Intersecting Systems

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Abstract

An intersecting system of type $(\exists, \forall, k, n)$ is a collection $\mathbb{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ pf pairwise disjoint families of k–subsets of an n–element set satisfying the following condition. For every ordered pair $\mathcal{F}_i$ and $\mathcal{F}_j$ of distinct members of $\mathbb{F}$ there exists an $A \in \mathcal{F}_i$ that intersects every $B \in \mathcal{F}_j$. Let $I_n(\exists, \forall, k)$ denote the maximum possible cardinality of an intersecting system of type $(\exists, \forall, n)$. Ahlswede, Cai and Zhang conjectured that for every $k \geq 1$, there exists an $n_0(k)$ so that $I_n(\exists, \forall, k) = \binom{n-1}{k-1}$ for all $n > n_0(k)$. Here we show that this is true for $k \leq 3$, but false for all $k \geq 8$. We also prove some related results.

1 Introduction

One of the basic results in extremal set theory is the Erdős–Ko–Rado (EKR) theorem [4]: if $\mathcal{F}$ is an intersecting family of k–element subsets of $N = \{1, 2, \dots, n\}$ (i.e. every two members of $\mathcal{F}$ have a non–empty intersection) and $n \geq 2k$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and this bound is attained. Several subsequent results generalize and strengthen this result.

Recently Ahlswede, Cai and Zhang [1] considered various problems that study extremal properties of a collection of families of the set N. They recognized that many classical problems dealing with families suggest interesting and challenging questions when one replaces the notion of a family of sets by one of a collection of families of sets.

In the present note we consider the analogous problem of the EKR theorem for such systems. Let $\mathbb{F}$ be a collection of pairwise disjoint families of k–subsets of N. We call $\mathbb{F}$ an *intersecting system of type* $(\exists, \forall, n)$ if any ordered pair of distinct families $\mathcal{F}$ and $\mathcal{F}'$ satisfies the following condition:

$$\exists F \in \mathcal{F} \text{ such that } \forall F' \in \mathcal{F}' \qquad (F \cap F' \neq \varnothing). \tag{1.1}$$

With $\mathbb{F}$ as above, we say that F is a set in $\mathcal{F}$ responsible for the family $\mathcal{F}'$. Let $I_n(\exists, \forall, k)$ denote the maximum cardinality $|\mathbb{F}|$ of an intersecting system of type $(\exists, \forall, k, n)$. It is easy to see that

$$I_n(\exists, \forall, k) \geq \binom{n-1}{k-1}.$$

Indeed, the intersecting family $\mathbb{F}(1)$ of type $(\exists, \forall, k, n)$, which contains each of the $\binom{n-1}{k-1}$ k–subsets that contain 1 as a one element family shows it. From the other side the following trivial upper bound holds.

Proposition 1.1.

$$I_n(\exists, \forall, k) \leq j\binom{n-1}{k-1}. \tag{1.2}$$

Proof. Let $\mathbb{F}$ be an intersecting system of type $(\exists, \forall, k, n)$ and let $\mathcal{F}$ be the smallest family in $\mathbb{F}$. Then every edge of all other families must intersect at least one element of $\mathcal{F}$. Therefore

$$\left|\bigcup \mathbb{F}\right| \leq |\mathcal{F}|k\binom{n-1}{k-1},$$

and due to the minimality of $|\mathcal{F}|$

$$|\mathbb{F}| \leq \frac{1}{|\mathcal{F}|}|\mathcal{F}|k\binom{n-1}{k-1}.$$

□

Ahlswede, Cai and Zhang [1] made the following conjecture.

Conjecture 1.1. *There exists a function $n_0(k)$ such that for every $n \geq n_0(k)$ the equality $I_n(\exists, \forall, k) = \binom{n-1}{k-1}$ holds.*

Our main result here is that the conjecture is false for all $k \geq 8$. On the other hand, it holds for all $k \leq 3$. In addition, we prove the assertion of the conjecture in several special cases for some value of $n_0(k)$.

2 Some special cases

Our goal in this section is to prove Conjecture 1.1 for $k \geq 3$ and to clarify the basic properties of the counterexample which will be given in the next section.

Proposition 2.1. *Let $\mathbb{F}$ be an intersecting system of type $(\exists, \forall, n)$ on an $n \geq k^5$–element underlying set and suppose the family $\mathcal{F}_1 \in \mathbb{F}$ has empty total intersection, that is*

$$\bigcap_{F \in \mathcal{F}_1} F = \varnothing. \tag{2.1}$$

Then $|\mathbb{F}| < \binom{n-1}{k-1}$.

Proof. We apply a version of the so–called *kernel method* introduced by Hajnal and Rothschild [6]. It is easy to see that (2.1) implies that there exist at most $k+1$ elements $F_1, \dots, F_l$ of $\mathcal{F}_1$ such that

$$F_1 \cap F_2 \cap \cdots \cap F_l = \varnothing. \tag{2.2}$$

Indeed, let $F_1 = \{i_1, \dots, i_k\}$ be an arbitrary member of $\mathcal{F}_1$. By the assumption there exist $F_{i_1}, \dots, F_{i_k}$ in $\mathcal{F}_1$ such that $i_j \notin F_{i_j}$ $(i = 1, \dots, k)$, implying (2.2).

Let $H = F_1 \cup \cdots \cup F_l$. Then, by (2.2), for any family $\mathcal{F}_2 \neq \mathcal{F}_1$ the set $F \in \mathcal{F}_2$ which is responsible for the family $\mathcal{F}_1$ must intersect the set H in at least two vertices. Therefore

$$\begin{aligned}
|\mathbb{F}| &\leq 1 + \left| \left\{ F \in \binom{N}{k} : F \text{ can be responsible for } \mathcal{F}_1 \right\} \right| \\
&\leq 1 + \binom{|H|}{2}\binom{n-2}{k-2} \leq 1 + \binom{k(k+1)}{2}\binom{n-2}{k-2} \\
&\leq 1 + k^4 \binom{n-2}{k-2} < \binom{n-1}{k-1},
\end{aligned}$$

where in the last step we applied the fact that $k^5 \leq n$. □

By the above proposition, we assume from now on that condition (2.1) does not apply to any family $\mathcal{F} \in \mathbb{F}$. Let *ker*$(\mathcal{F})$ denote the *kernel* of $\mathcal{F}$, i.e. the non–empty intersection of all its members.

Now we will show that each member of a 'big' intersecting system has to contain 'many elements'.

Call an intersecting system $\mathbb{F}$ of type $(\exists, \forall, k, n)$ *minimal* if there is no 'superfluous' element in any family. More precisely, for every set $F \in \mathcal{F} \in \mathbb{F}$ there exists a family $\mathcal{F}' \in \mathbb{F}$ such that only F is responsible in $\mathcal{F}$ for the family $\mathcal{F}'$.

Note that if a given intersecting system $\mathbb{F}$ is not minimal then one can get a minimal one with the same cardinality by repeatedly deleting a superflous element as long as there is one.

Proposition 2.2. *Let $\mathbb{F}$ be an intersecting system of type $(\exists, \forall, k, n)$ and assume that there is no family with empty total intersection. If $\mathbb{F}$ contains a family $\mathcal{F}$ satisfying $1 < |\mathcal{F}| \leq n/k^3$ and $n \geq 2k^4$ the $|\mathbb{F}| < \binom{n-1}{k-1}$.*

Proof. We may suppose that our system is minimal. For every $F \in \mathcal{F}$ there exists a set $F' \in \mathcal{F}'$ such that $F \cap F' = \varnothing$. Otherwise F can be responsible for every other family in $\mathbb{F}$, and therefore the system would not be minimal. Let the subsystem $\mathbb{F}(F)$ contain all the families for which F is responsible. Any family in $\mathbb{F}(F)$ contains an element which is responsible for the family $\mathcal{F}'$, and this set intersects F as well as F'. hence the number of k–sets in N intersecting F and F' is an upper bound for the cardinality of this subsystem. Thus

$$|\mathbb{F}(F)| \leq k^2 \binom{n-2}{k-2}. \tag{2.3}$$

Therefore

$$|\mathbb{F}| \leq 1 + |\mathcal{F}| k^2 \binom{n-2}{k-2} < \binom{n-1}{k-1}, \tag{2.4}$$

where we made use of the assumption on the size of the family $\mathcal{F}$. □

Proposition 2.3. *Let $\mathbb{F}$ be an intersecting system of type $(\exists, \forall, k, n)$ and assume that there is no family with empty total intersection. If $n \geq 2k^4$ and $\mathbb{F}$ contains a family $\mathcal{F}_1$ of cardinality $|\mathcal{F}_1| = 1$ then $|\mathbb{F}| \leq \binom{n-1}{k-1}$.*

Proof. Again we suppose that our system is minimal and by Proposition 2.2 we may assume that

$$\text{if } |\mathcal{F}| > 1 \text{ then } |\mathcal{F}| > n/k^3. \tag{2.5}$$

We apply the well–known theorem of Hilton and Milner [5] about non–trivially intersecting k–hypergraphs. Here we give a slightly weaker form of it which is useful for our purposes.

Theorem 2.1. *Let $k > 2$ and $n > 2k$ and let $\mathcal{H}$ be an intersecting k–uniform hypergraph on the set N such that*

$$\bigcap_{F \in \mathcal{H}} F = \varnothing.$$

Then

$$|\mathcal{H}| < HM(n,k) := k\binom{n-2}{k-2}.$$

□

Returning to the proof of Proposition 2.3 we consider two cases:

Case 1 $\quad 1 \leq |\{\mathcal{F} \in \mathbb{F} : |\mathcal{F}| = 1\}| \leq HM(n,k)$ (2.6)

Suppose $\mathcal{F}_1 = \{F\}$. Then F is responsible for every other family in $\mathbb{F}$ and hence intersects every set in every other family. Using the fact that

$$\left| \left\{ H \in \binom{N}{k} : H \cap F \neq \varnothing \right\} \right| \leq k \binom{n-1}{k-1},$$

we get

$$\left|\bigcup_{|\mathcal{F}|>1} \mathcal{F}\right| \leq k\binom{n-1}{k-1}.$$

Applying assumptions (2.5) and (2.6) we conclude

$$\begin{aligned}|\mathbb{F}| &\leq HM(n,k) + \frac{k^3}{n}k\binom{n-1}{k-1} \\ &\leq k\binom{n-2}{k-2} + \frac{k^4}{n}\binom{n-1}{k-1} \\ &< \frac{2k^4}{n}\binom{n-1}{k-1} \leq \binom{n-1}{k-1}.\end{aligned}$$

Case 2 $|\{\mathcal{F} \in \mathbb{F} : |\mathcal{F}| = 1\}| > HM(n,k)$.

By the Hilton–Milner theorem [5] the system of all one–element families, as a set system, is a trivially intersecting system, that is, there is a common vertex, say 1, in every set. Furthermore, every set in a one element family is responsible for every other family, and hence intersects every set in every other family. In particular, it meets every set in the larger families, i.e. in the families with more than one member. Therefore the elements of the larger families intersect every set in the one–element families, implying, by the Hilton–Milner theorem, that they all contain the vertex 1. Hence

$$\left|\bigcup_{\mathcal{F}\in\mathbb{F}} \mathcal{F}\right| \leq \binom{n-1}{k-1},$$

as claimed. □

Now we are rerady to prove Conjecture 1.1 for the case $k \leq 3$. For $k = 1$ the conjecture is trivial, and for $k = 2$ it is proved in [2]. Here we prove it for $k = 3$.

Theorem 2.2. *There exists an integer n_0 such that $I_n(\exists, \forall, \varepsilon) = \binom{n-1}{2}$ for every $n \geq n_0$.*

Proof. To simplify the presentation we assume, whenever this is needed, that n is sufficiently large, and use the asymptotic $o(1)$ notation. All our $o(1)$'s denote quantities that tend to 0 as n tends to infinity.

Let $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ be a minimal intersecting system of type $(\exists, \forall, \varepsilon, n)$. Our objective is to show that $m \leq \binom{n-1}{2}$. Suppose this is false. By Proposition 2.1, 2.2 and 2.3 we may assume that $|\mathcal{F}_i| > n/27$ and $ker(\mathcal{F}_i)$ is nonempty for all $1 \leq i \leq m$.

Proposition 2.4. *The number of families $\mathcal{F}_i$ whose kernel is of cardinality 1 is $o(n^2)$.*

Proof. Suppose this is false and so for some $\varepsilon > 0$ there are arbitraly large n for which there are εn^2 families $\mathcal{F}_i$ with $|ker(\mathcal{F}_i)| = 1$. By averaging, at least εn of these families have the same kernel, say $\{1\}$. Each of these families contains more than $n/27$ triples that contain 1. Hence there are less than $\binom{n}{2} - \varepsilon n^2/27$ other families that contain a triple containing 1. There are $\binom{n}{2} - \left(\binom{n}{2} - \varepsilon n^2/27\right) = \Omega(n^2)$ additional families; families that contain no triple that contains 1. Fix a family, say $\mathcal{F}_1$, whose kernel is $\{1\}$, and let F_1, F_2, F_3 be three triples of this family whose intersection is $\{1\}$. Each of the additional families must contain a triple that intersects F_1, F_2 nd F_3 and does not contain 1. There are only $\binom{6}{2}n = O(n)$ such triples, since they contain at least two vertices from $F_1 \cup F_2 \cup F_3 \setminus \{1\}$. Therefore there can be only $O(n)$ additional families, contradicting the fact that there are $\Omega(n^2)$ of them. This implies the assertion of Proposition 2.4. $\square$

Returning to the proof of Theorem 2.2, note, that by Proposition 2.4, there are $(1/2-o(1))n^2$ families $\mathcal{F}_i$ whose kernels are of cardinality 2. Let us restrict our attention to the collection $\mathcal{G}$ of these families. Note that since each family contains more than $n/27$ triples, no pair of points of N can be the kernel of more than 26 families in $\mathcal{G}$. Since every family in $\mathcal{G}$ contains many triples, it is obvious that for every ordered pair of two distinct families $\mathcal{F}_i$ and $\mathcal{F}_j$ in $\mathcal{G}$ there is an $F \in \mathcal{F}_i$ that intersects the kernel $ker(\mathcal{F}_j)$. Therefore, the union of elements in the triples of each $\mathcal{F}_i \in \mathcal{G}$ intersects all kernels of families in $\mathcal{G}$. By averaging, there is a family in $\mathcal{G}$ that contains at most

$$\frac{\binom{n}{3}}{(1/2-o(1))n^2} = (1/3+o(1))n$$

triples, and as they have a common kernel of size 2, the size of their union is at most $(1/3+o(1))n$. This union intersects all kernels of the families in $\mathcal{G}$, implying that there are at most $(5/18+o(1))n^2$ distinct kernels. Therefore, there is a kernel, say $\{1,2\}$, which is the kernel of two distinct families, say $\mathcal{F}_1$ and $\mathcal{F}_2$, of $\mathcal{G}$. However, these two families do not contain a common triple, implying that the intersection of their unions is precisely their common kernel $\{1,2\}$. Let X_1 be the union of all triples of $\mathcal{F}_1$, and let X_2 be the union of all triples of $\mathcal{F}_2$. As $|X_1| + |X_2| \leq n+2$ and $|X_1 \cap X_2| = 2$ it follows that there are only $(1/4+o(1))n^2$ pairs of elements of N that intersect both X_1 and X_2 (and are thus potential kernels to the other families of $\mathcal{G}$). Moreover, all the triples of those families contain one of these kernels, showing that no family of $\mathcal{G}$ contains a triple contained in $X_1 - \{1,2\}$ or in $X_2 - \{1,2\}$. Hence, the total number of triples in all the families in $\mathcal{G}$ is at most $(1/8+o(1))n^2$, implying that there is a family containing at most $(1/4+o(1))n$ triples whose union is of size at most $(1/4+o(1))n$. This union intersects all kernels of families in $\mathcal{G}$, showing that there are only at most $(7/32+o(1))n^2$ such kernels. Since no kernel is the kernel of more than 26 families, and since there are at least $(1/2-o(1))n^2$ families, we conclude that there are to pariwise disjoint kernels, say $\{a,b\}$ and $\{c,d\}$, where $\{a,b\}$ is a kernel of a family and $\{c,d\}$ is a kernel of at least three distinct families. However, each of these three families must contain a triple that intersects $\{a,b\}$ and hence must be either $\{a,c,d\}$ or $\{b,c,d\}$. This is a contradiction, as the three families are pairwise disjoint. Therefore, the assertion of the conjecture holds for $k=3$. $\square$

The counterexample we shall present in the next section consists of families whose kernels contain (at least) two elements. The following proposition shows that this is essential for any example (like the one we present in the next section) that has substantially more than $\binom{n-1}{k-1}$ families.

Proposition 2.5. *Let $k \geq 1$ be a fixed integer, let $\mathbb{F}$ be an intersecting system of type $(\exists, \forall, k, n)$, and suppose it contains a family $\mathcal{F}_1$ whose kernel consists of a single element. Then $|\mathbb{F}| < (1+o(1))\binom{n-1}{k-1}$.*

Proof. Let $\{1\}$ be the kernel of $\mathcal{F}_1$. Then, as in the proof of Proposition 2.1, there are $F_1, \ldots, F_k \in \mathcal{F}_1$ whose intersection is $\{1\}$. Define $H = \bigcup_{i=1}^{k} F_i$, and observe that $|H| \leq k^2$ and that every family $\mathcal{F}$ in $\mathbb{F}$ that does not contain any k–set that contains 1 must contain a k–set that intersects H by at least two elements. Therefore,

$$|\mathbb{F}| \leq \binom{n-1}{k-1} + \binom{k^2}{2}\binom{n-2}{k-2} \leq (1+o(1))\binom{n-1}{k-1},$$

as needed. □

3 A counterexample

In this section we show that Conjecture 1.1 is false for all $k \geq 8$. More precisely, we prove the following.

Theorem 3.1. *For every integer k and every $\varepsilon > 0$ there exists an n_0 so that for every $n > n_0$*

$$I_n(\exists, \forall, k) \geq (1-\varepsilon)\frac{(k-2)2^{k-1} + k\binom{k-1}{\lfloor k/2 \rfloor} - 2k + 2}{k2^{k-1}}\binom{n-1}{k-1}.$$

In particular, for all $k \geq 8$ and $n > n_0(k)$

$$I_n(\exists, \forall, k) > \binom{n-1}{k-1}.$$

The proof of this theorem combines some simple conbinatorial and probabilistic arguments with a result of Pippenger and Spencer [7] on coverings in uniform hypergraphs. If $H = (V, E)$ is an r–uniform hypergraph, which may contain multiple edges, let $D(H)$ denote the maximum degree of a vertex of H and let $d(H)$ denote the minimum degree of a vertex of H. Let $C(H)$ denote the maximum number of edges of H whose total intersection is of cardinality at least 2. A *covering* in H is a collection of edges whose union covers all vertices of H. Let $\phi(H)$ denote the maximum possible number of coverings into shich the edges of H may be partitioned. Here is then the theorem of Pippenger and Spencer [7] we shall need.

Theorem 3.2. *For every $r \geq 2$ and $\delta > 0$ there exists a $\delta' > 0$ such that if H is an r–uniform hypergraph satisfying $d(H) \geq (1-\delta')D(H)$ and $C(H) \leq \delta' D(H)$ then $\phi(H) \geq (1-\delta)D(H)$.*

We note that in [7] the theorem is stated with the additional assumption that the number of vertices of H is sufficiently large (as a function of δ and k), but this assumption is not needed, as the result for any hypergraph H follows from the result for a hypergraph with many vertices by applying it to a disjoint union of sufficiently many copies of H.

Proof of Theorem 3.1. Let k be a fixed integer. Throughout this proof, the notation $f = (1+o(1))g$ will always mean that the ratio f/g tends to 1 as n tends to infinity (when all the other parameters are fixed). Let $N = \{1, 2, \ldots, n\} = N_1 \cup N_2$ be a partition of N into two disjoint subsets of cardinalities $|N_1| = n_1 = \lfloor n/2 \rfloor$ and $|N_2| = n_2 = \lceil n/2 \rceil$. Each family in the system $\mathbb{F}$ that we construct will have a kernel of cardinality two containing an element in N_1 and another one in N_2. In addition, the union of the members of each family will either contain the whole of N_1 or the whole of N_2. Clearly, these two properties ensure that $\mathbb{F}$ is an intersecting system of the type we need. It thus remains to show that there exists such an $\mathbb{F}$ containing sufficiently many families.

Let us choose, for each k–subset F of N that intersects both N_1 and N_2, randomly and independently, a member $a = a(F) \in N_1 \cap F$ and a member $b = b(F) \in N_2 \cap F$, where a is chosen according to a uniform distribution among the elements in $N_1 \cap F$ and b is chosen in a similar manner. For each integer i, $1 \leq i \leq k-1$ and for each $a \in N_1$, and $b \in N_2$, define $\mathcal{F}(i,a,b)$ to be the set of all k–subsets F for which $|F \cap N_1| = i, a(F) = a$ and $b(F) = b$. The families of our system $\mathbb{F}$ will be obtained by splitting each of the sets $\mathcal{F}(i,a,b)$ into disjoint families, so that the union of each will cover either N_1 or N_2. The crucial idea is to cover N_1 if $i \geq k/2$ and to cover N_2 otherwise. In this way, we are always using the bigger part of each k–set to cover the corresponding N_i and hence obtain a large number of families. The precise argument requires an application of Theorem 3.2 and the following lemma.

Lemma 3.1. *The following statements hold almost surely (that is, with probability that approaches* 1 *as* n *tends to infinity):*

(i) *For every* i *satisfying* $k/2 \leq i \leq k-1$ *and for every two distinct elements* $a, a_1 \in N_1$ *and every* $b \in N_2$*, the number of members of* $\mathcal{F}(i,a,b)$ *that contain* a_1 *is*

$$(1+o(1))\frac{\binom{n_1-2}{i-2}\binom{n_2-1}{k-i-1}}{i(k-i)}.$$

(ii) *For every* i *satisfying* $k/2 \leq i \leq k-1$ *and for every three distinct elements* $a, a_1, a_2 \in N_1$ *and every* $b \in N_2$*, the number of members of* $\mathcal{F}(i,a,b)$ *that contain* a_1 *and* a_2 *is*

$$(1+o(1))\frac{\binom{n_1-3}{i-3}\binom{n_2-1}{k-i-1}}{i(k-i)}.$$

Proof. We describe the proof of (i). The Proof of (ii) is similar. There are precisely

$$\binom{n_1-2}{i-2}\binom{n_2-1}{k-i-1}$$

k–subsets F of N satisfying $|F \cap N_1| = i$ and containing a, a_1 and b. Foreach such subset F the probability that it lies in $\mathcal{F}(i,a,b)$, that is, the probability that $F(a) = a$ and $F(b) = b$ is $\frac{1}{i(k-1)}$. It follows that the expected number of members of $\mathcal{F}(i,a,b)$ containing a_1 is

$$g = \frac{\binom{n_1-2}{i-2}\binom{n_2-1}{k-i-1}}{i(k-i)}.$$

Notice that since k is fixed $g = \Theta(n^{k-3})$. Since the value in (i) is a binomial random variable, the standard tail estimates for the distribution of such a variable (see, for example, [3, Appendix A, Theorem A.4]) imply that the probability that it deviates from its expection g by at least, say, $n^{(k-3)/2+1}$, is at most $\exp\{-cn^2\}$, where $c>0$ is a constant depending only on k. Since there are less than kn^3 choices for i, a, a_1 and b, the probability that there exist i, a, a_1 and b for which the corresponding number deviates from its expectation g by at least $n^{(k-3)/2+1}$ tends to 0 as n tends to infinity, as needed. $\square$

Returning to the proof of Theorem 3.1, fix a choice for the various families $\mathcal{F}(i,a,b)$ satisfying the assertion of the lemma. Define for every $i \geq k/2$ and for every $a \in N_1$ and $b \in N_2$, an $(i-1)$–uniform hypergraph $\mathcal{H}=\mathcal{H}(i,a,b)$ as follows. The set of vertices of $\mathcal{H}$ is $N_1 - a$ and for each $F \in \mathcal{F}(i,a,b)$, $(F \cap N_1) - a$ is an edge of $\mathcal{H}$. (Note that $\mathcal{H}$ may have multiple edges, as $\mathcal{F}(i,a,b)$ may have members that differ only on the elements of N_2). Let $D(i,a,b)$ denote the maximum degree of $\mathcal{H}$, $d(i,a,b)$ the minimum degree and $C(i,a,b)$ the maximum number of edges of $\mathcal{H}$ containing a pair of vertices. By Part (i) of Lemma 3.1 for all admissible i, a, b, and for every $\delta' > 0$

$$d(i,a,b) \geq (1-\delta')D(i,a,b),$$

provided n is sufficiently large. Similarly, By Lemma 3.2, part (ii),

$$C(i,a,b) \leq \delta' D(i,a,b)$$

for all sufficiently large n. Therefore, by Theorem 3.2 we conclude that for every $\delta > 0$ the set of edges of $\mathcal{H}(i,a,b)$ can be partitioned into at least $(1-\delta)D(i,a,b)$ coverings, provided n is sufficiently large. The original k–subsets F correponding to the members of each such covering supply a family of k–subsets, all of which contain a and b, whose union covers N_1. This, together with the symmetric argument obtained by replacing the roles of N_1 and N_2 imply the following.

Lemma 3.2. *For every $\delta > 0$, $n > n_0(\delta)$ and for every i satisfying $k/2 \leq i \leq k-1$ and every $a \in N_1$ and $b \in N_2$ one can split the collection $\mathcal{F}(i,a,b)$ into at least*

$$(1-\delta)\frac{\binom{n_1-2}{i-2}\binom{n_2-1}{k-i-1}}{i(k-i)}$$

families, so that the union of each family covers N_1 and its intersection contains a and b. Similarly, for every $1 \leq i < k/2$ and every $a \in N_1$ and $b \in N_2$, it is possible to partition the collection $\mathcal{F}(i,a,b)$ into at least

$$(1-\delta)\frac{\binom{n_1-1}{i-1}\binom{n_2-2}{k-i-2}}{i(k-i)}$$

families, so that the union of each family covers N_2 and its intersection contains the elements a and b. $\square$

Therefore,

$$
\begin{aligned}
I_n(\exists,\forall,k) &\geq (1-\delta)\sum_{i=k/2}^{k-1}\frac{\binom{n_1-2}{i-2}\binom{n_2-1}{k-i-1}}{i(k-i)} + (1-\delta)\sum_{1\leq i<k/2}\frac{\binom{n_1-1}{i-1}\binom{n_2-2}{k-i-2}}{i(k-i)} \\
&= (1+o(1))\left(\frac{n}{2}\right)^2(1-\delta)\sum_{i=1}^{k-1}\frac{n^{k-3}\max\{i-1,k-i-1\}}{2^{k-3}i!(k-i)!} \\
&= (1+o(1))(1-\delta)\sum_{i=1}^{k-1}\frac{n^{k-1}\max\{i-1,k-i-1\}}{2^{k-1}i!(k-i)!} \\
&= (1+o(1))(1-\delta)\binom{n-1}{k-1}\sum_{i=1}^{k-1}\frac{\max\{i-1,k-i-1\}\binom{k}{i}}{k2^{k-1}}.
\end{aligned}
$$

To complete the proof we need the following simple fact.

Lemma 3.3. *For every integer $k \geq 2$*

$$\sum_{i=1}^{k-1}\max\{i,k-1\}\binom{k}{i} = k2^{k-1} + k\binom{k-1}{\lfloor k/2\rfloor} - 2k.$$

Proof. In the even case $k = 2m$,

$$
\begin{aligned}
&\sum_{i=1}^{k-1}\max\{i,k-1\}\binom{k}{i} = k\left\{\binom{k-1}{1}\right. \\
&\left.+\binom{k-1}{2}+\cdots+\binom{k-1}{m-1}+2\binom{k-1}{m}+\binom{k-1}{m+1}+\cdots+\binom{k-1}{k-2}\right\} \\
&= k\left\{2^{k-1}+\binom{k-1}{m}-2\right\}.
\end{aligned}
$$

The proof of the case when k is odd is similar. □

By substituting the formula in Lemma 3.3 into the estimation following Lemma 3.2 the assertion of Theorem 3.1 follows. □

Remark. The construction in the proof above can be modified to the case of families with larger kernels, simply by splitting N into r nearly equal classes $N_1, \ldots, N_r$ and by constructing families with kernels that contain one element from each N_i, so that the union of the sets in each family covers at least one set N_j. By taking, say, $r = \lfloor k/10/logk \rfloor$ one can prove that there exists an *absolute* constant $\mu > 0$ so that for every $k \geq 8$

$$I_n(\exists,\forall,k) \geq (1+\mu)\binom{n-1}{k-1}.$$

We omit the detailed computation.

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