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General Edge-isoperimetric Inequalities, Part II: a Local–Global Principle for Lexicographical Solutions

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1. INTRODUCTION

The lexicographical order $\mathscr{L}$ on a sequence space $\mathscr{X}^n = \{0, 1, \ldots, \alpha\}^n$, defined by $x^n <_{\mathscr{L}} y^n$ iff there exists a t such that $x_t < y_t$ and $x_s = y_s$ for $s < t$, is one of the most important and frequently encountered orders in combinatorial extremal theory. An early result in this area, Harper's solution of an edge-isoperimetric problem (EIP) in binary Hamming space ([13]) (generalized in [16] to non-binary cases and rediscovered many times; see, e.g, [6], [9] and [15]) says that first segments in $\mathscr{L}$ are optimal.

There are two kinds of EIP. They can be represented as extremal problems in graph theory. Let $G = (\mathscr{V}, \mathscr{E})$ be a graph. For any $A \subset \mathscr{V}$, define the set $\mathscr{B}(A)$ of all boundary edges, that is,

$$\mathscr{B}(A) = \{\{x, y\} \in \mathscr{E}\colon |\{x, y\} \cap A| = 1\} \tag{1.1}$$

and the set $\mathscr{L}(A)$ of all inner edges; that is,

$$\mathscr{I}(A) = \{\{x, y\} \in \mathscr{E}\colon x, y \in A\}. \tag{1.2}$$

1. BOUNDARY-EDGE-ISOPERIMETRIC PROBLEM (BEIP). For a given graph and positive integer m, find a set $A \subset \mathscr{V}$ of cardinality m with minimal possible value of $|\mathscr{B}(A)|$.

2. INNER-EDGE-ISOPERIMETRIC PROBLEM (IEIP). For a given graph and positive integer m, find a set $A \subset \mathscr{V}$ with maximal possible value of $|\mathscr{I}(A)|$.

Notice that, for regular graphs of degree d,

$$|\mathscr{B}(A)| + 2\,|\mathscr{I}(A)| = d\,|A|$$

and that therefore the two problems are equivalent in the sense that a solution of one of these problems is at the some time a solution of the other.

We concentrate here on EIP's of the Cartesian sum graphs

$$G^n = G_1 \times G_2 \times \cdots \times G_n = (\mathscr{X}^n, \mathscr{E}^n) \tag{1.3}$$

of graphs $G_t = (\mathscr{X}_t, \mathscr{E}_t)$; $t = 1, 2, \ldots, n$, where $\mathscr{X}^n = \mathscr{X}_1 \times \mathscr{X}_2 \times \cdots \times \mathscr{X}_n$ and for $x^n = (x_1, \ldots, x_n)$, $y^n = (y_1, \ldots, t_n) \in \mathscr{X}^n$ $\{x^n, y^n\} \in \mathscr{E}^n$ iff there exists a $t \in \{1, 2, \ldots, n\}$ such that, for all $t' \neq t$, $x_{t'} = y_{t'}$ and $\{x_t, y_t\} \in \mathscr{E}_t$. Then EIP's in Hamming, Manhattan and Lee metrics can be understood as EIP's of Cartesian sum graphs of complete graphs, paths and cycles, respectively. We speak of an (optimal) order for an EIP if the initial segments of this order always achieve the extremal value. Then Harper's Theorem and its generalization show that $\mathscr{L}$ is an order for the EIP in Cartesian sums of complete graphs. Notice that the regularity implies that the BEIP and the IEIP become the same here. Of course, $\mathscr{L}$ is not always optimal for these EIP's, and for many of these problems there is no order at all. One can find such examples for Manhattan and Lee spaces in [1], [4] and [8].

On the other hand, one can ask

$$\text{'Is } \mathscr{L} \text{ optimal for an EIP in } G^n\text{?'} \tag{1.4}$$

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Since there are $\prod_{t=1}^{n} |\mathscr{X}_t|$ orders on $\mathscr{X}^n$, one might expect the complexity of an algorithm deciding this question to be very high. Quite surprisingly, our main delivery, *a local-global principle,* shows that the problem is not NP-hard and not even P-hard. Actually, its complexity is independent of n!

For the convenience of the readers, we limit ourselves here in (1.3) to the case of identical factors, i.e. $G_t = G$ for $t = 1, 2, \dots, n$. Here we call G^n the nth power of G. It is not very hard to extend our main result to general Cartesian sum graphs. In another direction, our work is more general. We introduce a fairly large family of set functions on G^n, including 'boundary-edge' and 'inner-edge' functions. Our local–global principle says that $\mathscr{L}$ is an optimal order for the extremal problems of the functions of this family in nth power space exactly if it is optimal in the first and the second power spaces. This means that often the question (1.4) can be decided by a simple inspection!

In Section 2 we give the necessary definitions, state known facts from [2] and present a generalization of a lemma from [2]. Our main result (Theorem 1) is presented and proved in Section 3. Finally, as an example demonstrating the power of our local–global principle, we give an edge-isoperimetric theorem for the powers of complete bipartite graphs $C_{m,m}$ (Theorem 2) in Section 4.

2. Preliminaries

2.1. *Definitions and known facts.* We list all definitions and needed known facts in the first part of this section. The proofs of these facts are not very hard and can be found in [2]. For all $J \subset N \triangleq \{1, 2, \dots, n\}$, $x^n \in \mathscr{X}^n$, denote by x^J the subsequence of x^n obtained by deleting components x_t with $t \notin J$. $\mathscr{X}^J$ is defined analogously. Thus x^n and $\mathscr{X}^n$ can be rewritten as x^N and $\mathscr{X}^N$, respectively. Define, for any $A \subset \mathscr{X}^N$, the general slices

$$A_J(x^{N\setminus J}) \triangleq \{x^J \in \mathscr{X}^J : x^N \in A\} \qquad \text{for } x^{N\setminus J} \in \mathscr{X}^{N\setminus J} \tag{2.1}$$

and the projections

$$A_J \triangleq \bigcup_{x^{N\setminus J} \in \mathscr{X}^{N\setminus J}} A_J(x^{N\setminus J}). \tag{2.2}$$

For $J \subset N$, write the set of the lexicographically first m elements in $\mathscr{X}^J$ as $\mathscr{L}(\mathscr{X}^J, m)$. Then the general pushing down operations under $\mathscr{L}$ on $\mathscr{X}^J$ are defined by

$$D_J(\mathscr{L}, \mathscr{A}) = \bigcup_{x^{N\setminus J} \in A_{N\setminus J}} \{y^N : y^{N\setminus J} = x^{N\setminus J} \text{ and } \quad y^J \in \mathscr{L}(\mathscr{X}^J, |A_J(x^{N\setminus J})|)\} \tag{2.3}$$

for all $A \subset \mathscr{X}^N$.

When $J = \{t\}$ we also write $D_J = D_t$. $A \subset \mathscr{X}^N$ is a downset, if $y^n \in A$ implies $x^n \in A$ in the case $x_t \leqslant y_t$ for all t. In other words, a downset A of $\mathscr{X}^N$ is a set with $D_t(A) = A$ for all $t \in N$. For a given graph G, $|\mathscr{B}(\cdot)|$ in (1.1) and $|\mathscr{I}(\cdot)|$ in (1.2) are functions on the subsets of the vertex set and we write them as $B(G, \cdot)$ and $I(G, \cdot)$. We shall state their essential properties and study them as abstract set functions.

For this goal, we define the nth power function φ^n of a set function φ on $2^{\mathscr{X}} \triangleq \{A : A \subset \mathscr{X}\}$ as a function on $2^{\mathscr{X}^n}$:

$$\varphi^n(A) = \sum_{t=1}^{n} \sum_{x^{N\setminus\{t\}} \in \mathscr{X}^{N\setminus\{t\}}} \varphi(A_t(x^{N\setminus\{t\}})), \tag{2.4}$$

where we abbreviate $A_{\{t\}}(\cdot)$ as $A_t\{\cdot\}$.

One readily verifies that for the nth power G^n of a graph $G = (\mathscr{X}, \mathscr{E})$ in the sense of (1.3), $\varphi^n(\cdot) = -B(G^n, \cdot)$ (or $I(G^n, \cdot)$), if we let $\varphi(\cdot) = -B(G, \cdot)$ (or $I(G, \cdot)$). Thus

EIP's are reduced to maximizing, for given φ, $\varphi^n(A)$ over all $A \subset \mathscr{X}^n$ with fixed $|A|$. The following properties were proved in [2].

(1) $\max_{|A|=m} \varphi^n(A)$ is assumed on a downset of $\mathscr{X}^n$, if the following conditions are satisfied:

I (nestedness)—for all $k \in \mathscr{X} = \{0, 1, \dots, \alpha\}$, $A \subset \mathscr{X}$ with $|A| = k + 1$,

$$\varphi(A) \leqslant \varphi([k]) \qquad \text{where } [k] = \{0, 1, \dots, k\}; \tag{2.5}$$

II (submodularity)—for $A,\ B \subset \mathscr{X}$,

$$\varphi(A) + \varphi(B) \leqslant \varphi(A \cup B) + \varphi(A \cap B); \tag{2.6}$$

III—$\varphi(\phi) = 0$.

(We can always assume that III holds by replacing φ by φ', where $\varphi'(A) = \varphi(A) - \varphi(\phi)$. Furthermore, obviously III holds for $\varphi(\cdot) = -B(G, \cdot)$ or $I(G, \cdot)$ for all G.)

(2) For all graphs G, both $-B(G, \cdot)$ and $I(G, \cdot)$ satisfy II.

(3) II implies that, for any family $\{A_i\}_{i=1}^m$ of subsets of $\mathscr{X}$,

$$\sum_{i=1}^m \varphi(A_i) \leqslant \sum_{k=1}^m \varphi\biggl(\bigcup_{1 \leqslant i_1 \leqslant i_2 < \dots < i_k \leqslant m} \biggl(\bigcap_{j=1}^k A_{i_j}\biggr)\biggr). \tag{2.7}$$

(4) Let

$$\Delta_\varphi(k) = \varphi([k]) - \varphi[(k-1]), \tag{2.8}$$

(where we set $[-1]$ as empty set.) Then

$$\varphi^n(A) = \sum_{x^n \in A} \sum_{t=1}^n \Delta_\varphi(x_t), \tag{2.9}$$

if A is a downset.

Remarks. (1) Condition I (2.5) says exactly that $\mathscr{L}$ is an order for maximizing $\varphi^n(A)$ for fixed $|A|$, when $n = 1$. $-B(G, \cdot)$ ($I(G, \cdot)$) satisfies I, if G has nested solutions for BEIP (IEIP) after labeling the vertices properly.

(2) Condition II (2.6) is a key issue for pushing down to work. (2.7) is an extension of condition II to more than two sets.

(3) By known facts (1), (2) and (4), the EIP's of power graph G^n have been reduced to maximizing $\sum_{x^n \in A} f(x^n)$ over all downsets A of $\mathscr{X}^n$ for a sum-type function $f(x^n) \triangleq \sum_{t=1}^n \Delta_\varphi(x_t)$ (if G has nested solutions for the corresponding EIP's). The importance of the extremal values of $\sum_{x^n \in A} f(x^n)$ over the downsets of $\mathscr{X}^n$ was known to Ahlswede and Katona [3] 20 years ago, and the problems for $\mathscr{X} = \{0, 1\}$ were well studied there. In the sequel, we always assume that $|\mathscr{X}| \geqslant 3$. Actually, comparing Theorem 1 in Section 3 of the present paper with the solutions of the corresponding problems in the binary case in [3], one may immediately notice that the local–global principle does not extend to a binary alphabet. In the sequel, we consider maximizing $\varphi^n(A)$ (or $\varphi(A)$) for A of a given size, and speak of an order for φ^n (or φ).

2.2. *A lemma concerning general pushing down operations.* In this subsection we generalize Lemma 2 of [2] to general pushing down operations. In the following lemma, $\mathscr{L}$ can be replaced by any order. For any $J \subseteq N = \{1, 2, \dots, n\}$, $A \in \mathscr{X}^J$ and φ, we write

$$\varphi^J(A) = \sum_{j \in J} \sum_{x^{J \setminus \{j\}} \in \mathscr{X}^{J \setminus \{j\}}} \varphi(A_j(x^{J \setminus \{j\}})), \tag{2.10}$$

where $A_j(\cdot) = A_{\{j\}}(\cdot)$ is defined by (2.1).

Lemma 1. *If φ satisfies II and if $\mathscr{L}$ is an order for φ^J $(J \subset N)$ then, for all $A \subset \mathscr{X}^N$,*

$$|D_J(\mathscr{L}, A)| = |A| \tag{2.11}$$

and

$$\varphi^N(A) \leqslant \varphi^N(D_J(\mathscr{L}, A)), \tag{2.12}$$

where $D_J(\mathscr{L}, \cdot)$ is defined by (2.3) *and φ^n is written as φ^N.*

Proof. (2.11) follows from the definition of $D_J(\mathscr{L}, \cdot)$. Let $B = D_J(\mathscr{L}, A)$. Then, by (2.1), (2.4) and (2.10), for all $C \subset \mathscr{X}^N$ (in particular, for $C = A$ or B),

$$\begin{aligned}\varphi^N(C) &= \sum_{t \in J} \sum_{x^{N\setminus\{t\}} \in \mathscr{X}^{N\setminus\{t\}}} \varphi(C_t(x^{N\setminus\{t\}})) + \sum_{t \in N\setminus J} \sum_{x^{N\setminus\{t\}} \in \mathscr{X}^{N\setminus\{t\}}} \varphi(C_t(x^{N\setminus\{t\}})) \\ &= \sum_{x^{N\setminus J} \in \mathscr{X}^{N\setminus J}} \varphi^J(C_J(x^{N\setminus J})) + \sum_{x^J \in \mathscr{X}^J} \varphi^{N\setminus J}(C_{N\setminus J}(x^J)).\end{aligned} \tag{2.13}$$

By the definition of $D_J(\mathscr{L}, \cdot)$ in (2.3) and $B = D_J(\mathscr{L}, A)$, we have $B_J(x^{N\setminus J}) = \mathscr{L}(\mathscr{X}^J, |A_J(x^{N\setminus J})|)$ for all $x^{N\setminus J} \in \mathscr{X}^{N\setminus J}$. Thus, $\mathscr{L}$ being an (optimal) order for φ^J implies, for all $x^{N\setminus J} \in \mathscr{X}^{N\setminus J}$,

$$\varphi^J(A_J(x^{N\setminus J})) \leqslant \varphi^J(B_J(x^{N\setminus J})). \tag{2.14}$$

Denote $A_{N\setminus J}(x^J)$ $(B_{N\setminus J}(x^J))$ by A_k (B_k) if x^J is the lexicographically kth sequence in $\mathscr{X}^J$. We have for all $y^{N\setminus J} \in \mathscr{X}^{N\setminus J}$, $y^{N\setminus J} \in B_k$ exactly if there are at least k A_l's containing $y^{N\setminus J}$; or, in other words,

$$B_k = \bigcup_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant |\mathscr{X}|^{|J|}} \left(\bigcap_{j=1}^{k} A_{i_j} \right). \tag{2.15}$$

Furthermore, we observe that by (2.10) the submodularity II of φ implies the submodularity of $\varphi^{J'}$, and therefore φ and $\mathscr{X}$ in (2.7) can be replaced by $\varphi^{J'}$ and $\mathscr{X}^{J'}$ for all $J' \subset N$. Applying the resulting inequality and (2.15) to $J' = N\setminus J$, $m = |\mathscr{X}|^{|J|}$, $\sum_{x^J \in \mathscr{X}^J} \varphi^{N\setminus J}(A_{N\setminus J}(x^J))$, which equals $\sum_{i=1}^{m} \varphi^{N\setminus J}(A_k)$, and $\sum_{x^J \in \mathscr{X}^J} \varphi^{N\setminus J}(B_{N\setminus J}(x^J))$, which equals $\sum_{k=1}^{m} \varphi^{N\setminus J}(B_k)$, we obtain

$$\sum_{x^J \in \mathscr{X}^J} \varphi^{N\setminus J}(A_{N\setminus J}(x^J)) \leqslant \sum_{x^J \in \mathscr{X}^J} \varphi^{N\setminus J}(B_{N\setminus J}(x^J)), \tag{2.16}$$

which, together with (2.13) and (2.14), implies (2.12). □

3. A Neccessary and Sufficient Condition for the Lexicographical Order to be Optimal for Edge-isoperimetric Problems Satisfying I–III

Before we state and prove our main result (Theorem 1), here we first derive two auxiliary results (Lemmas 2 and 3).

Lemma 2. *Assume that $\mathscr{X} = \{0, 1, \ldots, \alpha\}$, $\alpha \geqslant 2$, and that $n > 2$. Now suppose that $A \subset \mathscr{X}^N$ and that, for all $J \subset N = \{1, 2, \ldots, n\}$, $J \neq N$,*

$$D_J(\mathscr{L}, A) = A, \tag{3.1}$$

then, for $a^N \in A$, the following sequences belong to A:

(i) *all* $x^N \in \mathscr{X}^N$ *with* $x_1 = a_1$ *and* $x^{N\setminus\{1\}} <_{\mathscr{L}} a^{N\setminus\{1\}}$;
(ii) $b^N = (a_1 - 1, \alpha, \ldots, \alpha, a_n)$, *if* $a_1 \geq 1$, *and all* x^N *with* $x_1 = a_1 - 1$ *and* $x^{N\setminus\{1\}} <_{\mathscr{L}} b^{N\setminus\{1\}}$;
(iii) *in case* $a_1 \geq 2$ *all* x^N *with* $x_1 \leq a_1 - 2$.

Proof. Since $D_{N\setminus\{1\}}(\mathscr{L}, A) = A$, (i) holds. Thus, by (i), for (ii) and (iii) it is sufficient to show that

$$b^N \in A, \qquad \text{if } a_1 - 1 \geq 0 \tag{3.2}$$

and

$$c^N = (a_1 - 2, \alpha, \ldots, \alpha) \in A, \qquad \text{if } a_1 - 2 \geq 0. \tag{3.3}$$

Now (3.2) follows from $a^N \in A$, $D_{N\setminus\{n\}}(A) = A$ and $(a_1 - 1, \alpha, \ldots, \alpha)(\in \mathscr{X}^{N\setminus\{n\}}) <_{\mathscr{L}} a^{N\setminus\{n\}}$.

Finally, (3.3) follows from (3.2), $D_{N\setminus\{2\}}(A) = A$ and, for $(a_1 - 2, \alpha, \ldots, \alpha)$, $(a_1 - 1, \alpha, \ldots, \alpha, a_n) \in \mathscr{X}^{N\setminus\{2\}}$ $(a_1 - 2, \alpha, \ldots, \alpha) <_{\mathscr{L}} (a_1 - 1, \alpha, \ldots, \alpha, a_n)$ (the second components are deleted from both b^N and c^N, and this is possible because $n > 2$).

Lemma 3. *Assume that* $|\mathscr{X}| \geq 3$ *and let* φ *satisfy I–III in Section* 2. *Then:*
(i) *If* $\mathscr{L}$ *is the optimal order for* φ^2, *then*

$$\Delta_\varphi(1) \leq \Delta_\varphi(2) \tag{3.4}$$

and, for $a, b, i \in \mathscr{X} = \{0, 1, \ldots, \alpha\}$ *with* $a < b \leq b + i \leq \alpha$ *and either* $a = 0$ *or* $b + i = \alpha$,

$$\sum_{j=0}^{i} \Delta_\varphi(a + j) \leq \sum_{j=0}^{i} \Delta_\varphi(b + j). \tag{3.5}$$

(ii) *If* $\mathscr{L}$ *is the order for* φ^m, *then, for* $a \in \mathscr{X}'$, $a \geq 1$ *and* $x_i \in \mathscr{X}$ $(i = 2, \ldots, m)$ *not all equal to* α,

$$\sum_{i=2}^{m} \Delta_\varphi(x_i) + \Delta_\varphi(a) \leq (m - 1)\Delta_\varphi(\alpha) + \Delta_\varphi(a - 1). \tag{3.6}$$

Proof. (i) If $\mathscr{L}$ is the order for φ^2, then, in particular,

$$\varphi^2(\{00,\ 01,\ 10\}) \leq \varphi^2(\{00,\ 01,\ 02\}) \tag{3.7}$$

and therefore (2.9) in property (4) of Section 2 gives (3.4).

While showing (3.5) we can assume that

$$a + i < b, \tag{3.8}$$

because otherwise we can delete the common terms on both sides of (3.5).

Now define the interval

$$(x^l, y^l) = \{z^l \in \mathscr{X}^l : x^l \leq_{\mathscr{L}} z^l \leq_{\mathscr{L}} y^l\}, \quad \langle y^l \rangle = \langle 0^l, y^l \rangle. \tag{3.9}$$

Case $a = 0$. By (3.8), $(\langle (1, b + i) \rangle \setminus \langle 1, b),\ (1, b + i)\rangle) \cup \langle (2, 0),\ (2, i)\rangle = \{(0, 0),$

$(0,1),\dots,(0,\alpha),\ (1,0),\dots,\ (1,b-1),\ (2,0),\dots,(2,i)\} = B$, say, is a downset and $|B| = |\langle(1, b+i)\rangle|$. Since $\mathscr{L}$ is the order for φ^2, we conclude with (2.9) that

$$
\begin{aligned}
0 &\leqslant \varphi(\langle(1, b+i)\rangle) - \varphi(B) \\
&= \sum_{x^2 \in \langle(1,b+i)\rangle \setminus B} (\Delta_\varphi(x_1) + \Delta_\varphi(x_2)) - \sum_{x^2 \in B \setminus \langle 1, b+i)\rangle} (\Delta_\varphi(y_1) + \Delta_\varphi(y_2)) \\
&= \Big((i+1)\Delta_\varphi(1) + \sum_{j=0}^{i} \Delta_\varphi(b+j)\Big) - \Big((i+1)\Delta_\varphi(2) + \sum_{j=0}^{i} \Delta_\varphi(j)\Big)
\end{aligned}
$$

or, equivalently,

$$
\sum_{j=0}^{i} \Delta_\varphi(b+j) - \sum_{j=0}^{i} \Delta_\varphi(j) \geqslant (i+1)(\Delta_\varphi(2) - \Delta_\varphi(1)). \tag{3.10}
$$

In this case, this and (3.4) imply (3.5).

Case $a \neq 0$ and $b + i = \alpha$. Instead of $\langle(1, b+i)\rangle$ and B, we now consider $\langle(2, a-1)\rangle$ and

$$
\begin{aligned}
B' &= (\langle(2, a-1)\rangle \setminus \langle(1, b), (1, \alpha)\rangle) \cup \langle(2, a), (2, a+1)\rangle \\
&= \{(0,0),\dots,(0,\alpha),(1,0),\dots,(1,b-1),(2,0),\dots,(2,a-1),\dots,(2,a+i)\}
\end{aligned}
$$

and by the previous argument we also obtain (3.5) in this case.

(ii) For $x^l, y^l \in \mathscr{X}^l$, now define the half-open interval

$$
(x^l, y^l) = \langle x^l, y^l\rangle \setminus \{y^l\}. \tag{3.11}
$$

Then, for $x^m = (a, x_2, \dots, x_m)$, $y^m = (a-1, \alpha, \dots, \alpha)$,

$$
\varphi(\langle 0^m, x^m)) \geqslant \varphi(\langle x^m\rangle \setminus \{y^m\}). \tag{3.12}
$$

Since both arguments in (3.12) are downsets, we can use (2.9) to estimate them. Then we delete the common terms on both sides and obtain (3.6). □

Quite surprisingly, we found the following result.

THEOREM 1. *If $|\mathscr{X}| \geqslant 3$, then for any set function $\varphi: 2^{\mathscr{X}} \to \mathbb{R}$ satisfying I–III and any integer $n \geqslant 2$, $\mathscr{L}$ on X^n is optimal for φ^n iff $\mathscr{L}$ on $\mathscr{X}^2$ is optimal for φ^2. (Condition I says that $\mathscr{L}$ is an optimal order for φ.)*

PROOF. Assume to the contrary that $\mathscr{L}$ is optimal for φ^2 but not for φ^n and that $n \geqslant 3$ is smallest with this property.

By Lemma 1 in Section 2 we can assume that, for all $J \neq N$,

$$
D_J(\mathscr{L}, A) = A \tag{3.13}
$$

for an optimal set A, and therefore A is a downset.

Moreover, among all optimal sets satisfying (3.13) we choose as A one which achieves the minimal value of $|A \Delta \langle\theta(|A|)\rangle|$, where $\theta(|A|)$ is the $|A|$th smallest element in $\mathscr{X}^n$ in lexicographical order (i.e. $\theta(A) = \mathscr{L}(\mathscr{X}^n, |A|\rangle)$. Since, by assumption, $\mathscr{L}$ is not the order of φ^n, we have

$$
|A \Delta \langle\theta(|A|)\rangle| > 0 \qquad \text{or} \qquad A \neq \langle\theta(|A|)\rangle. \tag{3.14}
$$

Now let a^n be the lexicographically last element of A. Then, obviously,

$$
A \nsubseteq \langle a^n\rangle. \tag{3.15}
$$

Since $\mathscr{L}$ is optimal for φ^{n-1}, $a_1 \neq 0$.

It immediately follows from Lemma 2 that all elements in $\langle a^n\rangle\backslash A$ are of the form

$$(a_1 - 1, \alpha, \ldots, \alpha, x), \qquad x > a_n. \tag{3.16}$$

Moreover, by (3.13), none of the a_i's equals α. Thus, by (3.16),

$$\langle a^n\rangle\backslash A = \{(a_1 - 1), \alpha, \ldots, \alpha, x): \beta \leqslant x \leqslant \alpha\} \tag{3.17}$$

for some $\alpha \geqslant \beta > a_n$.

Since A is a downset,

$$\{(a_1, \ldots, a_{n-1}, y): 0 \leqslant y \leqslant a_n\} \subset A. \tag{3.18}$$

Considering $a_n < \beta$, one can choose a, b and i such that

$$a = 0, \qquad a + i = a_n, \qquad b = \beta \qquad \text{and} \qquad \beta + i \leqslant \alpha \tag{3.19}$$

or

$$a > 0, \qquad a + i = a_n, \qquad b = \beta \qquad \text{and} \qquad \beta + i = \alpha. \tag{3.20}$$

In both cases, we have $a > b$. When $a_n \leqslant \alpha - \beta$, we choose (3.19) and otherwise (3.20).

Now we remove the 'top part'

$$\{(a_1, a_2, \ldots, a_{n-1}, y): y = a, a + 1, \ldots, a + i\} \subset A$$

from A and add

$$\{(a_1 - 1, \alpha, \ldots, \alpha, x): x = b, b + 1, \ldots, b + i\},$$

which is disjoint from A, to A and obtain a set A', with $|A'| = |A|$ and $|A'\Delta\theta(|A|)| < |A\Delta\theta(|A|)|$.

From the structure of A, as described in Lemma 2, it is not hard to see that A' also satisfies (3.13).

On the other hand, by (2.9) and Lemma 3,

$$\begin{aligned}
\varphi^n(A) - \varphi^n(A') &= \sum_{j=0}^{i} \Delta_\varphi(a + j) + (i + 1)\Big(\sum_{j=2}^{n-1} \Delta_\varphi(a_j) + \Delta_\varphi(a_1)\Big) \\
&\quad - \sum_{j=0}^{i} \Delta_\varphi(b + j) + ((n - 2)\Delta_\varphi(\alpha) + \Delta_\varphi(a_1 - 1))(i + 1) \\
&= \Big(\sum_{j=0}^{i} \Delta_\varphi(a + j) - \sum_{j=0}^{i} \Delta_\varphi(b + j)\Big) \\
&\quad + (i + 1)\Big(\Big(\sum_{j=2}^{n-1} \Delta_\varphi(a_j) + \Delta_\varphi(a_1)\Big) - ((n - 2)\Delta_\varphi(\alpha) + \Delta_\varphi(a_1 -)1)\Big) \leqslant 0,
\end{aligned}$$

in contradiction to the definition of A. □

4. An Application of Theorem 1 to Complete Bipartite Graphs

This last section demonstrates how useful the *local–global principle* of Theorem 1 is. We begin with the simple C_4 and then consider all complete bipartite graphs.

4.1. *The lexicographical order $\mathscr{L}$ is optimal for $I(C_4^n, \cdot)$ or, by regularity, equivalently for $B(C_4^n, \cdot)$.* We know from Theorem 1 that it suffices to show the property for $I(C_4^2, \cdot)$. This in turn is readily done by the following simple observation.

For all $A \subset \{0, 1, 2, 3\}^2$, $|A| \leqslant 8 = \frac{1}{2}|\{0, 1, 2, 3\}^2|$,

$$I(C_4^2, A) \leqslant \begin{cases} |A| - 1 & \text{if } |A| < 4, \\ 2\,|A| - 4 & \text{if } 4 \mid |A|, \\ 2\,|A| - 5 & \text{otherwise.} \end{cases}$$

4.2. *The complete bipartite graph* $C_{m,m}$. Let $\mathscr{X} = \{0, 1, \ldots, 2m-1\}$ be the vertex set with edges exactly between even and odd numbers. This defines $C_{m,m}$, which is a regular graph, so that boundary- and inner isoperimetric problems are equivalent.

We set $\varphi(\cdot) = I(C_{m,m}, \cdot)$ and readily verify that I–III hold, and that

$$\Delta_\varphi(l) = \left\lceil \frac{l}{2} \right\rceil \qquad \text{for } l = 0, 1, \ldots, 2m-1. \tag{4.1}$$

Theorem 2. *The first segments in lexicographical order $\mathscr{L}$ give optimal sets for the (inner and boundary) edge-isoperimetric problems for $C_{m,m}^2$, and therefore for $C_{m,m}^n$.*

Proof. Let us consider $C_{m,m}^2$. In order to proceed by induction on m, we introduce the notation $\Psi_m = \Delta_\varphi$.

Case $m = 2$. Notice that $C_4 = C_{2,2}$ and that the result was established above.

Case $m \to m+1$. For any $A \subset [2m+1]^2$, we have to show that replacement of A by the first $|A|$-segment of $[2m+1]^2$ under $\mathscr{L}$ cannot decrease $\varphi^2(A)$. We shall make use of the fact that the following three properties are equivalent:
(α) $A \subset [2m+1]^2 = \mathscr{X}^2$ is optimal among the subsets of cardinality $|A|$.
(β) $A^c = \mathscr{X}^2 \backslash A$ is optimal among the subsets of cardinality $|A^c| = |\mathscr{X}^2| - |A|$.
(γ) The set $\tilde{A} = \{(2m+1-x, 2m+1-y) : (x, y) \in A^c\}$ is optimal among the subsets of cardinality $|\mathscr{X}^2| - |A|$.
Notice that A is a downset iff $\tilde{A}$ is a downset.

Case 1. There is *no* $(x, y) \in A$ with $x \geqslant 2m$. Therefore we are able to rule out the last two columns $\{(x, y) : x = 2m, 2m+1\}$, which are useless.

Let 'top' and 'bottom' of the part remaining be

$$T = \{(x, y) : 0 \leqslant x \leqslant 2m-1,\ 2 \leqslant y \leqslant 2m+1\} \tag{4.2}$$

and

$$B = \{(x, y) : 0 \leqslant x \leqslant 2m-1,\ 0 \leqslant y \leqslant 2m-1\} \tag{4.3}$$

respectively.

Define O_T (resp. O_B) as the operator keeping the $A \backslash T$ (resp. $A \backslash B$) part unchanged and changing $A_T \triangleq A \cap T$ (resp. $A_B \triangleq A \cap B$) to the first $|A_T|$ (resp. $|A_B|$) elements of T (resp. B) in order $\mathscr{L}$.

Obviously, both $O_T(A)$ and $O_B(A)$ are downsets whenever A is a downset. Furthermore, by the induction hypothesis, for $B \subset \{0, 1, \ldots, 2m-1\}^2$,

$$\varphi^2(O_B(A)) \geqslant \varphi^2(A). \tag{4.4}$$

In fact, the induction hypothesis also implies that $\varphi^2(O_T(A)) \geqslant \varphi^2(A)$, since by (4.1), for $2 \leqslant y \leqslant 2m+1$, $\Psi_{m+1}(y) = 1 + \Psi_m(y-2)$ holds, so that we can shift two units down and transform T to $\{(x, y) : 0 \leqslant x \leqslant 2m-1,\ 0 \leqslant y \leqslant 2m-1\}$ and apply the induction hypothesis. However, $O_T(A) \neq A$ (resp. $O_B(A) \neq A$) implies that the operator O_T (resp. O_B)) strictly decreases the maximal element in the order $\mathscr{L}$ of A_T

(resp. A_B). Thus one can use O_T and O_B repeatedly finitely often, and obtain as resulting set A' a downset with

$$\varphi^2(A') \geqslant \varphi^2(A), \qquad O_T(A') = A' \qquad \text{and} \qquad O_B(A') = A'. \tag{4.5}$$

Thus, A'_T and A'_B must have the following forms:

$$A'_T = \{(x, y): 0 \leqslant x \leqslant a - 1,\ 2 \leqslant y \leqslant 2m + 1\} \cup \{(a, y): 2 \leqslant y \leqslant u\}$$
$$\text{for some } 0 \leqslant t \leqslant 2m - 1 \qquad \text{and} \qquad 2 \leqslant u \leqslant 2m + 1,$$

and

$$A'_B = \{(x, y): 0 \leqslant x \leqslant b - 1,\ 0 \leqslant y \leqslant 2m - 1\} \cup \{(x, b): 0 \leqslant y \leqslant v\}$$
$$\text{for some } t \leqslant b \leqslant 2m - 1 \qquad \text{and} \qquad 0 \leqslant v \leqslant \min(u, 2m - 1).$$

Now, considering that A' is a downset and $A' = A'_T \cup A'_B$, a simple calculation shows that in case A' is not the first segment of $\mathscr{X}^2 = [2m + 1]^2$, we have $a = b - 1$, $u \in \{2m - 1, 2m\}$, $v \in \{0, 1\}$, and A' must be one of the following four subsets described in (a)–(d) below. Write

$$A^* = \{(x, y): 0 \leqslant x \leqslant a - 1,\ 0 \leqslant y \leqslant 2m + 1\}.$$

Then:

(a) $A' = A^* \cup \{(a, y): 0 \leqslant y \leqslant 2m - 1\} \cup \{(a + 1, y): y = 0,1\}$;
(b) $A' = A^* \cup \{(a, y): 0 \leqslant y \leqslant 2m\} \cup \{(a + 1, y): y = 0,1\}$;
(c) $A' = A^* \cup \{(a, y): 0 \leqslant y \leqslant 2m - 1\} \cup \{(a + 1, 0)\}$;
(d) $A' = A^* \cup \{(a, y): 0 \leqslant y \leqslant 2m\} \cup \{(a + 1, 0)\}$.

Now let $K_1 = (a, 2m + 1)$, $K_2 = (a, 2m)$, $K_3 = (a + 1, 1)$ and $K_4 = (a + 1, 0)$. We make the following operations, which change A' to the first segment in $\mathscr{L}$:
case (a)—remove K_3 and K_4 from A' and add K_1 and K_2 to it;
case (b)—remove K_3 from A' and add K_1 to it;
case (c)—remove K_4 from A' and add K_2 to it;
case (d)—remove K_4 from A' and add K_1 to it.
Now the value of Ψ_{m+1} ($=\Delta_\varphi$) at K_i ($i = 1, 2, 3, 4$) is, by (4.1),

$$K_1: \Psi_{m+1}(a) + \Psi_{2m+1}(2m + 1) = \left\lceil \frac{a}{2} \right\rceil + m + 1,$$

$$K_2: \Psi_{2m+1}(a) + \Psi_{2m+1}(2m) = \left\lceil \frac{a}{2} \right\rceil + m,$$

$$K_3: \Psi_{2m+1}(a + 1) + \Psi_{2m+1}(1) = \left\lceil \frac{a + 1}{2} \right\rceil + 1,$$

$$K_4: \Psi_{2m+1}(a + 1) + \Psi_{2m+1}(0) = \left\lceil \frac{a + 1}{2} \right\rceil.$$

Remember that we assumed that $m \geqslant 2$. The operations in (a)–(d) cannot make things worse. This concludes the proof in the first case.

Moreover, by symmetry and the equivalences (α)–(γ), the following cases can be excluded too:

$$\{(x, y): 0 \leqslant x \leqslant 2m + 1,\ y \geqslant 2m\} \cap A = \varnothing, \qquad \{(x, y): x \leqslant 1,\ 0 \leqslant y \leqslant 2m + 1\} \subset A,$$
$$\text{and} \qquad \{(x, y): 0 \leqslant x \leqslant 2m + 1,\ y \leqslant 1\} \subset A. \tag{4.6}$$

Thus we can always assume that

$$(0, 2m),\ (2m, 0) \in A \qquad \text{and} \qquad (1, 2m+1),\ (2m+1, 1) \notin A. \tag{4.7}$$

Case 2. (4.7) holds, and at least one of the elements $(0, 2m+1)$ and $(2m+1, 0)$ is not in A. Assume that $(2m+1, 0) \notin A$ and, consequently, that $A \cap \{(x, y): x = 2m + 1\} = \emptyset$. Let $Cl_{2m} = \{(x, y) \in A: x = 2m\}$ and $R_{2m} = \{(x, y) \in A: y = 2m\}$.

Then (4.7) implies that $Cl_{2m} \neq \emptyset$ and $R_{2m} \neq \emptyset$.

W.l.o.g., we can assume that

$$|Cl_{2m}| \triangleq c \leqslant |R_{2m}|, \tag{4.8}$$

because otherwise we can exchange the roles of rows and columns and, if necessary, move the $(2m+1)$st column to the $(2m+1)$st row.

When $(0, 2m+1) \notin A$, i.e. there is no element of A in the last row at all, we remove $Cl_{2m} = \{(2m, 0),\ (2m, 1) \cdots (2m, c-1)\}$ from A and add $\{(0, 2m+1),\ (1, 2m+1) \cdots (c-1, 2m+1)\}$ to A.

By (4.8), the resulting set is a downset and (4.1) shows that it has a larger value of φ^2. If $(0, 2m+1) \in A$ and $|Cl_{2m}| < |B_{2m}|$, one can also remvoe Cl_{2m} from A, but add $\{(1, 2m+1) \cdots (c, 2m+1)\}$ to A. When $(0, 2m+1) \in A$ and $c = |Cl_{2m}| = |R_{2m}| \neq 1$, we remove $\{(2m, 1),\ (2m, 2) \cdots (2m, c-1)\}$ from A and add $\{(1, 2m+1) \cdots (c-1, 1)\}$ to it.

For all of the three subcases above, we change our A to a downset in Case 1 or a set of type (4.6) for Case 1. Finally, we assume that

$$(0, 2m+1) \in A, \qquad Cl_{2m} = \{(2m, 0)\}, \qquad R_{2m} = \{0, 2m)\}. \tag{4.9}$$

Thus $(0, 2m+1) \in A$ implies $(2m+1, 0) \notin \tilde{A}$, where $\tilde{A}$ is defined in (γ).

Now $(2m, 1) \notin A$ and A is a downset. This, together, implies that $(2m+1, 1) \notin A$ and therefore $(1, 2m),\ (0, 2m) \in \tilde{A}$. Similarly, since $(1, 2m) \notin A$, $(2m, 1),\ (2m, 0) \in \tilde{A}$.

Therefore, both the $(2m)$th row and the $(2m)$th column of $\tilde{A}$ have at least two elements. By (α)–(γ), instead of A we can consider $\tilde{A}$, which has been settled in the previous subcases of this case.

Case 3. $(0, 2m+1),\ (2m+1, 0) \in A$. Here $(2m+1, 0) \notin \tilde{A}$ and $(0, 2m+1) \notin \tilde{A}$. By (γ), we can consider $\tilde{A}$ and reduce our problem to Case 2.

Finally, the theorem now follows from Theorem 1.

Remark 4. The result does not extend to general $C_{m,m}$. Already, for $C_{2m,2m-1}$, $\mathscr{L}$ is not optimal for the inner isoperimetric problem. The first $4m-4$ elements of $[2m-1] \times [2m-2]$ in the order $\mathscr{L}$ are

$$\{(0, 0), (0, 1), \ldots, (0, 2m-1), (1, 0), \ldots, (1, 2m-3)\}.$$

However, $\{(0, 0), \ldots, (0, 2m-2), (1, 0), \ldots, (1, 2m-2)\}$ has more inner edges.

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