems $S \subset \mathscr{I}_n^k$ and $\mathscr{D} \subset \mathscr{B}_n^{k-1}$.

4. FROM LOCAL TO GLOBAL INTERSECTION OF INTERVALS AND INTERSECTING ANTICHAINS

The fact that parallel intervals are disjoint (Lemma 5) has a useful extension.

LEMMA 7. *For two non-disjoint intervals $\langle C_1, D_1 \rangle$ and $\langle C_2, D_2 \rangle$, with $D_1 \subset D_2$, necessarily*

$$\langle C_1, D_1 \rangle \leqslant \langle C_2, D_2 \rangle$$

or (equivalently)

$$[C_1, C_1 \cup D_1] \subset [C_2, C_2 \cup D_2].$$

PROOF. Recall the definition (1.4) of the partial order. By our assumption, for some $X \subset [n]$,

$$C_1 \subset X \subset C_1 \cup D_1, \qquad C_2 \subset X \subset C_2 \cup D_2$$

and therefore

$$C_1 \subset C_2 \cup D_2, \qquad C_2 \subset C_1 \cup D_1. \tag{4.1}$$

Since $D_1 \subset D_2$, we conclude first that

$$C_1 \cup D_1 \subset C_2 \cup D_2. \tag{4.2}$$

Since $C_2 \cap D_1 \subset C_2 \cap D_2 = \varnothing$ and $C_2 \subset C_1 \cup D_1$ we conclude further that

$$C_2 \subset C_1. \tag{4.3}$$

Finally, (4.2) and (4.3) say that

$$C_2 \subset C_1 \subset C_1 \cup D_1 \subset C_2 \cup D_2$$

and thus $[C_1, C_1 \cup D_1] \subset [C_2, C_2 \cup D_2]$. □

A well-known inequality of Bollobas [3] states that, for any intersecting antichain $\mathscr{F} \subset \mathscr{B}_n$,

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{|\mathscr{F} \cap B_n^k|}{\binom{n-1}{k-1}} \leqslant 1. \tag{4.4}$$

What is the Bollobas-type inequality for $\mathscr{I}_n$? The answer follows by simple reasoning. For an intersecting antichain $\mathscr{S} = \{\langle C_i, D_i \rangle: 1 \leqslant i \leqslant m\}$ in $\mathscr{I}_n$ by Lemma 7 necessarily $\{D_i: 1 \leqslant i \leqslant m\}$ is an antichain in $\mathscr{B}_n$. We obtain the following inequality.

THEOREM 3. *For an intersecting antichain $\mathscr{S}$ in $\mathscr{I}_n$,*

$$\sum_{k=1}^{n+1} \frac{|\mathscr{S} \cap \mathscr{I}_n^k|}{\binom{n}{k-1}} \leqslant 1.$$

Conversely, we can translate this inequality backwards. Thus the LYM-inequality for the Boolean lattice is exactly the Bollobas-type inequality for the Boolean interval lattice.

5. An Intersection Theorem for Chain Posets

We now introduce chain posets and prove for them an intersection property conjectured by Erdös *et al.* in [7]. There and also by Füredi (according to [7]), this conjecture has been verified in over large range of parameters. The methods used do not seem to be suitable to settle the conjecture. Our approach does this, and is very simple. In Section 6 we give an even simpler and more direct proof.

A strictly increasing sequence of subsets of $[n]$ and of length k is claled a k-chain. $\mathscr{C}_n^k$ denotes the set of all those chains and we define $\mathscr{C}_n = \bigcup_{k=1}^{n+1} \mathscr{C}_n^k$. The chain $\mathbb{C} = \{C_1 \subset C_2 \subset \cdots \subset C_l\}$ is contained in the chain $\mathbb{C}' = \{C_1' \subset C_2' \subset \cdots \subset C_{l'}'\}$, if $\{C_n: 1 \leqslant i \leqslant l\} \subset \{C_i': 1 \leqslant i \leqslant l'\}$. We denote this containment by '$\prec$'. Then $(\mathscr{C}_n, \prec)$ is a poset, which we call the poset of chains (on an n-set).

With the chain $\mathbb{C}$ we associate an interval $\operatorname{conv}(\mathbb{C}) = [C_1, C_l] \in \mathscr{I}_n$ and, conversely, with an interval $I \in \mathscr{I}_n$ we associate the set of chains

$$\mathscr{C}_n(I) = \{\mathbb{C} \in \mathscr{C}_n: \operatorname{conv}(\mathbb{C}) = I\}. \tag{5.1}$$

Furthermore, for any set of chains $\mathscr{C} \subset \mathscr{C}_n$ we consider the subset of chains

$$\mathscr{C}(I) = \{\mathbb{C} \in \mathscr{C}: \operatorname{conv}(\mathbb{C}) = I\}. \tag{5.2}$$

Similarly $\mathscr{C}_n^k(I)$ are the k-chains with convex hull I. For fixed k and n, $|\mathscr{C}_n^k(I)|$ depends only on $\rho(I) = r$, say, and shall be denoted by $q(r)$. Clearly, $q(r) = 0$, if $r < k$.

Now we consider intersecting chains. Two chains $\mathbb{C}$ and $\mathbb{C}'$ are intersecting if, for some pair (i, i'), $C_i = C_{i'}'$. We write $\mathbb{C} \overset{\lambda}{\sim} \mathbb{C}'$. A family of chains $\mathscr{C}$ is intersecting if $\mathbb{C} \overset{\lambda}{\sim} \mathbb{C}'$ for all $\mathbb{C}, \mathbb{C}' \in \mathscr{C}$. The maximal cardinality of such a family shall be $M(n)$. If only k-chains are permitted in $\mathscr{C}$, then we denote the maximal cardinality of $|\mathscr{C}|$ by $M(n, k)$.

We say that $\mathscr{C}$ is a *simple* intersecting family if, for some $X \subset [n]$, all chains in $\mathscr{C}$ have X as a member (or meet X).

CONJECTURE ESS & F. $M(n, k)$ is assumed by the simple intersecting family ${}_{\varnothing}\mathscr{C}_n^k$ of all k-chains meeting $\varnothing$ (or $[n]$). Since

$$ {}_{\varnothing}\mathscr{C}_n^k = \bigcup_{I=[\phi,B],B\in\binom{[n]}{r-1},r\geqslant k} \mathscr{C}_n^k(I) \tag{5.3}$$

and, therefore,

$$|{}_{\varnothing}\mathscr{C}_n^k| = \sum_{r=k}^{n+1} q(r)\binom{n}{r-1}, \tag{5.4}$$

the conjecture can be restated as

$$M(n, k) = \sum_{r=k}^{n+1} q(r)\binom{n}{r-1}. \tag{5.5}$$

THEOREM 4. *The intersecting family of all k-chains in $\mathscr{B}_n$ starting with the empty set $\varnothing$ has the maximal cardinality $M(n, k)$.*

PROOF. Let $\mathscr{C}$ be an intersecting family of k-chains of cardinality $|\mathscr{C}| = M(n, k)$. Introduce $\mathscr{I}_n(\mathscr{C}) = \{\mathrm{conv}(\mathbb{C}): \mathbb{C} \in \mathscr{C}\}$ and observe that (recalling (5.2)) $(\mathscr{C}(I))_{I\in\mathscr{I}_n(\mathscr{C})}$ is a partition of $\mathscr{C}$.

Now write

$$\mathscr{I}_n(\mathscr{C}) = \{[A_i, B_i]: i \in T\}.$$

Note also that the intersection property of the chains implies that $\mathscr{I}_n(\mathscr{C})$ is an intersecting system of intervals.

Therefore $A_i \subset B_j$ for all $i, j \in T$ and hence, for all $i \in T$,

$$A_i \subset \bigcap_{j\in T} B_j = \underline{B} \text{ (say)} \subset B_i,$$

i.e.

$$\underline{B} \in I \qquad \text{for all } I \in \mathscr{I}_n(\mathscr{C}).$$

This means that, in the terminology of Section 3,

$$\mathscr{I}_n(\mathscr{C}) \subset \mathscr{I}_n(\underline{B}) = \bigcup_{r=0}^{n+1} \mathscr{I}_n^r(\underline{B})$$

and

$$\begin{aligned} M(n, k) = |\mathscr{C}| &= \sum_{I\in\mathscr{I}_n(\mathscr{C})} |\mathscr{C}(I)| \leqslant \sum_{I\in\mathscr{I}_n(\mathscr{C})} q(\rho(I)) \\ &\leqslant \sum_{I\in\mathscr{I}_n(\underline{B})} q(\rho(I)) = \sum_{r=0}^{n+1}\sum_{I\in\mathscr{I}_n^r(\underline{B})} q(r) = \sum_{r=0}^{n+1}\binom{n}{r-1}q(r) \qquad \text{(by Lemma 4).} \quad \square \end{aligned}$$

REMARK 2. $q(r)$ equals the number of ways in which a set of $r - 1$ elements can be partitioned into a sequence of $k - 1$ non-empty subsets. This observation gave us the idea of constructing the more direct proof of Theorem 4 in Section 6.

6. A Direct Proof of Theorem 4

With the chain $\mathbb{A} = \{A_1 \subset A_2 \subset \cdots \cap A_k\} \in \mathscr{C}_n^k$, we associate the sequence of disjoint non-empty sets

$$C = A_1, \quad D_1 = A_2 \backslash A_1, \quad D_2 = A_3 \backslash A_2, \quad \ldots, \quad D_{k-1} = A_k \backslash A_{k-1}. \tag{6.1}$$

Conversely, from $\langle C, D_1, \ldots, D_{k-1}\rangle$ we can recover $\mathbb{A}$ via the equations

$$A_1 = C, \quad A_j = C \cup \left(\bigcup_{i=1}^{j-1} D_i\right) \qquad \text{for } j \geq 2. \tag{6.2}$$

Thus we have an alternate representation of chains: $\langle C, D_1, \ldots, D_{k-1}\rangle$.

We now study the intersection property of chains in this terminology. Recall that parallel intervals are disjoint (Lemma 5 in Section 3).

Therefore, for intervals $\langle C'; D_1\rangle \overset{\lambda}{\sim} \langle C; D_1\rangle$ implies that $C = C'$ and thus $\langle C'; D_1\rangle = \langle C; D\rangle$. This fact generalizes.

Lemma 8.

$$\langle C'; D_1, \ldots, D_{k-1}\rangle \overset{\lambda}{\sim} \langle C; D_1, \ldots, D_{k-1}\rangle \Rightarrow C = C'$$

and

$$\langle C'; D_1, \ldots, D_{k-1}\rangle = \langle C; D_1, \ldots, D_{k-1}\rangle.$$

Proof. There are j and j' with

$$C \cup \left(\bigcup_{i=1}^{i-1} D_i\right) = C' \cup \left(\bigcup_{i=1}^{j'-1} D_i\right). \tag{6.3}$$

We subtract $C \cup C'$ from both sides and obtain, using (6.1),

$$\bigcup_{i=1}^{j-1} D_i = \left(C \cup \left(\bigcup_{i=1}^{j-1} D_i\right)\right) \Big\backslash (C \cup C')$$
$$= \left(C' \cup \left(\bigcup_{i=1}^{j'-1} D_i\right)\right) \Big\backslash (C \cup C') = \bigcup_{i=1}^{j'-1} D_i.$$

Now, necessarily, $C = C'$. □

Proof of Theorem 4. By Lemma 8, for a family $\mathscr{C}$ of intersecting k-chains $|\mathscr{C}|$ does not exceed the cardinality $d(n, k)$ of the set $\mathscr{D}_n^k$ of all distinct sequences $(D_1, \ldots, D_{k-1})$ with $D_i \cap D_j = \emptyset$ $(i \neq j)$ and $D_i \subset [n]$ for $i = 1, \ldots, k-1$.

Thus $M(n, k) \leq d(n, k)$. Instead of determining $d(n, k)$ by counting we just observe that there is a bijection $\Psi: \mathscr{D}_n^k \to \{\langle \emptyset, D_1, \ldots, D_{k-1}\rangle : (D_1, \ldots, D_{k-1}) \in \mathscr{D}_n^k\}$. The image is exactly ${}_{\emptyset}\mathscr{C}_n^k$ and its optimality is thus proved.

Remark 3. Inspection of the proofs shows that the condition $D_i \neq \emptyset$ $(i = 1, \ldots, k-1)$ was not used. Therefore also the intersection problem for chains of length $\leq k$ has a solution in the set of thus restricted chains starting in $\emptyset$. In particular, this is also true for $k = n + 1$.

Remark 4. We have started to think about families of d-intersecting k-chains. Here the chains $A_1 \subset A_2 \subset \cdots \subset A_m$ and $A_1' \subset A_2' \subset \cdots \subset A_{m'}'$ are d-intersecting if there are indices $i_1 < \cdots < i_d$ and $j_1 < \cdots < j_d$ with $A_{i_l} = A_{j_l}'$ for $l = 1, 2, \ldots, d$.

Applying a shifting operator as in [7], one can show that there is an optimal family $\mathscr{F}$ with a *strong d-intersection property* saying that for any $\mathbb{C}$, $\mathbb{C}' \in \mathscr{F}$ there is a subset $S \subset \{0, 1, \ldots, n\}$, $|S| \geqslant d$, such that all $X_j = \{1, 2, \ldots, j\}$ with $j \in S$ are contained in both, $\mathbb{C}$ and $\mathbb{C}'$. Here $X_0 = \varnothing$.

This means that there is a set $\mathscr{S} = \{S(F): F \in \mathscr{F}\}$ of subsets of $\{0, 1, \ldots, n\}$ with

$$|S(F) \cap S(F')| \geqslant d$$

associated with F, such that for $j \in S(F)$, X_j is contained in F.

It seems natural to *conjecture* that for some $\varepsilon > 0$ and n large for $d \leqslant n(1-\varepsilon)$ there is an optimal family of d-intersecting chains all containing $X_0, X_1, \ldots, X_{d-1}$. The restriction on d is essential, because otherwise the guess is false.

To see this, let us assume that $n - d$ is bounded by a constant b. Then the number of chains in the family just specified is bounded by a function of b only. However, the family of chains containing $\{X_i\}_{i \in S}$, where S runs through all subsets of $\{0, 1, \ldots, n\}$ with cardinality at least $(n+d)/2$, is d-intersecting and increases with n. Our last contribution is in the spirit of this construction.

CONJECTURE. For all n,d and some $w \geqslant d$ there is an optimal d-intersecting family of chains, which contain at least $\lceil (w+d)/2 \rceil$ members of $X_0, X_1, \ldots, X_{w-1}$.

7. ANOTHER DESCRIPTION FOR $\mathscr{I}_n$

Consider ${}_{-}\mathscr{I}_n = \mathscr{I}_n \setminus \{I_\varnothing\}$. We express $C \subset [n]$ as a binary sequence $c^n = (c_1, \ldots, c_n)$ of length n. An interval $[A, B]$ can thus be described by a pair $[a^n, b^n]$, where $a_t \leqslant b_t$ for $t = 1, \ldots, n$. This pair in turn can be described by one sequence $z^n = (z_1, \ldots, z_n)$, with

$$z_t = 2 - (a_t + b_t), \qquad t = 1, 2, \ldots, n. \tag{7.1}$$

We also write $z^n = \varphi([A, B])$. Conversely, z^n determines $[A, B]$ uniquely.

Moreover, if $z^n = \varphi([A, B])$ and $z'^n = \varphi([A', B'])$, then

$$[A, B] \leqslant [A', B'] \Leftrightarrow \text{for every } t \in [n] \text{ with } z'_t \neq 1,\ z_t = z'_t \text{ holds.} \tag{7.2}$$

Therefore the poset ${}_{-}\mathscr{I}_n$ can be expressed as a product of posets. $[A, B] \in \mathscr{I}_n^{k+1}$, $k \geqslant 0$, means that, for $z^n = \varphi([A, B])$, $|\{z_t: z_t = 1, t \in [n]\}| = k$.

Two intervals $[A, B]$ and $[A', B'] \in {}_{-}\mathscr{I}_n^{k+1}$ are intersecting iff $|z_t - z'_t| \neq 2$ for all $t \in [n]$ and z^n and z'^n do not have the 1's in exactly the same k positions.

This again allows us to characterize the intersecting systems in $\mathscr{I}_n^{k+1}$ of maximal cardinality $\binom{n}{k}$.

We already have mentioned the system

$$\mathscr{I}_n^k(C) = \{\langle C \setminus D, D\rangle: D \in \mathscr{B}_n^k\}, \qquad C \subset [n]. \tag{7.3}$$

Here $\langle C \setminus D, D\rangle = [C \setminus D, C \cup D]$ and $\varphi([C \setminus D, C \cup D]) = z^n$, with

$$z_t = \begin{cases} 0 & \text{for } t \in C \setminus D, \\ 1 & \text{for } t \in D, \\ 2 & \text{for } t \notin C \cup D. \end{cases} \tag{7.4}$$

By Lemma 3, for a maximal intersecting system S, necessarily

$$S = \{\langle E_D, D\rangle: D \in \mathscr{B}_n^k\}.$$

Let T_i ($i = 0, 2$) be the components in which an i occurs for some z^n. Then

$$T_0 \cap T_2 = \emptyset.$$

Furthermore, $T_0 \cup T_2 = [n]$, because otherwise not all $D \in \mathcal{B}_n^k$ are used.

Define $C = T_0$ and observe that $E_D = C \setminus D$. Thus all optimal systems are of the form (7.3).

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