Europ. J. Combinatorics (1998) **19**, 649–661
Article No. ej980223

The Intersection Theorem for Direct Products

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1. INTRODUCTION

Before we state the intersection problem for direct products and our solution, we set up our notation and give a sketch of some key steps in the extremal theory of set intersections.

$\mathbb{N}$ denotes the set of positive integers and for $i, j \in \mathbb{N}$, $i < j$, the set $\{i, i+1, \dots, j\}$ is abbreviated as $[i, j]$.

For $k, n \in \mathbb{N}$, $k \le n$, we set

$$2^{[n]} = \{F : F \subset [1, n]\}, \binom{[n]}{k} = \{F \in 2^{[n]} : |F| = k\}.$$

Similarly, for a finite set Ω we use 2^{Ω} and $\binom{\Omega}{k}$.

A system of sets $\mathcal{A} \subset 2^{[n]}$ is called t-intersecting, if

$$|A_1 \cap A_2| \ge t \text{ for all } A_1, A_2 \in \mathcal{A},$$

and $I(n, t)$ denotes the set of all such systems.

We denote by $I(n, k, t)$ the set of all k-uniform t-intersecting systems, that is,

$$I(n, k, t) = \left\{\mathcal{A} \in I(n, t) : \mathcal{A} \subset \binom{[n]}{k}\right\}.$$

The investigation of the function

$$M(n, k, t) = \max_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}|, 1 \le t \le k \le n,$$

and the structure of maximal systems was initiated by Erdös, Ko, and Rado [6].

THEOREM 1.1 ([6]). *For $1 \le t \le k$ and $n \ge n_0(k, t)$ (suitable)*

$$M(n, k, t) = \binom{n-t}{k-t}.$$

The smallest $n_0(k, t)$ has been determined by Frankl [8] for $t \ge 15$ and subsequently by Wilson [15] for all t:

$$n_0(k, t) = (k - t + 1)(t + 1).$$

In the recent paper [1] all the remaining cases

$$2k - t < n < (k - t + 1)(t + 1)$$

have been settled by proving the General Conjecture of Frankl [8], which stated that for $1 \le t \le k \le n$

$$M(n,k,t) = \max_{0 \le i \le \frac{n-t}{2}} |\mathcal{F}_i|,$$

where

$$\mathcal{F}_i = \left\{ F \in \binom{[n]}{k} : |F \cap [1, t+2i]| \ge t+i \right\}, 0 \le i \le \frac{n-t}{2}. \tag{1.1}$$

THEOREM 1.2 ([1]). *For $1 \le t \le k \le n$ with*

(i) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right)$ *for some $r \in \mathbb{N} \cup \{0\}$, we have*

$$M(n,k,t) = |\mathcal{F}_r|$$

and $\mathcal{F}_r$ is—up to permutations—the unique optimum (by convention $\frac{t-1}{r} = \infty$ for $r = 0$).

(ii) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) = n$ *for $r \in \mathbb{N} \cup \{0\}$ we have*

$$M(n,k,t) = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$$

and an optimal system equals up to permutations—either $\mathcal{F}_r$ or $\mathcal{F}_{r+1}$.

A very special case of Theorem 1.2 establishes the validity of the long-standing so-called 4m-conjecture (see [7, p. 56] and survey [5]).

In connection with Theorem 1.2 we note that, using the ideas of [1], in [2] maximal nontrivial intersecting systems (see [12]) have been determined completely, and in [3] the problem of optimal anticodes in Hamming spaces has been solved.

The following problem, initiated by Frankl, arose in connection with a result of Sali [14]. Let $n = n_1 + \cdots + n_m, k = k_1 + \cdots + k_m$ and $\Omega = \Omega_1 \dot\cup \Omega_2 \dot\cup \cdots \dot\cup \Omega_m$ with $|\Omega_i| = n_i$. Define

$$\mathcal{H} = \left\{ F \in \binom{\Omega}{k} : |F \cap \Omega_i| = k_i \text{ for } i = 1, \dots, m \right\}.$$

For given integers $t_i, 1 \le t_i \le k_i, 1 \le i \le m$, we say that $\mathcal{A} \subset \mathcal{H}$ is $(t_1, \dots, t_m)$-interesting, if for every $A, B \in \mathcal{A}$ there exists an $i, 1 \le i \le m$, such that

$$|A \cap B \cap \Omega_i| \ge t_i \text{ holds.}$$

Denote the set of all such systems by $I(\mathcal{H}, t_1, \dots, t_m)$.

The problem is to determine

$$M(\mathcal{H}, t_1, \dots, t_m) = \max_{\mathcal{A} \in I(\mathcal{H}, t_1, \dots, t_m)} |\mathcal{A}|.$$

Later, instead of $I(\mathcal{H}, t_1, \dots, t_m)$(resp. $M(\mathcal{H}, t_1, \dots, t_m)$), we use the abbreviations $I(\mathcal{H})$ (resp. $M(\mathcal{H})$).

The case $t_1 = t_2 = \cdots = t_m = 1$ has been solved by Frankl [10].

THEOREM 1.3 ([10]). *Let $\frac{k_m}{n_m} \le \cdots \le \frac{k_1}{n_1} \le \frac{1}{2}$ and $t_1 = t_2 = \cdots = t_m = 1$, then*

$$M(\mathcal{H}) = \frac{k_1}{n_1} \cdot |\mathcal{H}|.$$

The proof is based on the eigenvalue method (the idea of which is due to A. J. Hoffman (see [11]) and developed by Lovász [13]). In the same paper [10] the following more general result has been stated without proof.

THEOREM 1.4 ([10]). *Let the integers n_i, k_i, t_i satisfy $n_i \geq (k_i - t_i + 1)(t_i + 1)$ for $i = 1, \dots, m$, then*

$$M(\mathcal{H}) = \max_i \frac{\binom{n_i - t_i}{k_i - t_i}}{\binom{n_i}{k_i}} |\mathcal{H}|.$$

In the present paper we determine $M(\mathcal{H})$ for all parameters. Our result is

THEOREM 1.5. *Let $n_i \geq k_i \geq t_i \geq 1$ for $i = 1, \dots, m$, then*

$$M(\mathcal{H}) = \max_i \frac{M(n_i, k_i, t_i)}{\binom{n_i}{k_i}} |\mathcal{H}|.$$

We emphasize that the combination of this Theorem and Theorem 1.2 gives an explicit value of $M(\mathcal{H})$.

The proof of the Theorem is purely combinatorial and heavily (but not only!) based on ideas and methods from [1]. An essential ingredient is a result from [4].

REMARKS. (1) We can always assume that $n_i > 2k_i - t_i$ for all $i = 1, \dots, m$, because otherwise obviously $M(\mathcal{H}) = |\mathcal{H}|$.

(2) With the set $\mathcal{H}$, having parameters $n_i \geq k_i \geq t_i, n_i > 2k_i - t_i$, we consider any 'twin' set

$$\mathcal{H}' = \binom{\Omega_1}{k_1'} \times \binom{\Omega_2}{k_2'} \times \cdots \times \binom{\Omega_m}{k_m'},$$

where either $k_1' = k_i$ or $k_i' = n_i - k_i$ and the 'intersection numbers' are $t_i' = t_i$, if $k_i' = k_i$, and $t_i' = n_i - 2k_i + t_i$, if $k_i' = n_i - k_i$.

Clearly $M(\mathcal{H}) = M(\mathcal{H}')$ holds.

2. LEFT COMPRESSED SETS, GENERATING SETS AND THEIR PROPERTIES

We recall first some well-known and also more recent concepts, which can be found in [1]. Then we give extensions to direct products and basic properties of generating sets.

DEFINITION 2.1. Let $B_1 = \{i_1, \dots, i_k\} \in \binom{[n]}{k}$, $i_1 < i_2 < \cdots < i_k$, and $B_2 = \{j_1, \dots, j_k\} \in \binom{[n]}{k}$, $j_1 < j_2 < \cdots < j_k$. We write $B_1 \prec B_2$ iff $i_s \leq j_s$ for all $1 \leq s \leq k$, that is, B_1 can be obtained from B_2 by left-pushing. Denote by $\mathcal{L}(B_2)$ the set of all sets obtained this way from B_2. Also set $\mathcal{L}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{L}(B)$ for any $\mathcal{B} \subset 2^{[n]}$.

DEFINITION 2.2. $\mathcal{B} \subset 2^{[n]}$ is said to be left compressed or stable iff $\mathcal{B} = \mathcal{L}(\mathcal{B})$. We also recall the well-known exchange operation S_{ij}, defined for any family $\mathcal{B} \subset 2^{[n]}$.

DEFINITION 2.3. Set

$$S_{ij}(B)=\begin{cases}\{i\}\cup(B\backslash\{j\}) & \text{if } i\notin B,\ j\in B,\ \{i\}\cup(B\backslash\{j\})\notin\mathcal{B},\\ B & \text{otherwise}\end{cases}$$

and $S_{ij}(\mathcal{B})=\{S_{ij}(B):B\in\mathcal{B}\}$.

DEFINITION 2.4. Denote by $LI(n,k,t)$ the set of all stable systems from $I(n,k,t)$. It is known (from the shifting technique [8]) that

$$M(n,k,t)=\max_{\mathcal{B}\in I(n,k,t)}|\mathcal{B}|=\max_{\mathcal{B}\in LI(n,k,t)}|\mathcal{B}|.$$

DEFINITION 2.5. For any $B\in 2^{[n]}$ we define the upset $\mathcal{U}(B)=\{B'\in 2^{[n]}:B\subset B'\}$ and for $\mathcal{B}\subset 2^{[n]}$ we define

$$\mathcal{U}(\mathcal{B})=\bigcup_{B\in\mathcal{B}}\mathcal{U}(B).$$

Furthermore, recall the concept of generating sets [1].

DEFINITION 2.6. For any $\mathcal{B}\subset\binom{[n]}{k}$ a set $g(\mathcal{B})\subset\bigcup_{i\le k}\binom{[n]}{i}$ is called a generating set of $\mathcal{B}$, if

$$\mathcal{U}(g(\mathcal{B}))\cap\binom{[n]}{k}=\mathcal{B}.$$

Furthermore, $G(\mathcal{B})$ is the set of all generating sets of $\mathcal{B}(G(\mathcal{B})\neq\emptyset$, because $\mathcal{B}\in G(\mathcal{B}))$.

DEFINITION 2.7. For $B\subset[1,n]$ denote the greatest element of B by $s^+(B)$, and for $\mathcal{B}\subset 2^{[n]}$ set

$$s^+(\mathcal{B})=\max_{B\in\mathcal{B}}s^+(B).$$

DEFINITION 2.8. Let $\mathcal{B}\subset\binom{[n]}{k}$ be left compressed, i.e., $\mathcal{B}=\mathcal{L}(\mathcal{B})$. For any generating set $g(\mathcal{B})\in G(\mathcal{B})$ consider $\mathcal{L}(g(\mathcal{B}))$ and introduce its set of minimal (in the sense of set-theoretical inclusion) elements $\mathcal{L}_*(g(\mathcal{B}))$. Also define

$$G_*(\mathcal{B})=\{g(\mathcal{B})\in G(\mathcal{B}):\mathcal{L}_*(g(\mathcal{B}))=g(\mathcal{B})\}.$$

DEFINITION 2.9. For $\mathcal{B}\in LI(n,k,t)$ we set

$$s_{\min}(G(\mathcal{B}))=\min_{g(\mathcal{B})\in G(\mathcal{B})}s^+(g(\mathcal{B})).$$

Now we extend these definitions to a direct product of uniform sets $\mathcal{H}=\binom{\Omega_1}{k_1}\times\cdots\times\binom{\Omega_m}{k_m}$ in a natural way. To simplify notation we associate each block $\Omega_i=\{w_1^i,w_2^i,\ldots,w_{n_i}^i\}$ with $[1,n_i]$ for $i=1,\ldots,m$.

DEFINITION 2.10. For an $A=(A_1,\ldots,A_m)\in\prod_{i=1}^m 2^{\Omega_i}$ with $A_i=A\cap\Omega_i, i=1,\ldots,m$, we define

$$\mathcal{L}(A)=\left\{A'=(A'_1,A'_2,\ldots,A'_m)\in\prod_{i=1}^m 2^{\Omega_i}:A'_i\in\mathcal{L}(A_i),i=1,\ldots,m\right\}.$$

We set also $\mathcal{L}(\mathcal{A})=\bigcup_{A\in\mathcal{A}}\mathcal{L}(A)$.

DEFINITION 2.11. We say that $\mathcal{A} \subset \mathcal{H}$ is left compressed or stable, if $\mathcal{A} = \mathcal{L}(\mathcal{A})$. In other words $\mathcal{A}$ is stable, if it is stable under exchange operations S_{ij} with $i < j$ inside each block. The generating sets of an $\mathcal{A} \subset \mathcal{H}$ and notions $G(\mathcal{A}), \mathcal{L}(g(\mathcal{A})), \mathcal{L}_*(g(\mathcal{A})), G_*(\mathcal{A})$ one defines similarly.

DEFINITION 2.12. For $A = (A_1, \dots, A_m) \in \prod_{i=1}^m 2^{\Omega_i}$, $A_i = A \cap \Omega_i$, denote the greatest element of A_i (in $\Omega_i = [1, n_i]$) by $s_i^+(A)$, and for $\mathcal{A} \subset \prod_{i=1}^m 2^{\Omega_i}$ set

$$s_i^+(\mathcal{A}) = \max_{A \in \mathcal{A}} s_i^+(A), \; i = 1, \dots, m.$$

For $A = (A_1, \dots, A_m), B = (B_1, \dots, B_m) \in \prod_{i=1}^m 2^{\Omega_i}$, $A_i = A \cap \Omega_i, B_i = B \cap \Omega_i (i = 1, \dots, m)$ we write $A \prec B$ if $A_i \prec B_i$ (inside the set Ω_i) for all $i = 1, \dots, m$.

DEFINITION 2.13. For $\mathcal{A} \in LI(\mathcal{H})$ we set

$$s_{i\,\min}(G(\mathcal{A})) = \min_{g(\mathcal{A}) \in G(\mathcal{A})} s_i^+(g(\mathcal{A})), \; i = 1, \dots, m.$$

We start with simple, but important properties of generating sets.

LEMMA 2.1. *Let $\mathcal{A} \in I(\mathcal{H})$. For any $B, C \in g(\mathcal{A}) \in G(\mathcal{A})$ there exists an $1 \le i \le m$ such that*

$$|B \cap C \cap \Omega_i| \ge t_i.$$

LEMMA 2.2. *Let $A \in \mathcal{L}_*(g(\mathcal{A}))$. Then for any $B \in \prod_{i=1}^m 2^{\Omega_i}$ with $B \prec A$, either $B \in \mathcal{L}_*(g(\mathcal{A}))$ or there exists a $B' \in \mathcal{L}_*(g(\mathcal{A}))$ such that $B' \subset B$.*

LEMMA 2.3. *Let $\mathcal{A} \subset \mathcal{H}, \mathcal{L}(\mathcal{A}) = \mathcal{A}$, and $g(\mathcal{A}) \in G_*(\mathcal{A})$. Choose $E = (E_1, \dots, E_m) \in g(\mathcal{A})$ such that for some $1 \le i \le m, s_i^+(E) = s_i^+(g(\mathcal{A}))$, and denote by $\mathcal{A}_E$ the set of elements of $\mathcal{A}$, which are only generated by E. Then*

(i) for every $A \in \mathcal{A}_E$

$$A \cap [1, s_i^+(g(A))] = E_i$$

(ii) $|\{(A \cap \Omega_i) : A \in \mathcal{A}_E\}| = \binom{n_i - s_i^+(E)}{k_i - |E_i|}.$

LEMMA 2.4. *Let $\mathcal{A} \in LI(\mathcal{H}), g(\mathcal{A}) \in G_*(\mathcal{A})$ and let $E = (E_1, \dots, E_m), F = (F_1, \dots, F_m) \in g(\mathcal{A})$ have the properties*

(1) $|E_i \cap F_i| \ge t_i$ for some $1 \le i \le m$, and $|E_j \cap F_j| < t_j$ for all $j \ne i$ and
(2) $u \notin E_i \cup F_i, v \in E_i \cap F_i$ for some $u, v \in \Omega_i$ with $u < v$. Then

$$|E_i \cap F_i| \ge t_i + 1.$$

LEMMA 2.5. *(Pigeon hole principle with weight function). For $\mathcal{B} \subset \binom{[n]}{k}, \mathcal{B}^j \triangleq \{B \in \mathcal{B} : j \notin B\}$ let*

$$f : \mathcal{B} \to \mathbb{R}^+.$$

Then there exists an $i \in [1, n]$, such that

$$\sum_{B \in \mathcal{B}^i} f(B) \ge \frac{n-k}{n} \sum_{B \in \mathcal{B}} f(B).$$

The proof is readily established by counting in two ways.

3. The Main Auxiliary Results

LEMMA 3.1. *Let* $\mathcal{A} \in L(\mathcal{H})$ *with* $|\mathcal{A}| = M(\mathcal{H})$ *and let*

$$n_i > (k_i - t_i + 1)\left(2 + \frac{t_1 - 1}{r_i + 1}\right) \tag{3.1}$$

for some $i \in [1, m]$ *and* $r_i \in \mathbb{N} \cup \{0\}$. *Then*

$$s_{i\,\min}(G(\mathcal{A})) \leq t_i + 2r_i, \text{ if } t_i \geq 2 \tag{3.2}$$

and

$$s_{i\,\min}(G(\mathcal{A})) \leq 1, \text{ if } t_i = 1. \tag{3.3}$$

PROOF. We can assume that

$$n_i \geq 2k_i - t_i + 2, \tag{3.4}$$

because for $t_i = 1$ this is the condition (3.1) and in the case $n_i = 2k_i - t_i + 1 (t_i > 1)$ we have from (3.1) $r_i \geq k_i - t_i + 1$, and hence (3.2) holds. We are going to prove only (3.2), because the proof of (3.3) is just a step-by-step repetition. The proof is more complex than its predecessor in [1]. However, being based to a large extent on the same ideas and methods, we can omit some details. W.l.o.g. we prove the lemma for $i = 1$.

Let us have for some $g(\mathcal{A}) \in G_*(\mathcal{A})$

$$s_1^+(g(\mathcal{A})) = s_{1\,\min}(G(\mathcal{A}))$$

and let us assume in the opposite to (3.2) that

$$s_1^+(g(\mathcal{A})) = \ell > t_1 + 2r_1. \tag{3.5}$$

We shall show that under the assumptions (3.1) and (3.5) there exists an $\mathcal{A}' \in I(\mathcal{H})$ with $|\mathcal{A}'| > |\mathcal{A}| = M(\mathcal{H})$, which is a contradiction.

Towards this end we start with the partition

$$g(\mathcal{A}) = g_0(\mathcal{A}) \mathbin{\dot\cup} g_1(\mathcal{A}),$$

where

$$g_0(\mathcal{A}) = \{B \in g(\mathcal{A}) : s_1^+(B) = \ell\} \text{ and } g_1(\mathcal{A}) = g(\mathcal{A}) \backslash g_0(\mathcal{A}).$$

Obviously, for every $B \in g_0(\mathcal{A})$ and $C \in g_1(\mathcal{A})$

$$|(B \backslash \{\ell\}) \cap C \cap \Omega_i| \geq t_i$$

holds for some $i, 1 \leq i \leq m$ (see Lemma 2.1).

As $G(\mathcal{A}) \in G_*(\mathcal{A})$, we observe that omission of ℓ from any $E \in g_0(\mathcal{A})$ destroys the intersection property, that is, there exists an $F \in g_0(\mathcal{A})$, such that $|(E \backslash \{\ell\}) \cap F \cap \Omega_i| < t_i$ for all $i, 1 \leq i \leq m$.

The elements in $g_0(\mathcal{A})$ have an important property, which follows immediately from Lemma 2.4.

(P_1) For any $E = (E_1, \dots, E_m), F = (F_1, \dots, F_m) \in g_0(\mathcal{A})$ with $|E_1 \cap F_1| = t_1$, and $|E_i \cap F_i| < t_i$ for $i = 2, \dots, m$ necessarily

$$|E_1| + |F_1| = \ell + t_1.$$

Moreover, we have also the property
(P_2) For any $E, F \in g_0(\mathcal{A})$ with $|E_1| + |F_1| \neq \ell + t_1$

$$|(E\backslash\{\ell\}) \cap (F\backslash\{\ell\}) \cap \Omega_i| \geq t_i$$

holds for some i, $1 \leq i \leq m$.

Partition now $g_0(\mathcal{A})$ in the form

$$g_0(\mathcal{A}) = \bigcup_{0\leq i\leq \ell} \mathcal{R}_i$$

with $\mathcal{R}_i = \{F \in g_0(\mathcal{A}) : |F_1| = i\}$ and consider the set

$$\mathcal{R}'_i = \{F' : F' = F\backslash\{\ell\}; F \in \mathcal{R}_i\}.$$

Thus $|\mathcal{R}'{}_i| = |\mathcal{R}_i|$ and for any $F' \in \mathcal{R}'_i$ $|F'_1| = i - 1$.

We shall prove that (under conditions (3.1) and (3.5)) all $\mathcal{R}_i$ (and hence $\mathcal{R}'_i$) are empty. As $n_1 > 2k_1 - t_1$, we notice that the equation $|E_1| + |F_1| = t_1 + \ell$ for

$$E = (E_1, \dots, E_m), F = (F_1, \dots, F_m) \in g_0(\mathcal{A})$$

implies $|E_1| > k_1 - (n_1 - \ell), |F_1| > k_1 - (n_1 - \ell)$.

Suppose that $\mathcal{R}_i \neq \emptyset$ (equivalently $\mathcal{R}'_i \neq \emptyset$) for some i. We distinguish two cases (a) $i \neq \frac{t_1+\ell}{2}$ and (b) $i = \frac{t_1+\ell}{2}$.

Case (a): We consider generating sets

$$f_1 = g_1(\mathcal{A}) \cup (g_0(\mathcal{A})\backslash(\mathcal{R}_i \cup \mathcal{R}_{\ell+t_1-i})) \cup \mathcal{R}'_i \tag{3.6}$$

and

$$f_2 = g_1(\mathcal{A}) \cup (g_0(\mathcal{A})\backslash(\mathcal{R}_i \cup \mathcal{R}_{\ell+t_1-i})) \cup \mathcal{R}'_{\ell+t_1-i}.$$

We know from properties (P_1) and (P_2) that f_1 and f_2 satisfy Lemma 2.1. Hence, we have

$$\mathcal{B}_i = (\mathcal{U}(f_i) \cap \mathcal{H}) \in I(\mathcal{H}) \text{ for } i = 1, 2.$$

The desired contradiction will take the form

$$|\mathcal{A}| < \max_{i=1,2} |\mathcal{B}_i|. \tag{3.7}$$

The negation of (3.7) is

$$|\mathcal{A}| - |\mathcal{B}_i| \geq 0 \text{ for } i = 1, 2. \tag{3.8}$$

Let z (resp. y) be the number of those elements of $\mathcal{A}$, which are generated only by $\mathcal{R}_i$ (resp. $\mathcal{R}_{t_1+\ell-i}$), and let z'(resp. y') be the number of those elements of $\mathcal{B}_1$(resp. $\mathcal{B}_2$), which are generated only by $\mathcal{R}'_i$(resp. $\mathcal{R}'_{t_1+\ell-i}$).

From Lemma 2.3 it follows that for some $z_1, y_1 \in \mathbb{N}$,

$$z = z_1 \cdot \binom{n_1 - \ell}{k_1 - i} \text{ and } y = y_1 \cdot \binom{n_1 - \ell}{k_1 - \ell - t_1 + i} \tag{3.9}$$

and similarly we obtain

$$z' \geq z_1 \cdot \binom{n_1 - \ell + 1}{k_1 - \ell + 1} \text{ and } y' \geq y_1 \cdot \binom{n_1 - \ell + 1}{k_1 - \ell - t_1 + i + 1}. \tag{3.10}$$

(Actually, equalities hold, but they are not needed here.) Hence (3.8) is equivalent to

$$z + y - z' \geq 0, z + y - y' \geq 0. \tag{3.11}$$

Using (3.9), (3.10) one obtains

$$(n_1 + t_1 - k_1 - i)(n_1 - \ell - k_1 + i) \leq (k_1 - i + 1)(k_1 - \ell - t_1 + i + 1).$$

However, this is false, because $n_1 \geq 2k_1 - t_1 + 2$ and consequently $n_1 + t_1 - k_1 - i > k_1 - i + 1$ as well as $n_1 - \ell - k_1 + i > k_1 - \ell - t_1 + i + 1$.

Case (b): $i = \frac{\ell + t_1}{2}$. Let

$$\mathcal{T} = \left\{ E_1 \subset [1, \ell] \subset \Omega_1 : (E_1, \ldots, E_m) \in \mathcal{R}_{\frac{t_1+\ell}{2}} \right\}$$

and consider the partition

$$\mathcal{R}_{\frac{t_1+\ell}{2}} = \bigcup_{T \in \mathcal{T}} Q(T),$$

where

$$Q(T) = \left\{ E = (E_1, \ldots, E_m) \in \mathcal{R}_{\frac{t_1+\ell}{2}} : E_1 = T \right\},$$

and the partition

$$\mathcal{R}'_{\frac{t_1+\ell}{2}} = \bigcup_{T \in \mathcal{T}} Q'(T),$$

where

$$Q'(T) = \left\{ E = (E_1, \ldots, E_m) \in \mathcal{R}'_{\frac{t_1+\ell}{2}} : E_1 \cup \{\ell\} = T \right\}$$

Let $z(Q(T))$ be the number of elements of $\mathcal{A}$, which are generated only by elements from $Q(T)$. By Lemma 2.3 (ii) these numbers can be written in the form

$$z(Q(T)) = \binom{n_1 - \ell}{k_1 - \frac{t_1+\ell}{2}} \cdot z_1(Q(T)) \text{ for some } z_1(Q(T)) \in \mathbb{N}. \tag{3.12}$$

Further, let $z(R_{\frac{t_1+\ell}{2}})$ be the number of elements of $\mathcal{A}$, which are generated only by elements from $R_{\frac{t_1+\ell}{2}}$. Using Lemma 2.3 (i) and (3.12) we have

$$z\left(R_{\frac{t_1+\ell}{2}}\right) = \sum_{T \in \mathcal{T}} z(Q(T)) = \binom{n_1 - \ell}{k_1 - \frac{t_1+\ell}{2}} \cdot \sum_{T \in \mathcal{T}} z_1(Q(T)). \tag{3.13}$$

Now by Lemma 2.5 there exists a $j \in [1, \ell - 1]$ and a $\mathcal{T}' \subset \mathcal{T}$ such that $j \notin T$ for all $T \in \mathcal{T}'$ and

$$\sum_{T \in \mathcal{T}'} (Q(T)) \geq \frac{\ell - t_1}{2(\ell - 1)} \cdot z\left(R_{\frac{t_1+\ell}{2}}\right). \tag{3.14}$$

Let

$$\mathcal{R}^* = \bigcup_{T \in \mathcal{T}'} Q'(T) \subset \mathcal{R}'_{\frac{t_1+\ell}{2}},$$

and consider a new generating set

$$f = \left(g(\mathcal{A}) \backslash \mathcal{R}_{\frac{t_1+\ell}{2}}\right) \cup \mathcal{R}^*.$$

By Lemma 2.4 we have

$$(\mathcal{U}(f) \cap \mathcal{H}) = \mathcal{B} \in I(\mathcal{H}).$$

We show now that under condition (3.1)

$$|\mathcal{B}| > |\mathcal{A}| \tag{3.15}$$

holds, which will lead to the contradiction.

Indeed, let $z(\mathcal{R}^*)$ be the number of elements of $\mathcal{B}$, which are generated only by the elements from $\mathcal{R}^*$. Equivalent to (3.15) is

$$z(R^*) > z\left(\mathcal{R}_{\frac{t_1+\ell}{2}}\right). \tag{3.16}$$

The following relation similar to (3.13) can easily be verified.

$$z(R^*) \geq \binom{n_1 - \ell + 1}{k_1 - \frac{t_1+\ell}{2} + 1} \cdot \sum_{T \in \mathcal{T}'} z_1(Q'(T)). \tag{3.17}$$

(Actually, equality holds here.)

Now (3.16) and hence (3.15) easily follow from (3.13), (3.14), (3.17), and condition (3.1).

Inspection of the proof of Lemma 3.1 shows, that the following, slightly different statement also holds.

LEMMA 3.2. *Let $\mathcal{I} = \{i \in [1, m] : t_i \geq 2\}$ and let $n_i \geq (k_i - t_i + 1)\left(2 + \frac{t_1-1}{r_1+1}\right)$ for $i \in \mathcal{I}$ and $n_i > 2k_i$ for $i \in [1, m]\backslash\mathcal{I}$. Then there exists an $\mathcal{A} \in L(\mathcal{H})$ with maximal cardinality $|\mathcal{A}| = M(\mathcal{H})$, such that*

$$s_{i\,\min}(g(\mathcal{A})) \leq t_i + 2r_i \text{ for } i \in \mathcal{I}$$

and

$$s_{i\,\min}(g(\mathcal{A})) \leq 1 \text{ for } i \in [1, m]\backslash\mathcal{I}.$$

We recall the exchange operation S_{ij} (see Definition 2.3).

DEFINITION 3.1. We say that $\mathcal{B} \subset 2^{[n]}$ is invariant on $T \subset [1, n]$, if

$$S_{ij}(\mathcal{B}) = \mathcal{B} \text{ for all } i, j \in T.$$

LEMMA 3.3. *Let $\mathcal{A} \in LI(\mathcal{H})$, $|\mathcal{A}| = M(\mathcal{H})$, be an optimal set from Lemma 3.2, $i \in \mathcal{I} = \{i \in [1, m] : t_i \geq 2\}$ and let*

$$(k_i - t_i + 1)\left(2 + \frac{t_i - 1}{r_i + 1}\right) \leq n_i < (k_i - t_i + 1)\left(2 + \frac{t_i - 1}{r_i}\right). \tag{3.18}$$

Then $\mathcal{A}$ is invariant on $[1, t_i + 2r_i] \subset \Omega_i$.

PROOF. It suffices to prove the lemma for the first block with $t_1 \geq 2$. We know from Lemma 3.2 that

$$s_{1\,\min}(G(\mathcal{A})) \leq t_1 + 2r_1.$$

From the definition of generating sets we also know that $\mathcal{A}$ is invariant on $[t_1 + 2r_1 + 1, n_1]$. Consider now the 'twin' to set $\mathcal{H}$

$$\mathcal{H}' = \binom{[n_1]}{n_1 - k_1} \times \binom{[n_2]}{k_2} \times \cdots \times \binom{[n_m]}{k_m}$$

with intersection numbers $t_1' = n_1 - 2k_1 + t_1, t_2' = t_2, \ldots, t_m' = t_m$ and a new set

$$\mathcal{A}' = \{(A_1, \ldots, A_m) \in \mathcal{H}' : (\Omega_1\backslash A_1, A_2, \ldots, A_m) \in \mathcal{A}\}.$$

Clearly, $\mathcal{A}' \in I(\mathcal{H})$ and $|\mathcal{A}'| = |\mathcal{A}| = M(\mathcal{H}) = M(\mathcal{H}')$ (see the Remark in the Introduction). It is also clear that $\mathcal{A}'$ is right-compressed in Ω_1.

The right side of condition (3.18) gives the relation

$$n_1 > (k_1' - t_1' + 1)\left(2 + \frac{t_1' - 1}{r_1' + 1}\right)$$

for $k_1' = n_1 - k_1$, $t_1' = n_1 - 2k_1 + t_1$ and $r_1' = k_1 - t_1 - r_1$.

From the left–right symmetry and Lemma 3.1 we conclude that there exists a generating set $g(\mathcal{A}')$ such that for every $E = (E_1, \dots, E_m) \in g(\mathcal{A}')$ necessarily $E_1 \subset [t_1 + 2r_1 + 1, n_1]$. Consequently $\mathcal{A}'$ is invariant on $[1, t_1 + 2r_1]$ and this means that $\mathcal{A}$ has the same property.

LEMMA 3.4. *Let $\mathcal{A} \in LI(\mathcal{H})$, $|\mathcal{A}| = M(\mathcal{H})$, be an optimal set from Lemma 3.3, and let $g(\mathcal{A}) \in G_*(\mathcal{A})$. Then for every $E = (E_1, \dots, E_m) \in g(\mathcal{A})$ either $|E_i| = t_i + r_i$ or $|E_i| = 0$ for $i \in \mathcal{I} = \{i \in [1, m] : t_1 \geq 2\}$, and*

$$|E_i| \leq 1 \text{ for } i \in [1, m] \backslash \mathcal{I}.$$

PROOF. Again it suffices to show that the statement holds for the first block Ω_1.

Moreover, we assume that $1 \in \mathcal{I}$, that is, $t_1 \geq 2$, because for $t_1 = 1$ the statement holds, according to Lemma 3.1.

From Lemma 3.2 we know that, for every $E = (E_1, \dots, E_m) \in g(\mathcal{A})$, necessarily $E_1 \subset [1, t_1 + 2r_1]$. Moreover, it follows from the proof of case (a) in Lemma 3.1: if $t_1 + 2r_1 \in E_1$, then necessarily $|E_1| = t_1 + r_1$.

Suppose now that there exists an $F = (F_1, \dots, F_m) \in g(\mathcal{A})$ with $|F_1| \neq 0$ and $|F_1| \neq t_1 + r_1$. We have $t_1 + 2r_1 \notin F_1$. Two cases can occur: $|F_1| > t_1 + r_1$ and $0 < |F_1| < t_1 + r_1$. Here we treat only the first case, because the second can be done by similar arguments.

Let $\mathcal{A}_F \subset \mathcal{A}$ be the set of those elements of $\mathcal{A}$, which are generated only by $F = (F_1, \dots, F_m) \in g(\mathcal{A})$. As $g(\mathcal{A}) \in G_*(\mathcal{A})$, then clearly $\mathcal{A}_F \neq \emptyset$.

Moreover, as $k_1 - |F_1| < n_1 - t_1 - 2r_1$ (this follows from $|F_1| > t_1 + r_1$) there exists an $A = (A_1, \dots, A_m) \in \mathcal{A}_F$, such that $A_1 \cap [1, t_1 + 2r_1] = F_1$. Recall now the exchange operation and consider $A_1' = S_{j, t_1 + 2r_1}(A_1)$ for $j \in F_1$.

According to Lemma 3.3 we have $A' = (A_1', A_2, \dots, A_m) \in \mathcal{A}$ as well. Let $F' = (F_1', \dots, F_m') \in g(\mathcal{A})$ be an element, which generates A', that is, $A' \in \mathcal{U}(F')$.

Clearly, $t_1 + 2r_1 \in F_1'$, because otherwise $A \in \mathcal{U}(F')$ as well, and this would contradict the definition of the set $\mathcal{A}_F$. On the other hand, if $t_1 + 2r_1 \in F_1'$, then necessarily $|F_1'| = t_1 + r_1 < |F_1|$ and this again leads to a contradiction with Lemma 2.2.

4. FURTHER PREPARATIONS

The following statement summarizes our findings in previous sections.

LEMMA 4.1. *Let $r_i, i \in \mathcal{I} = \{j \in [1, m] : t_j \geq 2\}$, be integers uniquely determined in (3.18) and let us set $r_i = 0$ for $i \in [1, m] \backslash \mathcal{I}$. Then there exists an $\mathcal{A} \in I(\mathcal{H})$ with $|\mathcal{A}| = M(\mathcal{H})$, such that for any $A = (A_1, \dots, A_m)$, $B = (B_1, \dots, B_m) \in \mathcal{A}$ there is an $i \in [1, m]$ for which both,*

$$|A_i \cap [1, t_i + 2r_i]| \geq t_i + r_i \text{ and } |B_i \cap [1, t_i + 2r_i]| \geq t_i + r_i$$

hold.

PROOF. Let $\mathcal{A} \subset \mathcal{H}$ be an optimal $(t_1, \ldots, t_m)$-intersecting system for which the statements of Lemma 3.3 and 3.4 hold. According to Lemma 3.2 the system $\mathcal{A}$ has a generating set $g(\mathcal{A}) \in G(\mathcal{A})$, such that for each $E = (E_1, \ldots, E_m) \in g(\mathcal{A})$ one has $E_i \subset [1, t_i + 2r_i] \subset \Omega_i (i = 1, \ldots, m)$. On the other hand, Lemma 3.5 says that the cardinality of $E_i (i = 1, \ldots, m)$ is either $t_i + r_i$ or 0. Therefore, for any $E = (E_1, \ldots, E_m), F = (F_1, \ldots, F_m) \in g(\mathcal{A})$, to guarantee $(t_1, \ldots, t_m)$-intersection (see Lemma 2.1), there must exist an $i \in [1, m]$ such that $|E_i| = |F_i| = t_i + r_i$.

Let now $r_i, i = 1, \ldots, m$, be integers defined in Lemma 4.1. For a $\mathcal{C} \subset \mathcal{H}$ we consider the following mappings: $\varphi = (\varphi_1, \ldots, \varphi_m) : \mathcal{C} \to \{0, 1\}^m$, where for $C = (C_1, \ldots, C_m) \in \mathcal{C}, C_i = \Omega_i \cap C$

$$\varphi_i(C_i) = \begin{cases} 1, & \text{if } |C_i \cap [1, t_i + 2r_i]| \geq t_i + r_i \\ 0, & \text{otherwise,} \end{cases}$$

$\varphi(C) = (\varphi_1(C_1), \ldots, \varphi_m(C_m))$ and we set $\Phi(\mathcal{C}) = \{\varphi(C) : C \in \mathcal{C}\}$.

For any $\mathcal{C} \subset \mathcal{H}$ and $B \in \{0, 1\}^m$ we define the weight $w(B, \mathcal{C})$:

$$w(B, \mathcal{C}) = |\{C \in \mathcal{C} : \varphi(C) = B\}|.$$

Clearly,

$$\sum_{B \in \{0,1\}^m} w(B, \mathcal{C}) = |\mathcal{C}|.$$

It is also clear that for any $B = (b_1, \ldots, b_m) \in \{0, 1\}^m$ one has

$$w(B, \mathcal{C}) \leq \prod_{i=1}^{m} w(b_i), \tag{4.1}$$

where

$$w(b_i) = \begin{cases} |\mathcal{F}_{r_i}|, & \text{if } b_i = 1 \\ \binom{n_i}{k_i} - |\mathcal{F}_{r_i}|, & \text{if } b_i = 0, \end{cases} \tag{4.2}$$

and the $\mathcal{F}_{r_i}$s are defined in (1.1).

Now let $\mathcal{C} \subset \mathcal{H}$ be a set such that $\Phi(\mathcal{C}) \in I(m)$, where $I(m)$ is the set of all intersecting families in $2^{[m]}$ (to avoid an extra notation we identified $2^{[m]}$ with $\{0, 1\}^m$). Obviously $\mathcal{C} \in I(\mathcal{H})$.

Let $I(\mathcal{H}, \Phi)$ be the set of all such systems from $\mathcal{H}$ and denote

$$M(\mathcal{H}, \Phi) = \max_{\mathcal{C} \in I(\mathcal{H}, \Phi)} |\mathcal{C}|.$$

Clearly,

$$M(\mathcal{H}, \Phi) \leq M(\mathcal{H}) \tag{4.3}$$

and for any $\mathcal{C} \in I(\mathcal{H}, \Phi)$ with $|\mathcal{C}| = M(\mathcal{H}, \Phi)$ necessarily

$$w(B, \mathcal{C}) = \prod_{i=1}^{m} w(b_i) \text{ for all } B \in \Phi(\mathcal{C}) \subset \{0, 1\}^m.$$

It follows from Lemma 4.1 that the opposite to (4.3) also holds.

Moreover, for any $\mathcal{B} \in I(m)$ with arbitrary weights $h : B \to \mathbb{N}$ satisfying $h(B) \leq \prod_{i=1}^m w(b_i)$ for $B \in \mathcal{B}$, one can find a $\mathcal{C} \in I(\mathcal{H}, \Phi)$ with $\Phi(\mathcal{C}) = \mathcal{B}$ and $w(B, \mathcal{C}) = h(B)$, $B \in \mathcal{B}$. Therefore one has the following

LEMMA 4.2.

$$M(\mathcal{H}) = \max_{\mathcal{B} \in I(m)} \sum_{B \in \mathcal{B}} w(B),$$

where for $B = (b_1, \ldots, b_m)$, $w(B) = \prod_{i=1}^m w(b_i)$, *and the* $w(b_i)$ *are defined in (4.2).*

We need a special case of a result from Ahlswede and Cai [4]:

Let $u = \{u_1 \geq u_2 \geq \cdots \geq u_m\}$ be positive reals and let $\mathcal{B} \in I(m)$ be an intersecting family in $2^{[m]}$. Define

$$u(B) = \prod_{i \in B} u_i \text{ for } B \subset [1, m]$$

and $W(\mathcal{B}) = \sum_{B \in \mathcal{B}} u(B)$ for $\mathcal{B} \subset 2^{[m]}$. We set

$$\alpha(m, u) \triangleq \max_{\mathcal{B} \in I(m)} W(\mathcal{B}).$$

THEOREM 4.1 ([4]). *(In a special case.) Let* $u_1 < 1$, *then*

$$\alpha(m, u) = W(\mathcal{B}(u_1)),$$

where

$$\mathcal{B}(u_1) = \{B \subset [1, m] : 1 \in B\}.$$

Finally we need the following statement, which can easily be proved.

PROPOSITION 4.1. *For all* $n_i \geq k_i \geq t_i \geq 1$ *with* $n_i > 2k_i - t_i$ *for* $i = 1, \ldots, m$

$$M(\mathcal{H}) \leq \frac{|\mathcal{H}|}{2} \textit{holds}.$$

Moreover, if $n_i = 2k_i$, $t_i = 1$ *for some* $i \in [1, m]$, *then*

$$M(\mathcal{H}) = \frac{|\mathcal{H}|}{2}.$$

5. PROOF OF THEOREM 1.5

We say that $\mathcal{B} \subset 2^{[m]}$ is a 'star', if $\mathcal{B} = \{B \in 2^{[m]} : j \in B\}$ for some $j \in [1, m]$.

According to Lemma 4.2, the proof of Theorem 1.5 can be finished by showing that the maximum in $\max_{\mathcal{B} \in I(m)} \sum_{B \in \mathcal{B}} w(B)$ is assumed for a 'star'.

Of course, it is equivalent to show that $\max_{\mathcal{B} \in (m)} \sum_{B \in \mathcal{B}} \beta(B)$ is assumed for a 'star', where for $B = (b_1, \ldots, b_m) \in \{0, 1\}^m$

$$\beta(B) = \prod_{i=1}^m \beta(b_i)$$

and

$$\beta(b_i) = \frac{w(b_i)}{\binom{n_i}{k_i} - |\mathcal{F}_{r_i}|},$$

that is

$$\beta(b_i) = \begin{cases} \frac{|\mathcal{F}_{r_i}|}{\binom{n_i}{k_i}}, & \text{if } b_i = 1 \\ 1, & \text{if } b_i = 0. \end{cases}$$

As for any $n \geq k \geq t \geq 1, n > 2k - t$

$$M(n, k, t) \geq \frac{\binom{n}{k}}{2} \tag{5.1}$$

holds, we conclude that

$$\beta(b_i) \leq 1 \text{ for all } i \in [1, m].$$

Moreover, since equality in (5.1) is achieved iff $n = 2k, t = 1$, then, according to Proposition 4.1, we can assume that

$$\beta(b_i) < 1 \text{ for all } i \in [1, m].$$

Now we apply Theorem 4.1 with respect to the reals $u_i = \beta(b_i), i = 1, \ldots, m$, to complete the proof of Theorem 1.5.

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Received 14 April 1997 and accepted 7 April 1998

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