A counterexample to Kleitman's conjecture concerning an edge–isoperimetric problem

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Abstract

Kleitman's conjecture concerning the "Kleitman–West problem" is false for 3–element subsets.

1 The problem

We are concerned here with the graph

$$G = G_{n,k} = (\mathcal{V}_{n,k}, \mathcal{E}_{n,k}), \quad \text{where} \ \ \mathcal{V}_{n,k} = \binom{[n]}{k}$$

and $\mathcal{E}_{n,k} = \big\{\{A, B\} : A, B \in \mathcal{V}_{n,k} \text{ with } |A \cap B| = k-1\big\}$.

For any set of vertices $\mathcal{A} \subset \binom{[n]}{k}$ we introduce the set of boundary edges

$$\mathcal{B}(\mathcal{A}) = \big\{\{A, B\} \in \mathcal{E}_{n,k} : A \in \mathcal{A} \ \text{ and } \ B \in \mathcal{A}^c\big\}, \tag{1.1}$$

where $\mathcal{A}^c = \binom{[n]}{k} \smallsetminus \mathcal{A}$ is the complement of $\mathcal{A}$.

The edge–isoperimetric problem consists in determining for every positive integer N the quantity

$$b(N) = b_{n,k}(N) = \min \left\{ |\mathcal{B}(\mathcal{A})| : \mathcal{A} \subset \binom{[n]}{k}, |\mathcal{A}| = N \right\} \tag{1.2}$$

and the corresponding optimal configurations.

The problem was stated in [2] and analyzed there for $k = 2$. It is known as the Kleitman–West problem (see e.g. pages 60–61 of [6], pages 370–371 of [7] and page 1298 of [3]; Larry Harper [5] gave it this name, because West told him that he had heard it from Kleitman).

2 The conjecture

First observe that instead of considering the boundary of $\mathcal{A}$ one can look at the inner edges of $\mathcal{A}$ and define

$$\mathcal{I}(\mathcal{A}) = \big\{\{A, B\} \in \mathcal{E}_{n,k} : A, B \in \mathcal{A}\big\}, \tag{2.1}$$

because by regularity of the graph G we have

$$\begin{aligned} i(N) = i_{n,k}(N) &= \max \left\{ |\mathcal{I}(\mathcal{A})| : \mathcal{A} \subset \binom{[n]}{k}, |\mathcal{A}| = N \right\} \\ &= \frac{1}{2}\big(N \ k(n-k) - b(N)\big). \end{aligned}$$

Also, $\mathcal{I}(\mathcal{A}^c) = \mathcal{E}_{n,k} \smallsetminus \big(\mathcal{I}(\mathcal{A}) \cup \mathcal{B}(\mathcal{A})\big)$ and therefore $i\left(\binom{n}{k} - N\right) = \binom{n}{k} - i(N) - b(N)$.

Finally, by complementation in $[n]$

$$i_{n,k}(N) = i_{n,n-k}(N).$$

Therefore it suffices to consider $k \leq \frac{n}{2}$.

Now, Kleitman wrote in [7], pages 370–371, (where his w, $\mathcal{X}$, and $\mathcal{S}$ are our N, $\mathcal{A}$, and $\binom{[n]}{k}$): "There is an obvious conjecture: Suppose $2k \leq n$; if $w \leq \binom{n-1}{k-1}$ take only sets containing the first element; if $\binom{n}{k} > 2w > 2\binom{n-1}{k-1}$ take all sets containing the first element. Since the condition is symmetric between $\mathcal{X}$ and $\mathcal{S} \smallsetminus \mathcal{X}$ and hence between w and $\binom{n}{k} - w$, this construction handles one element completely; leaving a problem that can be handled recursively by the same construction on the remaining $n-1$ element set. The proposed optimal configurations do not "nest" inside one another here, which interferes with many methods of proof." The same conjecture can be found also in [6] and [3].

Remark: For $k = 2$ the conjecture is true by Theorem AK, the main result of [2].

3 An auxiliary result

The Lemma below can be found as Theorem 1 in [1] and was obtained independently, but earlier, in [4]. Still, we prove it, because the arguments are short.

Denote by $\mathcal{L}(n,k) = (\mathcal{S}_{n,k}; \leq)$ the lattice defined by

$$\mathcal{S}_{n,k} = \big\{(x_1, \dots, x_k) : 1 \leq x_1 < x_2 \dots < x_k \leq n, x_i \in \mathbb{Z}^+\big\}$$

and $(x_1, \dots, x_k) \leq (x'_1, \dots, x'_k) \Leftrightarrow x_i \leq x'_i (1 \leq i \leq k)$.

For $x^k = (x_1, \dots, x_k) \in \mathcal{S}_{n,k}$, the rank of x^k is defined as $|x^k| = \sum\limits_{i=1}^{k} x_i$ and for $W \subset \mathcal{S}_{n,k}$, let $\|W\| = \sum\limits_{x^k \in W} |x^k|$. In addition we let $A = \{x_1, \dots, x_k\} \in \binom{[n]}{k}$, with elements labelled in increasing order, correspond to $x^k = \Phi(A) = (x_1, \dots, x_k) \in \mathcal{S}_{n,k}$, and similarly, $\mathcal{A} \subset \binom{[n]}{k}$ to $\Phi(\mathcal{A}) = \big\{\Phi(A) : A \in \mathcal{A}\big\}$.

Using for $\mathcal{A}$ and $1 \leq i < j \leq n$ the following "pushing to the left" or so–called switching operator $S_{i,j}$, which is frequently employed in combinatorial extremal theory,

$$S_{i,j}(A) = \begin{cases} \big(A \smallsetminus \{j\}\big) \cup \{i\} & \text{if } \ (A \smallsetminus \{j\}) \cup \{i\} \notin \mathcal{A}, j \in A, \ \text{ and } \ i \notin A \\ A & \text{otherwise,} \end{cases}$$

one can prove by standard arguments that for every fixed N there is an $\mathcal{A} \subset \binom{[n]}{k}$, $|\mathcal{A}| = N$, which is left–compressed (that is stable under all switching operations) and satisfies $|I(\mathcal{A})| = i(N)$. It is also easy to see that such left–compressed families correspond to downsets in $\mathcal{L}(n,k)$.

Lemma. *For fixed $N \in \mathbb{Z}^+$, maximizing $|\mathcal{I}(\mathcal{A})|$ for $\mathcal{A} \subset \binom{[n]}{k}$, $|\mathcal{A}| = N$, is equivalent to finding a downset W in $\mathcal{L}(n,k)$ with $|W| = N$ and maximal $\|W\|$.*

Proof: Assume that $\mathcal{A} \subset \binom{[n]}{k}$, $|\mathcal{A}| = N$, and $W = \Phi(\mathcal{A})$ is a downset in $\mathcal{L}(n,k)$.

For every $x^k \in W$ there are exactly

$$(x_{i+1} - x_i - 1)\binom{k-i}{k-1-i} = (x_{i+1} - x_i - 1)(k-i) \tag{3.1}$$

y^k's with $y^k \leq x^k$, whose first i components coincide with those of x^k and whose $i+1$–st component does not, and for which A and B satisfy $|A \cap B| = k-1$, if $x^k = \Phi(A)$ and $y^k = \Phi(B)$. (Here $x_0 = 0$.)

By (3.1), for $x^k = \Phi(A)$ fixed, there is a total of

$$\sum_{i=0}^{k-1}(x_{i+1} - x_i - 1)(k-i) = \sum_{i=1}^{k}(k-i+1)x_i - \sum_{i=0}^{k-1}(k-i)x_i - \sum_{i=0}^{k-1}(k-i)$$

$$= \sum_{i=1}^{k} x_i - \binom{k+1}{2} = |x^k| - \binom{k+1}{2} \tag{3.2}$$

B's with $\Phi(B) = y^k \leq x^k$, $\{A, B\} \in \mathcal{E}_{n,k}$, and with $\Phi(B) \in \mathcal{A}$, because $\Phi(\mathcal{A})$ is a downset. Consequently

$$|\mathcal{I}(\mathcal{A})| = \sum_{x^k \in W} |x^k| - \binom{k+1}{2}|\mathcal{A}| = \|W\| - \binom{k+1}{2}N. \tag{3.3}$$

Thus the Lemma follows, because $\mathcal{A}$ can be assumed to be left–compressed.

4 A counterexample for $k = 3$

Let $k = 3$, $N = \left[\binom{n}{3} - \binom{n-2}{3}\right] - (n-3) - (n-2)$, and let n be sufficiently large. Then the conjecture gives a configuration $\mathcal{A} = \mathcal{A}_1 \setminus (\mathcal{A}_2 \cup \mathcal{A}_3)$, where

$\mathcal{A}_1 = \big\{A : A \cap \{1,2\} \neq \phi\big\}, \mathcal{A}_2 = \big\{\{2,3,n\}, \{2,4,n\}, \ldots, \{2,n-2,n\}, \{2,n-1,n\}\big\}$,
$\mathcal{A}_3 = \big\{\{2,3,n-1\}, \{2,4,n-1\}, \ldots, \{2,n-2,n-1\}, \{2,n-4,n-2\}, \{2,n-3,n-2\}\big\}$

and where $W = \Phi(\mathcal{A}) = \mathcal{S}_1 \setminus (\mathcal{S}_2 \cup \mathcal{S}_3)$, with $\mathcal{S}_1 = \Phi(\mathcal{A}_1) = \big\{(x,y,z) \in \mathcal{S}_{n,3} : x = 1,2\big\}$, $\mathcal{S}_2 = \Phi(\mathcal{A}_2) = \big\{(2,3,n), (2,4,n), \ldots, (2,n-2,n), (2,n-1,n)\big\}$, and $\mathcal{S}_3 = \Phi(\mathcal{A}_3) = \big\{(2,3,n-1), (2,4,n-1), \ldots, (2,n-2,n-1), (2,n-4,n-2), (2,n-3,n-2)\big\}$. It is obviously better than the other candidate given by the conjecture (of "complement form" but with the same cardinality) when n is large enough (c.f. Theorem AK and the Lemma). On the other hand, we define $\mathcal{A}' = \mathcal{A}_1 \setminus (\mathcal{A}_2 \cup \mathcal{A}_3')$ for $\mathcal{A}_3' = \big\{\{1,2,n\}, \{1,3,n\}, \{1,4,n\}, \ldots, \{1,n-2,n\}, \{1,n-1,n\}\big\}$. Then $\Phi(\mathcal{A}')$ is a downset in $\mathcal{L}(n,k)$ and for $W' = \Phi(\mathcal{A}')$ and $\mathcal{S}_3' = \Phi(\mathcal{A}_3')$,

$$\begin{aligned}
&\|W'\| - \|W\| = \|\mathcal{S}_3\| - \|\mathcal{S}_3'\| = \big[(2+3+n-1) + (2+4+n-1) + \cdots + \\
&(2+n-2+n-1) + (2+n-4+n-2) + (2+n-3+n-2)\big] \\
&- \big[(1+2+n) + (1+3+n) + (1+4+n) + \cdots + (1+n-2+n) + (1+n-1+n)\big] \\
&= n - 10 > 0, \quad \text{if} \quad n > 10.
\end{aligned} \tag{4.1}$$

(3.3) and (4.1) show that $|\mathcal{I}(\mathcal{A}')|$ can be arbitrarily much larger than $|\mathcal{I}(\mathcal{A})|$, if n is arbitrarily large. So the conjecture is false for $k = 3$.

5 Final remarks

This example first appeared in [1], where in the case $k = 3$ we solve the isoperimetric problem for "good" parameters N, i.e. $N = \binom{n}{3} - \binom{m}{3} \leq \frac{\binom{n}{3}}{2}$ for some $m \in \mathbb{Z}^+$ or $\binom{m'}{3} \geq \frac{\binom{n}{3}}{2}$ for some $m' \in \mathbb{Z}^+$.

We arrived at the edge–isoperimetric problem via [2] and while writing on it in [1] did not know Kleitman's conjecture about it. We remark that our positive results in [1] are obtained by embedding $\mathcal{S}_{n,3}$ into $\mathbb{R}^3$ and then treating the continuous version by analytic methods. Subsequently we became aquainted with Harper's closely related paper [4], solving a continuous version of the "Kleitman–West problem".

It may be of interest to point out the difference between [1] and [4]: a.) In the limiting process $n - k \to \infty$ of [4] the unit cubes get concentrated at single points, so that the objects may be smoothly embedded in continuous spaces, whereas in [1] the volume of the unit cube is comparable to the volumes of the objects for fixed n, which makes things more complicated. b.) Harper applies variational methods and we just use elementary calculus, which causes a further complexity of our proof. Nevertheless, it may be possible to greatly simplify Harper's proof along our lines.

We do not wish to publish the preprint [1] in its present form, as we feel it is incomplete. However, several requests have made it clear to us that the counterexample is of great interest for people working in this area, and therefore we are making it known without further delay.

6 References

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