SPLITTING PROPERTIES IN PARTIALLY ORDERED SETS AND SET SYSTEMS

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Abstract: It was shown in [1] that in any "dense" finite poset $\mathcal{P} = (P, <)$ (e.g. in the Boolean lattice) every maximal antichain S may be partitioned into disjoint subsets S_1 and S_2, such that the union of the upset of S_1 with the downset of S_2 yields the entire poset: $U(S_1) \cup D(S_2) = P$.
Under suitable denseness assumptions we establish splitting properties in great generality for infinite posets, directed graphs and set systems. We show also that for countable posets the conjecture (4.4) of [1] is not true. The poset of squarefree integers serves as an example.
It seems also to be of interest that already for the finite Boolean lattice there are antichains which splitt cardinalitywise only in an extremely unbalanced way.
Finally we introduce new notions of splitting, called Y–splitting, λ–splitting and X–splitting. For instance in a Y–splitting $\{S_1, S_2\}$ in addition to the property above we have also that $U(S_1) \cup D(S_1) \cup S_2 = P$. We establish first results in a challenging new area.

BASIC DEFINITIONS FOR POSETS

Downsets, upsets, generators, antichains

Let $\mathcal{P} = (P, <)$ be a partially ordered set (poset) and let H be a subset of P. The *downset* $D(H)$ of the subset H is

$$D(H) = \big\{x \in P : \exists s \in H (x \leq s)\big\}. \tag{1.1}$$

The upset $U(H)$ of H is

$$U(H) = \big\{x \in P : \exists s \in H (s \leq x)\big\}. \tag{1.2}$$

We introduce also the sets

$$D^*(H) = \{x \in P : \exists s \in H(x < s)\} \tag{1.3}$$

and

$$U^*(H) = \{x \in P : \exists s \in H(s < x)\}. \tag{1.4}$$

A subset $G \subset P$ is called a *generator* of $\mathcal{P}$, if

$$U(G) \cup D(G) = P. \tag{1.5}$$

A generator G of $\mathcal{P}$ is called *minimal*, if no proper subset of G is a generator of $\mathcal{P}$.
A subset $S \subset P$ is called *antichain or Sperner system*, if no two elements of S are comparable. An antichain S is maximal (or saturated) if for every antichain $S' \subset P$, $S \subset S'$ implies $S = S'$. It is easy to see that an antichain S is maximal iff it is a generator of $\mathcal{P}$. We also remark that a minimal generator of $\mathcal{P}$ is not necessary an antichain.

A splitting property and notions of denseness

We say that $H \subset P$ has the *splitting property*, if there exists an $H_1 \subset H$ with

$$U(H_1) \cup D(H \smallsetminus H_1) = P. \tag{1.6}$$

Of course, for H to have the splitting property it is necessary that H is a generator of $\mathcal{P}$. We say that $\mathcal{P}$ has the splitting property, if every maximal antichain has the splitting property.
Now we introduce notions of denseness in $\mathcal{P}$ for $H \subset P$.
If for every *open interval* $< x, y >= \{z \in P : x < z < y\}$ with endpoints $x, y \in P \smallsetminus H$:
(d_1) $\quad \langle x, y\rangle \cap H \neq \phi \Rightarrow |\langle x, y\rangle \cap P| \geq 2$,
then we call H d_1*–dense* in $\mathcal{P}$,
(d_2) $\quad \langle x, y\rangle \cap H \neq \phi \Rightarrow |\langle x, y\rangle \cap H| \geq 2$,
then we call H d_2*–dense* in $\mathcal{P}$.
Furthermore, if for every open interval $\langle x, y\rangle$ with endpoints $x, y \in P$:
(d_2^*) $\quad \langle x, y\rangle \cap H \neq \phi \Rightarrow |\langle x, y\rangle \cap H| \geq 2$,
then we call H d_2^**–dense* in $\mathcal{P}$.
Clearly, a d_2^*–dense set is also d_2–dense and a d_2–dense set is also d_1–dense.
Remarks:

- In the special case $H = P$ in [1] for d_2^*–denseness the term "$\mathcal{P}$ is weakly dense" is used. Also, $\mathcal{P}$ is strongly dense, if for any non–empty interval $\langle x, y\rangle$ and any $z \in \langle x, y\rangle$ there is a $z' \in \langle x, y\rangle$ *incomparable* with z. For finite P the notions coincide. Then $\mathcal{P}$ is said to be dense.
- If H is an antichain, then d_2–dense coinsides with d_2^*–dee and they are the same as "the antichain H is dense in $\mathcal{P}$".

Finally it is convenient to have the following notation:
For $H, G \subset P$ we write $H >\!\!|\!\!< G$ iff for all $h \in H$ and all $g \in G$ elements h and g are incomparable. For $s, s' \in P$ and $G \subset P$ we also write $s >\!\!|\!\!< s'$ instead of $\{s\} >\!\!|\!\!< \{s'\}$ and $s >\!\!|\!\!< G$ instead of $\{s\} >\!\!|\!\!< G$.
Similarly, we write

$$U(s) = U(\{s\}), U^*(s) = U^*(\{s\}), D(s) = D(\{s\}), D^*(s) = D^*(\{s\}). \quad (1.7)$$

REDUCTION OF GENERATORS TO ANTICHAINS

We begin with an auxiliary result.
Lemma 1 For any poset $\mathcal{P}$ let $C \subset P$ be a set such that every element $c \in C$ is comparable with at least one other element c' of C. Then

(i) there exists a $C_1 \subset C$ such that for $C_2 = C \smallsetminus C$, we have the properties: $\forall a \in C_1\ \exists b \in C_2$ such that $a > b$, $\forall b \in C_2\ \exists a \in C_1$ such that $b < a$.

(ii) there exists a $C_1 \subset C$ with $D(C) \cup U(C) = D(C_1) \cup U(C_2)$.

Proof: (i) Let $A \subset C$ be a maximal antichain in C. Its existence is guaranteed by Zorn's Lemma. By the maximality of the antichain A

$$C \subset D^*(A) \cup U^*(A) \cup A.$$

We write A in the form

$$A = A_{\max} \cup A_{\min} \cup A_0,$$

where

$$A_{\max} = \{a \in A : \nexists c \in C \text{ with } c > a\}, A_{\min} = \{a \in A : \nexists c \in C \text{ with } c < a\},$$
$$A_0 = A \smallsetminus (A_{\max} \cup A_{\min}).$$

By our assumption on C $A_{\max} \cap A_{\min} = \phi$ and also one of the sets $D^*(A)$ and $U^*(A)$ is not empty. W.l.o.g. we can assume that $D^*(A) \neq \phi$ and consider the sets

$$C_1 = \big(A_{\max} \cup U^*(A) \cup A_0\big) \cap C, \quad (2.1)$$

$$C_2 = \big(A_{\min} \cup D^*(A)\big) \cap C, \quad (2.2)$$

which clearly satisfy $C_2 = C \smallsetminus C_1$.
One also readily verifies that they can serve as sets whose existence is claimed in (i) and (ii).
Let now $G \subset P$ be a generator of $\mathcal{P}$. Partition it into $G = G_1 \dot{\cup} G_2$, where

$$G_1 = \{g \in G : \exists g' \in G, g' \neq g \text{ and } g >\!\!|\!\!< g'\}, \quad (2.3)$$

and $G_2 = G \setminus G_1$. Obviously G_2 is an antichain in $\mathcal{P}$.
We consider the poset $\mathcal{P}' = (P', <)$, where

$$P' = P \setminus \big(D(G_1) \cup U(G_1)\big). \tag{2.4}$$

Since G is a generator of $\mathcal{P}$, G_2 is a generator and hence maximal antichain in $\mathcal{P}'$. This and Lemma 1 yield the following result on reduction.
Proposition 1. Let $G \subset P$ be a generator of $\mathcal{P}$ and let G_1, G_2 be defined as above, $G_1 \dot\cup G_2 = G$. Now G has the splitting property in $\mathcal{P}$ iff the maximal antichain G_2 in $\mathcal{P}'$ has the splitting property in $\mathcal{P}'$.
The next and last result on reduction is readily verified.
Proposition 2. Let G be any d_1–dense (resp. d_2-dense) subset of P (not necessarily a generator) and define G_1, G_2 and $\mathcal{P}'$ as in (2.3) and (2.4). Then G_2 is d_1–dense (resp. d_2–dense) in the poset $\mathcal{P}'$.

SPLITTING OF D_1–DENSE ANTICHAINS

Under the weakest of our density assumptions and further regularity conditions we present next a splitting result for not necessarily finite posets.

Theorem 1 Let $\mathcal{P}$ be a poset and let $S \subset P$ be a maximal antichain, which is d_1–dense.
Additionally, we assume that

(i) in $D^*(S)$ exists an antichain $\underline{S}$ with $D(\underline{S}) = D^*(S)$

(ii) in $U^*(S)$ exists an antichain $\overline{S}$ with $U(\overline{S}) = U^*(S)$

(iii) S carries a well–ordering μ with the property: for all $u \in \overline{S}$ the set $A(u) = \{s \in S : s < u\}$ has a maximal element according to μ.

Then S has the splitting property.

Proof: For every $d \in \underline{S}$ we consider the set

$$B(d) = \{s \in S : d < s\}. \tag{3.1}$$

Let $f(d)$ be its minimal element according to μ. We consider $S_1 = \bigcup_{d \in \underline{S}} \{f(d)\}$ and prove that it gives the desired splitting. Since S is a maximal antichain, of course

$$D(S) \cup U(S) = P.$$

From condition (i) and the construction of S_1 we get

$$D(S_1) = D(S) \setminus (S \setminus S_1).$$

It remains to prove that

$$U(S \setminus S_1) = U(S) \setminus S_1.$$

By condition (ii) for this it suffices to show that

$$\overline{S} \subset U(S \setminus S_1).$$

Suppose then, to the opposite, that for some $u \in \overline{S}$ we have $u \notin U(S \setminus S_1)$. We consider the set

$$A(u) = \{s \in S : s < u\}. \tag{3.2}$$

Since $u \notin U(S \setminus S_1)$, necessarily $A(u) \subset S_1$. Let $s_0 \in A(u)$ be according to μ the maximal element of $A(u)$, which exists by (iii). From the construction of S_1 it follows that $s_0 = f(d_0)$ for some $d_0 \in \underline{S}$.
We consider now the open interval $\langle d_0, u\rangle$, which contains $s_0 \in S$. Since S is d_1–dense there is a $t \in P$ with $t \neq s_0$ and $d_0 < t < u$.
Furthermore, since $d_0 \in \underline{S}$ and by (i) $\underline{S}$ is antichain with $D(\underline{S}) = D^*(S)$, we know that $t \notin D^*(S)$. Symmetrically, by (ii), $t \notin U^*(S)$, and hence $t \in S$. Now we have $t \in A(u)$, since $t < u$, and $t \in B(d_0)$, since $d_0 < t$. However, s_0 is the maximal element of $A(u)$ in the well–ordering μ. Hence, s_0 is not the minimal element in $B(d_0)$ according to μ. Therefore, $s_0 \neq f(d_0)$, which is a contradiction.

Corollary 1 Let S be a maximal antichain in a finite poset $\mathcal{P}$. If S is d_1–dense in $\mathcal{P}$, then S has the splitting property.
Remark 3: Theorem 2.1 of [1] is a special case of this Corollary and also Theorem 3.1 of [1] easily follows. Actually in case of finite posets the proof above closely resembles the second proof of [1].
An instructive infinite poset is $\mathcal{Z} = (Z, <)$, where Z is the set of 0–1–sequences and for two sequences $a = (a_1, a_2, \dots), b = (b_1, b_2, \dots) \in Z$ $a \leq b$ exactly if $a_i \leq b_i$ for all $i = 1, 2, \dots$. Clearly, any subset $H \subset Z$ is d_1–dense.

Corollary 2 Let $S \subset Z$ be a maximal antichain, whose members have at most k ones. Then S has the splitting property.

Proof: The maximal elements in $D^*(S)$ form an antichain $\underline{S}$ and the minimal elements in $U^*(S)$ form an antichain $\overline{S}$. They guarantee (i) and (ii). Since for $u \in \overline{S}$ $A(u)$ is finite, also (iii) holds.

THE LATTICE OF SQUARE–FREE NUMBERS DOES NOT HAVE THE SPLITTING PROPERTY

Let $Z^* \subset Z = \{0,1\}^\infty$ be the set of all 0–1–sequences with finitely many ones. Those sequences can be identified with the sequences of exponents in the prime number representation of square–free numbers $\mathbb{N}^*$. The order relation in Z, and thus in Z^* says in terms of $\mathbb{N}^*$: for $a, b \in \mathbb{N}^*$ $a \leq b$ iff $a \mid b$ (a divides b). According to this relation the upset of $H \subset \mathbb{N}^*$ is the set of multiples of H

$$M(H) = \{n \in \mathbb{N}^* : \ell | n \text{ for some } \ \ell \in H\} \tag{4.1}$$

and the downset is the set of divisors of H

$$D(H) = \{n \in \mathbb{N}^* : n|\ell \text{ for some } \ \ell \in H\}. \tag{4.2}$$

Theorem 2 The poset of square–free numbers does not have the splitting property.

Remark 4: $\mathbb{N}^*$ is a countable and strongly dense poset. Therefore Theorem 2 refutes Conjecture 4.4 of [1].

Proof of Theorem 2: We construct a maximal antichain S without the splitting property as follows:
We choose an arbitrary $T_1 \in \mathbb{N}$ and consider the set

$$A_1 = \{n \in \mathbb{N}^* : T_1 < n \leq 2\, T_1\}.$$

Next we choosse any $T_2, T_2 > 8\, T_1^2$, and define the set

$$A_2 = \big\{n \in \mathbb{N}^* : n \in (T_2, 2\, T_2] \smallsetminus M(A_1)\big\}.$$

Inductively, for every $k > 1$ we choose $T_k, T_k > 8\, T_{k-1}^2$, and define the set

$$A_k = \left\{ n \in \mathbb{N}^* : n \in (T_k, 2T_k] \smallsetminus M\left(\bigcup_{i=1}^{k-1} A_i\right)\right\}.$$

Finally we define

$$S = \bigcup_{i=1}^{\infty} A_i. \tag{4.3}$$

- Clearly, numbers in A_i are incomparable and $a \in A_i$, $b \in A_j$ $(i < j)$ are incomparable, because we have excluded the multiples of A_i in the definition of A_j and $b > a$. Thus S is an antichain (also called primitive sequence in Number Theory).

- We show next that S is maximal, that is, $\mathbb{N}^* = M(S) \cup D(S)$. If this is not the case, then an $\alpha \in \mathbb{N}^*$ with $\alpha \notin M(S) \cup D(S)$ and, particularly, $\alpha \notin (T_i, 2\, T_i]$ for $i = 1, 2, \ldots$ exists. Hence $2\, T_k < \alpha \leq T_{k+1}$ for some $k \in \mathbb{N}$ or $2 \leq \alpha \leq T_1$. It follows from Bertrand's postulate that there exists a prime $p \in \mathbb{P}$ (the set of all primes) such that

  $$\frac{T_{k+2}}{\alpha} < p \leq 2\frac{T_{k+2}}{\alpha} \text{ or, equivalently, } \ T_{k+2} < \alpha \cdot p \leq 2 \cdot T_{k+2}.$$

  Since $T_{k+2} > 8\, T_{k+1}^2$ and $2\, T_k < \alpha \leq T_{k+1}$, we conclude that

  $$2\, T_{k+1} < p < T_{k+2}.$$

  Hence $p > \alpha$ and $\alpha \cdot p \in \mathbb{N}^*$.

Now, if $\alpha \cdot p \in M(S)$ or (equivalently) $\alpha'|\alpha \cdot p$ for some $\alpha' \in S$ ($\alpha' \leq 2\ T_{k+1}$), then, since $p \in \mathbb{P}$ and $p > 2\ T_{k+1}$ we have $\alpha'|\alpha$ and hence $\alpha \in M(S)$, a contradiction.

On the other hand, if $\alpha\ p \notin M(S) \setminus S$ then the conditions $T_{k+2} < \alpha\ p \leq 2\ T_{k+2}$, $\alpha\ p \in I\!N^*$ yield $\alpha\ p \in S$. But then $\alpha \in D(S)$, again a contradiction.

- Finally we show that the maximal antichain S does not have the splitting property.

  Let us assume to the opposite that for some $S_1 \subset S$

$$D(S_1) \cup M(S \setminus S_1) = I\!N^*.$$

Necessarily $S_1 \neq \phi$, because for example all squarefree integers from $[1, T_1)$ and all primes from $(2\ T_k, T_{k+1}]$, $k \in I\!N$, are not in $M(S)$.

Let then $\beta \in S_1$ and $T_k < \beta \leq 2\ T_k$ for some $k \in \mathbb{N}$. From Bertrand's postulate we know that there is a prime q with $2\ T_k < q \leq 4\ T_k$. Consider the integer $\beta \cdot q$. Obviously $\beta \cdot q \in I\!N^*$ and since $T_{k+1} > 8\ T_k^2$ we have

$$2\ T_k < \beta \cdot q < T_{k+1}.$$

Clearly, $\beta \cdot q \notin D(S)$, because S is an antichain and $\beta \in S$.

On the other hand $\beta \cdot q \in M(S \setminus S_1)$ would imply $\beta'|\beta q$ for some $\beta' \in S \setminus S_1$ and then $\beta' \leq 2\ T_k$, because $\beta \cdot q < T_{k+1}$, and hence $\beta'|\beta$, because $2\ T_k < q$. But then β', β are in the antichain S and at the same time comparable. This contradiction implies that for the integer $\beta \cdot q \in \mathbb{N}^*$

$$\beta q \notin D(S_1) \cup M(S \setminus S_1).$$

ON THE SPLITTING RATIO OF MAXIMAL ANTICHAINS IN THE BOOLEAN POSET $\mathcal{L}^N = \{0,1\}^N$

To fix ideas, let us consider the maximal antichain $S = \binom{[n]}{\ell}$ in $\mathcal{L}^n$. For a splitting $S = S_1 \dot\cup S_2$ necessarily $D(S_1) \supset \binom{[n]}{\ell-1}$ and $U(S_2) \supset \binom{[n]}{\ell+1}$, $1 \leq \ell \leq n-1$, and therefore

$$\begin{aligned} \frac{\ell}{\ell+1}\binom{n}{\ell} &\geq |S_1| \geq \tfrac{1}{\ell}\tbinom{n}{\ell-1} = \tfrac{1}{n-\ell+1}\tbinom{n}{\ell}, \\ \frac{n-\ell}{n-\ell+1}\binom{n}{\ell} &\geq |S_2| \geq \tfrac{1}{n-\ell}\tbinom{n}{\ell+1} = \tfrac{1}{\ell+1}\tbinom{n}{\ell}. \end{aligned}$$

Thus

$$\ell \geq \frac{|S_1|}{|S_2|} \geq \frac{1}{n-\ell} \tag{5.1}$$

or $\max\left(\frac{|S_1|}{|S_2|}, \frac{|S_2|}{|S_1|}\right) \leq \max(\ell, n-\ell) \leq n$.
So the ratio of the cardinalities is at most linear in n. However, we construct antichains whose splitting ratios $\rho(n) = \min\left\{\frac{|S_2|}{|S_1|} : \{S_1, S_2\} \text{ is a splitting of } \mathcal{L}^n\right\}$ satisfy for large n

$$\rho(n) \geq 2^{\varepsilon n} \text{ for some constant } \varepsilon. \tag{5.2}$$

Construction: For a $k \in I\!N$, $2|k$, let $L = L_k \subset \binom{[k]}{\frac{k}{2}}$ be a code with minimal Hamming distance ≥ 4 and with a maximal number of codewords. We consider the poset $\mathcal{P}_k = \{0,1\}^k \smallsetminus U(L)$ and define $E = E_k$ as the set of all maximal elements in $\mathcal{P}_k$. Every element of E has at least $\frac{k}{2}$ ones.
For $n = k \cdot r \in I\!N$ partition $[n]$ into r blocks $R_1, R_2, \dots, R_r$ each of cardinality k.
We denote by I_t, $1 \leq t \leq r$, the 0–1–sequence of length n, which has ones exactly in the positions from block R_t. For any $\ell \in L$, $e \in E$ and $t, 1 \leq t \leq r$, we denote by ℓ_t and q_t the 0–1–sequences of length n, which have zeros in the blocks R_i, $i \neq t$, and ℓ resp. e in the block R_t.
Define $L_t^* = \{\ell_t : \ell \in L\}$ and $E_t^* = \{e_t : e \in E\}$. We consider now $S = A \cup B \subset \{0,1\}^n$, where
$A = \{a \in \{0,1\}^n : a \wedge I_t \in L_t^* \text{ for all } 1 \leq t \leq r\}$
and
$B = \{b \in \{0,1\}^n : \exists t \in \{1, \dots, r\} \text{ with } b \wedge I_t \in E_t^* \text{ and } b \wedge T_{t'} = I_{t'} \text{ for } t' \neq t\}$.
One can verify that S is a maximal antichain and by Corollary 2 possesses the splitting property.
We observe that $A \subset \binom{[n]}{\frac{n}{2}}$ and consider the set

$$X = U(A) \cap \binom{[n]}{\frac{n}{2}+1}.$$

It satisfies $X \cap D(B) = \phi$, because S is antichain and for any $x \in X$ there exists exactly one $a \in A$ with $a < x$, since $a_1, a_2 \in A$ implies $d_H(a_1, a_2) \geq 4$. Hence, for every splitting $S = S_1 \dot\cup S_2$, $D(S_1) \cup U(S_2) = \{0,1\}^n$ we always have $A \subset S_2$.
Therefore, using a familiar lower bound on $|L|$,

$$|S_2| \geq |A| = |L|^{\frac{n}{k}} > \left(\frac{c\binom{k}{\frac{k}{2}}}{k^2}\right)^{\frac{n}{k}}$$

and

$$|S_1| \leq |B| = \frac{n}{k} \cdot |E| < \frac{n}{k} \cdot 2^k.$$

Now $\frac{|S_2|}{|S_1|} \geq 2^{\varepsilon(c)n}$ for large n, if we choose $k \sim \sqrt{n}$.

THE SET–THEORETICAL FORMULATION OF THE SPLITTING PROPERTY, D_2–DENSENESS

Let $\mathcal{P}$ be a poset and let $S \subset P$ be a maximal antichain in $\mathcal{P}$. Consider the families of sets $\mathcal{A}, \mathcal{B} \subset 2^S$ defined by

$$\mathcal{A} = \{A(u) : u \in \mathcal{U}^*(S)\},\ \mathcal{B} = \{B(d) : d \in D^*(S)\}. \tag{6.1}$$

Here we use again the definitions (3.1) and (3.2) for $A(u)$ and $B(d)$.
The splitting property of S can equivalently be written in the set–theoretic formulation: There exists a partition of S; $S = S_1 \dot\cup S_2$; such that

$$S_1 \cap A \neq \phi \text{ for all } A \in \mathcal{A} \text{ and } S_2 \cap B \neq \phi \text{ for all } B \in \mathcal{B}. \tag{6.2}$$

We can forget now how $\mathcal{A}, \mathcal{B}$ originated in (6.1) from $(\mathcal{P}, S)$ and can consider abstractly any set S and two families $\mathcal{A}, \mathcal{B}$ of subsets of S and ask whether they have the splitting property (6.2).
Of course any abstract system $(S, \mathcal{A}, \mathcal{B})$ can be viewed as coming via (6.1) from a suitable poset. The new language creates new associations. For instance in [2] for any set system $\mathcal{M} \subset 2^S$ a so called *B–property* was introduced, which means that S has a partition $S = S_1 \dot\cup S_2$ with

$$H \cap S_1 \neq \phi \text{ and } H \cap S_2 \neq \phi \text{ for all } H \in \mathcal{M}. \tag{6.3}$$

Obviously, if $\mathcal{M} = \mathcal{A} \cup \mathcal{B}$ has the B–property, then S possesses the splitting property with respect to $\mathcal{A}, \mathcal{B}$, but the converse is not always true.
In the following special situation it is easy to establish the B–property.

Proposition 3. Let S be an infinite set and let $\mathcal{M} \subset 2^S$ be countable, $\mathcal{M} = \{H_1, H_2, \dots,\}$; and let every $H_i \in \mathcal{M}$ be infinite. Then $\mathcal{M}$ has the B–property.

Proof: Since $|H_i| = \infty$ for $i = 1, 2, \dots$, we can sequentially choose two different elements $h_i, g_i \in H_i$ for $i = 1, 2, \dots$ such that $h_i \neq h_j$, $h_i \neq g_j$, $g_i \neq g_j$ $(i \neq j)$. Now we define

$$S_1 = \{h_1, h_2, \dots\} \text{ and } S_2 = S \setminus S_1.$$

Here we consider for the first time the property d_2–dense for a maximal antichain $S \subset \mathcal{P}$. We study it right away in the new setting. The set S is d_2–dense for the set systems $\mathcal{A}, \mathcal{B} \subset 2^S$, if for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$ necessarily

$$|A \cap B| \neq 1. \tag{6.4}$$

We also say that $\mathcal{A}, \mathcal{B}$ have property d_2.

Theorem 3 Let $\mathcal{A}, \mathcal{B} \subset 2^S$ have property d_2, let $\phi \notin \mathcal{A} \cup \mathcal{B}$ and let both, $\mathcal{A}$ and $\mathcal{B}$, be countable. Then S has the splitting property for $(\mathcal{A}, \mathcal{B})$.

Proof: First note that this theorem is not a consequence of Proposition 3, where we require all members of $\mathcal{A}$ and $\mathcal{B}$ to be infinite.

Let now $\mathcal{A} = \{A_1, A_2, \ldots\}$, $\mathcal{B} = \{B_1, B_2, \ldots\}$ and by property d_2 $|A_i \cap B_j| \neq 1$ for all $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$. Then we can choose $a_1 \in A_1$ and $b_1 \in B_1$; $a_1 \neq b_1$. We remove all sets from $\mathcal{A}$ which contain a_1 and all sets from $\mathcal{B}$, which contain b_1. We remove also the element a_1 from every set in $\mathcal{B}$ and the element b_1 from every set in $\mathcal{A}$. We denote the remaining sets by $\mathcal{A}^1$ and $\mathcal{B}^1$. Now verify that $\phi \notin \mathcal{A}^1 \cup \mathcal{B}^1$ and $\mathcal{A}^1, \mathcal{B}^1$ have again property d_2!
We note also that the set system $\mathcal{A}^1$ (as well as $\mathcal{B}^1$) is ordered according to the ordering of $\mathcal{A}$, i.e. $\mathcal{A}^1 = \{A_1^1, A_2^1, \ldots\}$ $A_k^1 = A_m \smallsetminus \{a_1\}$ is followed by $A_t^1 = A_\ell \smallsetminus \{a_1\}$ for $k < t$ iff $m < \ell$.
Now we choose $a_2 \in A_1^1$, $b_2 \in B_1^1$, $a_2 \neq b_2$ and construct set systems $\mathcal{A}^2, \mathcal{B}^2$, etc. Continuation of this procedure leads to the subsets of $S : S_1 = \{a_1, a_2, \ldots\}$ and $S_2 = \{b_1, b_2, \ldots, \}$. They splitt $\mathcal{A}, \mathcal{B}$.
Next we show how important it is that in Theorem 3 both, $\mathcal{A}$ and $\mathcal{B}$, are countable.

Example 2: *(S countable, $\mathcal{A}, \mathcal{B} \subset 2^S$, $\phi \notin \mathcal{A} \cup \mathcal{B}$, $\mathcal{A}, \mathcal{B}$ have property d_2 (and even a stronger property), $\mathcal{A}$ is countable, $\mathcal{B}$ is non–countable, but S does not have the splitting property.)*
$S = I\!N$, $\mathcal{A} = \{A \subset I\!N : |A^c| < \infty\}$, where A^c is the complement of A, $\mathcal{B} = \{B \subset I\!N : |B| = \infty\}$. Clearly for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$|A \cap B| = \infty \text{ (stronger than } d_2).$$

Suppose that $S = S_1 \dot{\cup} S_2$ and that

$$S_1 \cap A \neq \phi \quad \forall\, A \in \mathcal{A} \text{ and } S_2 \cap B \neq \phi \quad \forall\, B \in \mathcal{B}. \tag{6.5}$$

In case $|S_1| < \infty$ we have $S_1^c \in \mathcal{A}$ and hence $S_1 \cap S_1^c = \phi$ violates the first relation in (6.5). In case $|S_1| = \infty$ we have $S_1 \in \mathcal{B}$ and hence $S_2 \cap S_1 = \phi$ violates the second relation.

SPLITTING OF SETS WITH PROPERTY D_2, MINIMAL REPRESENTATIVE SETS AND MINIMAL COVERINGS

The results of the last Section gave the motivation for introducing a further concept.
Let S be a set and $\mathcal{M} \subset 2^S$. The set $R \subset S$ is a *representative set* for $\mathcal{M}$, if

$$R \cap H \neq \phi \text{ for all } H \in \mathcal{M}. \tag{7.1}$$

A representative set for $\mathcal{M}$ $R \subset S$ is *minimal*, if no proper subset $R' \subset R$ is representative set for $\mathcal{M}$.

Theorem 4 For a set S and $\mathcal{A}, \mathcal{B} \subset 2^S$ with property d_2 and $\phi \notin \mathcal{A} \cup \mathcal{B}$ let also $\mathcal{A}$ (or $\mathcal{B}$) have a minimal representative set.
Then S has the splitting property.

Proof: We show that we can choose as S_1 in the partition of S the minimal representative set $R \subset S$ of $\mathcal{A}$.

Since by definition $R \cap A \neq \phi$ for all $A \in \mathcal{A}$ and it remains to be seen that there does not exist a $B_0 \in \mathcal{B}$ with $(S \setminus R) \cap B_0 = \phi$, or equivalently $B_0 \subset R$. Assume the opposite.
We choose an arbitrary $b \in B_0$ and consider the set $R' = R \setminus \{b\}$. Since R' is not representative for $\mathcal{A}$ there is an $A \in \mathcal{A}$ with $A \cap R \neq \phi$ and $A \cap R' = \phi$. Therefore $A \cap R = \{b\}$ and since $b \in B_0$, $B_0 \subset R$ we have $|A \cap B_0| = 1$. This contradicts d_2.

Remark 5: The existence of minimal representatives is not necessary for the splitting property.

Example 3: Let $S = \{s_1, s_2, s_3, \ldots\}$ be any infinite countable set and $\mathcal{A} = \mathcal{B} = \big\{S, S \setminus \{s_1\}, S \setminus \{s_1, s_2\}, \ldots\big\}$.
Since $|A \cap B| = \infty$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have property d_2. Neither $\mathcal{A}$ (nor $\mathcal{B}$) has a minimal representative. However, for every infinite $S_1 \subset S$, for which $S \setminus S_1$ is also infinite, we have a splitting of $\mathcal{A}$ and $\mathcal{B}$. Moreover, in this case the existence of a splitting follows from Proposition 3.
Minimal representative sets are related to minimal coverings:
The set $\mathcal{M} \subset 2^X$ is a *covering* of the set X, if $\bigcup_{H \in \mathcal{M}} = X$, and it is a *minimal covering* if no proper subset is a covering of X.
Now, let $S \subset \mathcal{P}$ be a maximal antichain in the poset $\mathcal{P}$. Recall the definitions of $\mathcal{U}^*(s)$ and $D^*(s)$ for $s \in S$ in Section 1 and consider the systems of sets

$$\mathcal{U} = \big\{U^*(s) : s \in S\big\}, \mathcal{D} = \big\{D^*(s) : s \in S\big\}.$$

Since $\bigcup_{s \in S} U^*(s) = \mathcal{U}^*(S)$ and $\bigcup_{s \in S} D^*(s) = D^*(S)$, the systems $\mathcal{U}$ and $\mathcal{D}$ are coverings of $U^*(S)$ and $D^*(S)$ resp.
The following statement is immediately proved by inspection.

Proposition 4. Let $S \subset \mathcal{P}$ be a maximal antichain in the poset $\mathcal{P}$ and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{U}$, and $\mathcal{D}$ be the associated set systems. Thus $\mathcal{A}$ (resp. $\mathcal{B}$) has a minimal representative set iff $\mathcal{U}$ (resp. $\mathcal{D}$) contains a minimal covering of $U^*(S)$ (resp. $D^*(S)$).
From here we get an equivalent formulation of Theorem 4.

Theorem 4' Let $S \subset \mathcal{P}$ be a maximal antichain in the poset $\mathcal{P}$ with property d_2 and let the associated set system $\mathcal{U}$ (resp. $\mathcal{D}$) have a minimal covering of $\mathcal{U}^*(S)$ (resp. $\mathcal{D}^*(S)$). Then S possesses the splitting property.

Klimo [2] has studied minimal coverings and proved the following result.

Theorem [2] Let $\mathcal{M} \subset 2^X$ be a covering of X.

(i) Suppose that there is a well–ordering μ of $\mathcal{M}$ with the property: for all $x \in X$ the sets $\{H \in \mathcal{M} : x \in H\}$ have a maximal element according to μ. Then $\mathcal{M}$ contains a minimal covering of X.

(ii) Suppose that for all $H \in \mathcal{M}$ $|H| \leq k$ for some $k \in \mathbb{N}$, then $\mathcal{M}$ contains a minimal covering of X.

Remark 6: As explained in [2], this Theorem implies that a point–finite covering $\mathcal{M}$ of X (i.e. $\forall\ x \in X\ |\{H \in \mathcal{M} : x \in H\}| < \infty$) contains a minimal covering of X.

From Theorems 4, 4', [2] and Proposition 4 we obtain

Corollary 3 Let S be a set, $\mathcal{A}, \mathcal{B} \subset 2^S$, $\phi \notin \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A}, \mathcal{B}$ have property d_2.

(i) Let μ be a well–ordering of S such that every $A \in \mathcal{A}$ has a maximal element according to μ. Then S has the splitting property.

(ii) Suppose that for some $k \in \mathbb{N}$ every element of S is contained in at most k sets from $\mathcal{A}$, then S has the splitting property.

Remark 7: An immediate consequence of this Corollary is, that for $\mathcal{A}, \mathcal{B}$ with property d_2 and all $A \in \mathcal{A}$ finite S has the splitting property.

NEW AND STRONGER SPLITTING PROPERTIES

We say that S, a maximal antichain in the poset $\mathcal{P}$, has a Y*–splitting*, if for some partition $S = S_1 \dot{\cup} S_2$

$$U^*(S_1) \cup D^*(S_1) = U^*(S) \cup D^*(S) \tag{8.1}$$

and

$$U^*(S_2) = U^*(S). \tag{8.2}$$

Symmetrically, we say that S has a λ*–splitting*, if for some partition $S = S_1 \dot{\cup} S_2$

$$D^*(S_2) = D^*(S) \tag{8.3}$$

and (8.1) holds.

Finally, S has an X*–splitting*, if for some partition $S = S_1 \dot{\cup} S_2$

$$U^*(S_1) \cup D^*(S_1) = U^*(S_2) \cup D^*(S_2) = U^*(S) \cup D^*(S). \tag{8.4}$$

Clearly, all these properties imply the familiar splitting property.

We begin their exploration with one of the basic posets, namely $\mathcal{Z} = \{0,1\}^\infty$. At first we analyse d_2–dense antichains S for this poset. For this we look for $b \in S$ at intervalls $\langle c, a \rangle$ with $b \in S \cap \langle c, a \rangle$ and

$$\begin{aligned} a &= b_1 b_2 \dots b_{i-1}\ 1\ b_{i+1} \dots b_{j-1}\ 1\ b_{j+1} \dots \\ b &= b_1 b_2 \dots b_{i-1}\ 1\ b_{i+1} \dots b_{j-1}\ 0\ b_{j+1} \dots \\ c &= b_1 b_2 \dots b_{i-1}\ 0\ b_{i+1} \dots b_{j-1}\ 0\ b_{j+1} \dots\ . \end{aligned}$$

Clearly $c \in D^*(S)$, $a \in U^*(S)$ and $c < b < a$. Since S is b_2–dense, we must have

$$b' = b_1 b_2 \dots b_{i-1}\ 0\ b_{i+1} \dots b_{j-1}\ 1\ b_{j+1} \dots \in S.$$

Thus property d_2 implies the

Exchange property: S is closed under exchanging any two positions in its elements.

So, if S contains an element $s = (s_1, s_2, \ldots)$ with finitely many, say k, ones, then necessarily

$$S = \binom{\mathbb{N}}{k}. \tag{8.5}$$

We know from Remark 7 that this S has the splitting property. Actually we can choose $S_1 = \big\{s = (s_1, s_2, \ldots) \in S : s_1 = 1\big\}$ and $S_2 = S \smallsetminus S_1$.
Next we consider $\mathcal{Z}^* \subset \mathcal{Z}$, the poset of all 0–1–sequences with finitely many ones, $\mathcal{O}^* \subset \mathcal{Z}$, the poset of all 0–1–sequences with finitely many zeros, and

$$\mathcal{P}_\infty = \mathcal{Z} \smallsetminus (\mathcal{Z}^* \cup \mathcal{O}^*) \tag{8.6}$$

the poset of all 0–1–sequences with infinitely many ones and infinitely many zeros.

Proposition 5. Every maximal antichain in $\mathcal{P}_\infty$ is uncountable.

Proof: Cantor's diagonal argument shows that countability is contradictory.

Theorem 5

(i) In the poset $\mathcal{Z}^*$ every maximal d_2–dense and non–trivial $\Big(S \neq \binom{\mathbb{N}}{0}\Big)$ antichain S has a λ–splitting.

(ii) In the poset $\mathcal{P}_\infty$ every maximal d_2–dense antichain S has an X–splitting.

Proof:
(i) We have already demonstrated that for some k $S = \binom{\mathbb{N}}{k}$.

Case k even:
We choose $S_1 = \Big\{a = (a_1, a_2, \ldots) \in \binom{\mathbb{N}}{k} : \sum_{i=1}^\infty i\ a_i \equiv 0 \mathbf{mod} 2\Big\}$. and $S_2 = S \smallsetminus S_1$. Verification of the λ–splitting:
For $b = (b_1, b_2, \ldots) \in \binom{\mathbb{N}}{k+1}$ either $\sum_{i=1}^\infty i\ b_i \equiv 1 \mathbf{mod} 2$ and then $b \in U^*(S_1)$, because for some odd i_0 $b_{i_0} = 1$ and its replacement by 0 produces an $a \in S_1$, or $\sum_{i=1}^\infty i\ b_i \equiv 0 \mathbf{mod} 2$ and then $b \in U^*(S_1)$, because $k+1$ being odd enforces $b_{i_0} = 1$ for some even i_0 and its replacement by 0 produces an $a \in S_1$. Similarly we show that $D^*(S_1) = D^*(S_2) = D^*(S)$.

Case k odd:
Define $\mathbb{N}_1 = \{n \in \mathbb{N} : 2 \nmid n\}$, $T = \binom{\mathbb{N}}{k}$ and let $T = T_1 \dot\cup T_2$ be a splitting (guaranteed by Corollary 2) of $\mathcal{Z}_1^*$, the poset of all 0–1–sequences with finitely many ones in the positions $\mathbb{N}_1$ and zeros in the positions $\mathbb{N} \smallsetminus \mathbb{N}_1$.
Now we take

$$L_1 = S_1 \cup T_1 \text{ and } L_2 = \binom{\mathbb{N}}{k} \smallsetminus L_1$$

and again verify the λ–splitting.

(ii) Let $S \subset \mathcal{P}_\infty$ be a maximal and d_2–dense antichain. We have to show that there is a partition $S = S^1 \dot\cup S^2$ with

$$U^*(S) \cup D^*(S) = U^*(S^1) \cup D^*(S^1) = U^*(S^2) \cup D^*(S^2). \tag{8.7}$$

By the *exchange property* S is uniquely partitioned into equivalence classes $\{S_i\}_{i\in I}$ such that every class $S_i(i \in I)$ consists of those elements of S which can be obtained from each other by finitely many exchanges.
Clearly, $S_i(i \in I)$ is countable and hence by Proposition 5 the set of indices I must be uncountable.
Now we consider the sets

$$\begin{aligned}\overline{S}_i \quad &= \big\{a = (a_1, a_2, \dots) \in \mathcal{P}_\infty : \exists\ s = (s_1, s_2, \dots) \in S_i \text{ with } \ s_\ell = 0, a_\ell = 1 \\ &\qquad \text{for some } \ \ell \in I\!N \text{ and } \ a_j = s_j \ \text{ for } \ j \neq \ell\big\}\end{aligned}$$

and

$$\begin{aligned}\underline{S}_i \quad &= \big\{a = (a_1, a_2, \dots) \in \mathcal{P}_\infty : \exists\ s = (s_1, s_2, \dots) \in S_i \text{ with } \ s_\ell = 1, a_\ell = 0 \\ &\qquad \text{for some } \ \ell \in I\!N \text{ and } \ a_j = s_j \ \text{ for } \ j \neq \ell\big\}.\end{aligned}$$

Let $\overline{S}$ and $\underline{S}$ be the "parallel levels" of S, that is, $\overline{S} = \bigcup_{i\in I} \overline{S}_i$ and $\underline{S} = \bigcup_{i\in I} \underline{S}_i$.
It is clear that a partition $S = S^1 \dot\cup S^2$ satisfies (8.7) exactly if

$$\overline{S} \cup \underline{S} \subset U(S^1) \cup D(S^1) \text{ and } \ \overline{S} \cup \underline{S} \subset U(S^2) \cup D(S^2). \tag{8.8}$$

We observe that $\overline{S}$ and $\underline{S}$ are maximal antichains in $\mathcal{P}_\infty$ and their equivalence classes are $\{\overline{S}_i\}_{i\in I}$ and $\{\underline{S}_i\}_{i\in I}$ resp.
Moreover, for $u \in \overline{S}_i$ and $d \in \underline{S}_i$ the sets $A(u) = \{s \in S : s < u\}$ and $B(d) = \{s \in S : s > d\}$ are contained in S_i. For every $i \in I$ we consider now the systems of sets

$$\mathcal{A}_i = \big\{A(u) : u \in \overline{S}_i\big\}, \mathcal{B}_i = \big\{B(d) : d \in \underline{S}_i\big\}, \text{ and } \ \mathcal{M}_i = \mathcal{A}_i \cup \mathcal{B}_i.$$

We observe that $\mathcal{M}_i \subset 2^{S_i}$, $\mathcal{M}_i$ is countable and every subset of $\mathcal{M}_i$ is infinite.
By Proposition 3 $\mathcal{M}_i$ has property B. This is equivalent to the following: there exists a partition $S_i = S_i^1 \cup S_i^2$ such that $\overline{S}_i \cup \underline{S}_i \subset U^*(S_i^1) \cup D^*(S_i^1)$ and $\overline{S}_i \cup \underline{S}_i \subset U^*(S_i^2) \cup D^*(S_i^2)$. Finally we choose

$$S^1 = \bigcup_{i\in I} S_i^1 \text{ and } \ S^2 = \bigcup_{i\in I} S_i^2.$$

In conclusion we return to our best friend, the Boolean poset $\{0,1\}^n$. Under an exchange property its maximal antichains are of the form $S = \binom{[n]}{k}$.

Theorem 6 If there exists a partition $S = S_1 \dot\cup S_2$ for $S = \binom{[n]}{k} \subset \{0,1\}^n$ such that

$$U^*(S_1) = U^*(S_2) = U^*(S),$$

then S has a Y–splitting.
Proof: We consider the set of partitions

$$\mathcal{V}(S) = \big\{(S_1,S_2) : S_1 \dot\cup S_2 = S, U(S_1^*) = U(S_2^*) = U^*(S)\big\}.$$

Let $(S_1', S_2') \in \mathcal{V}(S)$ be extremal in the sense that $S_1' \subset S_1$, $S_1' \neq S_1$ implies $(S_1, S \setminus S_1) \notin \mathcal{V}(S)$. It suffices to show that $D^*(S_1') = D^*(S)$.
Suppose, in the opposite, that there exists an $\alpha \in \binom{[n]}{k-1}$ with $\alpha \notin D^*(S_1')$.
Hence, the elements $\beta_1, \beta_2, \ldots, \beta_{n-k+1} \in \binom{[n]}{k}$ with $\beta_i > \alpha$ are from the set S_2'. But then $\big(S_1' \cup \{\beta_1\}, S_2' \setminus \{\beta_1\}\big) \in \mathcal{V}(S)$, because $\gamma > \beta_1$ implies also $\gamma > \beta_i$ for some $i \geq 2$.

SPLITTING PROPERTIES FOR DIRECTED GRAPHS

We consider directed graphs $\mathcal{G} = (V, \mathcal{E})$ with multiple edges, that is, both edges, (v_1, v_2) and (v_2, v_1) can be in $\mathcal{E}$.
They can be viewed as generalizations of posets, because with every poset $\mathcal{P} = (P, <_p)$ we can associate a graph $G(\mathcal{P}) = \big(P, \mathcal{E}(<_p)\big)$ as follows:

$$\text{For } \; v_1, v_2 \in P \qquad (v_1, v_2) \in \mathcal{E}(<_p) \Leftrightarrow v_1 <_p \; v_2. \tag{9.1}$$

In such a graph there are no directed cycles, so the class of directed graphs is wider than the class of posets.
If S is an antichain in $\mathcal{P}$, then for $s_1, s_2 \in S$

(a) there is no edge in $G(\mathcal{P})$ between s_1 and s_2

(b) there is no directed path in $G(\mathcal{P})$ from s_1 to s_2.

For $G(\mathcal{P})$ properties (a) and (b) are the same. However, for general graphs they are different. If for a set $S \subset V$ (a) holds, then we call S an antichain, and if (the stronger) (b) holds, we call S a pathwise or (shortly) p–antichain.
We extend now the notion of a *dense poset* in the sense of [1], discussed in Section 1, to graphs. We use abbreviations like $a \rightsquigarrow b$ (resp. $a \not\rightsquigarrow b$), if there is (resp. is not) a directed path from a to b.
We say that $G = (V, \mathcal{E})$ *is* p–dense, if for every directed path $[a_1, a_2, \ldots, a_t]$ of length $t-1 \geq 2$ and every a_i $(2 \leq i \leq t-1)$ there exists a directed path $a_t \rightsquigarrow a_i$, a directed path $a_i \rightsquigarrow a_1$ or there exists a b_i on a directed path from a_1 to a_t and p–independent of a_i.
All notions of splitting in the previous Section 8 can be extended. However, we consider here only the original concept of [1].
Let S be a maximal p–antichain, then S possesses a *p–splitting of* $\mathcal{G}$, if there is a partition $S = S_1 \dot\cup S_2$ with

$$U(S_1) \cup D(S_2) = V,$$

where

$$U(S_1) = \{v \in V : \exists\; s \rightsquigarrow v \text{ for some } \; s \in S\},$$
$$D(S_2) = \{v \in V : \exists\; v \rightsquigarrow s \text{ for some } \; s \in S\}.$$

Here is our generalization of the main result in [1].

Theorem 7 Let $\mathcal{G}$ be a finite p–dense, directed graph, then every maximal p–antichain S in $\mathcal{G}$ possesses a splitting of $\mathcal{G}$.
Sketch of proof:
We follow the idea of the first proof of Theorem 3.1 in [1], which is by induction on $|V|$.
If $s \in S$ is needed for "up" to u and for "down" to d, then for the chain $d \rightsquigarrow s \rightsquigarrow u$ by p–denseness either we find a chain $u \rightsquigarrow d$ and we have a contradiction, because d can be attained in $U(S)$ (does not use full strength of (c)!), or by (d) there is a v with $d \rightsquigarrow v \rightsquigarrow u$ and $s \not\rightsquigarrow v$, $v \not\rightsquigarrow s$.
In this case independence of s from S would contradict maximality of S, so we have either for some $s_1 \in S$ $s_1 \rightsquigarrow v$ or for some $s_2 \in S$ $v \rightsquigarrow s_2$.
Therefore either $s_1 \rightsquigarrow u$ or $d \rightsquigarrow s_2$ and in any case a contradiction to the definition of s.
It remains to discuss the case where some $U(s)$ (or $D(s)$) is removed from the graph. As in [1] we show by inspection that the induced graph on $V \setminus U(s)$ is p–dense.
Remark 8: It is interesting to analyse number–theoretic examples such as $G = (V, \mathcal{E})$, where $V \subset I\!N$ and for $m, n \in V$ $(m,n) \in \mathcal{E}$ iff g.c.d $\{m,n\} = 1$ and $m < n$.
We thank Peter Erdös for proposing the study of splitting properties in infinite posets.

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