Maximal Antichains Under Dimension Constraints

R. Ahlswede, H. Aydinian, and L.H. Khachatrian

University of Bielefeld, Department of Mathematics, P.O. Box 100131, D-33501 Bielefeld, Germany

Abstract

We consider $A(n,k) = \max_{\mathcal{A}}\{|\mathcal{A}| : \dim(\mathcal{A}) \le k,\ \mathcal{A} \subset \{0,1\}^n \text{ is an antichain}\}$, where the dimension is taken from the linear span of $\mathcal{A}$ in $\mathbb{R}^n$, we conjecture the exact value of $A(n,k)$ and we prove this conjecture for all n and $k \le \left\lfloor \frac{n}{2} \right\rfloor + 1$ or $k = n-1$.

This is a contribution to the program of systematic investigation of extremal problems under dimension constraints, which was recently presented by the authors.

Key words: antichain, LYM property, dimension constraint, 0–1 matrices.

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1 Introduction

For $i, j \in \mathbb{N}$, $i < j$ the set $\{i, i+1, \ldots, j\}$ is denoted by $[i,j]$ and $[n]$ stands for $[1,n]$. We also use the notation $2^{[n]} = \Big\{F : F \subset [n]\Big\}$, $E(n) = \{0,1\}^n$, $\binom{n}{w} = \{F \in 2^{[n]} : |F| = w\}$, and $E(n,w) = \{x^n \in E(n) : x^n \text{ has } w \text{ ones}\}$.

In our paper [1] we solved a seemingly basic geometrical extremal problem. For the set $E(n,w)$ of vertices of weight w in the unit cube of $\mathbb{R}^n$ we determined $M(n,k,w) \triangleq \max\Big\{|U \cap E(n,w)| : U \text{ is a } k\text{–dimensional subspace of } \mathbb{R}^n\Big\}$.

Theorem AAK.

(a) $M(n,k,w) = M(n,k,n-w)$

(b) For $w \le \frac{n}{2}$ we have

$$M(n,k,w) = \begin{cases} \binom{k}{w}, & \text{if (i) } 2w \le k \\ \binom{2(k-w)}{k-w} 2^{2w-k}, & \text{if (ii) } k \le 2w < 2(k-1) \\ 2^{k-1}, & \text{if (iii) } k-1 \le w \end{cases}$$

The key sets giving the values of $M(n,k,w)$ in these three cases are

(i) $S_1 = E(k,w) \times \{0\}^{n-k}$
(ii) $S_2 = E\Big(2(k-w), k-w\Big) \times \{10,01\}^{2w-k} \times \{0\}^{n-2w}$
(iii) $S_3 = \{10,01\}^{k-1} \times \{1\}^{w-k+1} \times \{0\}^{n-k-w+1}$.

We note that this result is valid for any field of characteristic zero. However the problem is open for the vector spaces over finite fields (except for some partial cases stated in [1]).

This work can be viewed as the beginning of a very challenging program of research in extremal combinatorial theory, which recently has been described in [2]. (Already now it has led to new problems, new connections between problems, new proof methods, good hope for applications.)

We reconsider the basic combinatorial structures such as antichains, intersecting systems etc., in the light of what we call "dimension constraints". Here we adress antichains. $\mathcal{F} \subset 2^{[n]}$ is called an antichain if $F_1 \not\subset F_2$ holds for all $F_1, F_2 \in \mathcal{F}$. Correspondingly $\mathcal{F} = \{F_1, \dots, F_s\}$ is called a chain of size s if $F_1 \subset \dots \subset F_s$.

The corresponding notions are exdended to $(0,1)$–vectors in a natural way.

We ask now for the maximal size $A(n,k) \triangleq \max_{\mathcal{A}}\{|\mathcal{A}| : \dim(\mathcal{A}) \le k,\ \mathcal{A} \subset E(n)$ is an antichain$\}$.

It would be interesting to have also LYM–type inequalities (see e.g. [3]).

Conjecture:

$$A(n,k) = M\left(n, k, \left\lfloor \frac{n}{2} \right\rfloor\right).$$

Here are our partial results.

Theorem.

(i) $A(n,n-1) = M\left(n, n-1, \left\lfloor \frac{n}{2} \right\rfloor\right) = \begin{cases} 2\binom{n-2}{\frac{n-2}{2}}, & \text{if } 2 \mid n \\ \binom{n-1}{\frac{n-1}{2}}, & \text{if } 2 \nmid n. \end{cases}$

(ii) For $n \ge 2k-2$

$$A(n,k) = M\left(n,k,\left\lfloor \tfrac{n}{2} \right\rfloor\right) = 2^{k-1}.$$

2 Proof of (i)

Let $\mathcal{A}$ be an antichain with $\dim(\mathcal{A}) = n-1$, and let $\operatorname{span}(\mathcal{A}) \triangleq \mathcal{U}$ be defined by

$$\mathcal{U} = \left\{ (x_1,\ldots,x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} b_i x_i = 0 \right\},$$

for some real $b_1,\ldots,b_n$.

Thus $\mathcal{A}$ is an antichain chosen from the set of $(0,1)$–solutions to the equation

$$\sum_{i=1}^{n} b_i x_i = 0. \tag{2.1}$$

W.l.o.g. we may assume that $b_1,\ldots,b_\ell > 0$ and $b_{\ell+1},\ldots,b_n \le 0$ for some $1 \le \ell \le n-1$.

Partition $[n]$ into two parts $[n] = [1,\ell] \cup [\ell+1,n]$.

Think now about elements of $\mathcal{A}$ as elements of $2^{[n]}$ avoiding a new notation and represent each $E \in \mathcal{A}$ by a pair (E_1,E_2), where $E_1 = E \cap [1,\ell]$, $E_2 = E \cap [\ell+1,n]$.

Then it easily follows from (2.1) that any two elements (E_1,E_2) and (F_1,F_2) of $\mathcal{A}$ have the following property:

(Q) If E_1 and F_1 form a chain then E_2 and F_2 form an antichain.

An element $(E_1,E_2) \in \mathcal{A}$ is called (i,j)–configuration if $|E_1| = i$, $|E_2| = j$. Note that $j \neq 0$ otherwise the $(0,1)$–vector corresponding to (E_1,E_2) (the characteristic vector) does not satisfy (2.1). Denote by α_{ij} the number of (i,j) configurations in $\mathcal{A}$. Clearly

$$\sum_{(i,j)\in I} \alpha_{ij} = |\mathcal{A}|, \tag{2.2}$$

where $I \subset \{0,1,\ldots,\ell\} \times [n-\ell]$ is the set of different kinds of configurations in $\mathcal{A}$.

Recall further the notion of maximal chains. A chain in $2^{[n]}$ of size $n+1$ is called a maximal chain. Let P be the set of all ordered pairs $(\mathcal{C}_1,\mathcal{C}_2)$ such that $\mathcal{C}_1$ is a maximal chain in $2^{[\ell]}$ and $\mathcal{C}_2$ is a maximal chain in $2^{[\ell+1,n]}$. Then we have $|P| = \ell!(n-\ell)!$.

Denote by $f(i,j)$ the number of all pairs of maximal chains passing through a given (i,j)–configuration $(E_1, E_2) \in \mathcal{A}$. Clearly $f(i,j) = |\{(\mathcal{C}_1, \mathcal{C}_2) \in P : E_1 \in \mathcal{C}_1, E_2 \in \mathcal{C}_2\}| = i!(\ell - i)!j!(n - \ell - j)!$.

Notice that the property (Q) implies that for every pair $(\mathcal{C}_1, \mathcal{C}_2) \in P$ there is at most one pair $(E_1, E_2) \in \mathcal{A}$ with $E_1 \in \mathcal{C}_1, E_2 \in \mathcal{C}_2$. Therefore we have

$$\sum_{(i,j)\in I} f(i,j)\alpha_{ij} \le |P|$$

or $\sum\limits_{(i,j)\in I} i!(\ell - i)!j!\ (n - \ell - j)!\alpha_{ij} \le \ell!(n-\ell)!$.

Hence

$$\sum_{(i,j)\in I} \frac{\alpha_{ij}}{\binom{\ell}{i}\binom{n-\ell}{j}} \le 1. \tag{2.3}$$

Since $\binom{\ell}{i}\binom{n-\ell}{j} \le \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor}$ in view of (2.2) the inequality (2.3) implies

$$|\mathcal{A}| \le \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor}. \tag{2.4}$$

Consequently

$$A(n, n-1) \le \max_{1\le \ell < n} \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor}. \tag{2.5}$$

It can be easily shown (see [1]) that

$$\max_{1\le \ell < n} \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor} = 2\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}.$$

On the other hand in view of Theorem AAK

$$M\left(n, n-1, \left\lfloor \frac{n}{2} \right\rfloor\right) = 2\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}.$$

This means that $A(n, n-1) \le M\left(n, n-1, \left\lfloor \frac{n}{2} \right\rfloor\right)$.

The corresponding antichain $\mathcal{A} \subset E\left(n, \left\lfloor \frac{n}{2} \right\rfloor\right)$, with $\dim(\mathcal{A}) = n-1$, attaining the bound is

$$\mathcal{A} = E\left(n-2, \left\lfloor \frac{n-2}{2} \right\rfloor\right) \times E(2,1).$$

□

Remark: It is not hard to describe all optimal nonisomorphic antichains. Suppose $\mathcal{A}$ is a maximal antichain defined by (2.1). Consider first

Case $2 \mid n$.

Since $\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor} < 2\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}$ for $\ell \neq 0, 2, n-2$ we conclude that $\ell = 2$ (or equivalently $\ell = n-2$). In view of (2.3) we have

$$\frac{|\mathcal{A}|}{2\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}} = \sum_{(i,j)\in I} \frac{\alpha_{ij}}{2\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}} \leq \sum_{(i,j)\in I} \frac{\alpha_{ij}}{\binom{2}{i}\binom{n-2}{j}} \leq 1. \tag{2.6}$$

Observe now that $\mathcal{A}$ contains only $\left(1, \frac{n-2}{2}\right)$–configurations. This is clear since otherwise we would have strict inequality in the second relation of (2.6), a contradiction to the optimality of $\mathcal{A}$. This means that the only optimal antichain, up to permutations of coordinates, is

$$\mathcal{A} = E(2,1) \times E\left(n-2, \frac{n-2}{2}\right).$$

Correspondingly in (2.1) we have $b_1 = b_2 = \frac{n-2}{2}$, $b_3 = \cdots = b_n = -1$.

Case $2 \nmid n$.

Since $2\binom{n-2}{\frac{n-3}{2}} = \binom{n-1}{\frac{n-1}{2}}$ we have two possibilities for $\ell : \ell = 1$ or $\ell = 2$.

Suppose $\ell = 2$. Using the same arguments as before we conclude that $\mathcal{A}$ can have only two types of configurations: $\left(1, \frac{n-3}{2}\right)$ or $\left(1, \frac{n-1}{2}\right)$. Hence in this case we have the following optimal nonisomorphic antichains:

$$\mathcal{A} = E(2,1) \times E\left(n-2, \frac{n-3}{2}\right), \mathcal{A} = E(2,1) \times E\left(n-2, \frac{n-1}{2}\right),$$
$$\mathcal{A} = \left(\{1,0\} \times E\left(n-2, \frac{n-3}{2}\right)\right) \cup \left(\{0,1\} \times E\left(n-2, \frac{n-1}{2}\right)\right).$$

The corresponding values for $b_1, \ldots, b_n$ are

$b_1 = b_2 = \frac{n-3}{2}$, $b_3 = \cdots = b_n = -1$; $b_1 = b_2 = \frac{n-1}{2}$, $b_3 = \cdots = b_n = -1$;

$b_1 = \frac{n-3}{2}$, $b_2 = \frac{n-2}{2}$, $b_3 = \cdots = b_n = -1$.

Let now $\ell = 1$. Then $\mathcal{A}$ consists of $\left(0, \frac{n-1}{2}\right)$ or $\left(1, \frac{n-1}{2}\right)$ configurations.

If $\mathcal{A}$ consists of only one type of configurations we have

$$\mathcal{A} = \{0\} \times E\left(n-1, \frac{n-1}{2}\right), \text{ or } \mathcal{A} = \{1\} \times E\left(n-1, \frac{n-1}{2}\right).$$

Correspondingly we have $b_1 = 1$, $b_2 = \cdots = b_n = 0$, or $b_1 = \frac{n-1}{2}$, $b_2 = \cdots = b_n = -1$.

Finally if $\mathcal{A}$ contains both types of configurations one can easily observe that

$$\mathcal{A} = \left(\{00\} \times E\left(n-2, \frac{n-1}{2}\right)\right) \cup \left(\{11\} \times E\left(n-2, \frac{n-1}{2} - 1\right)\right)$$

with $b_1 = 1$, $b_2 = -1$, $b_3 = \cdots = b_n = 0$.

3 Proof of (ii)

In view of Theorem AAK we have

$$A(n,k) \geq M\left(n, k, \left\lfloor \frac{n}{2} \right\rfloor\right) = 2^{k-1}.$$

Thus it remains to show that $\mathcal{A}(n,k) \leq 2^{k-1}$. Let $\text{span}(\mathcal{A}) \triangleq \mathcal{U}$ (a k–dimensional subspace of $\mathbb{R}^n$) be the row space of a $k \times n$ matrix G.

In the sequel we essentially use an auxiliary result from [1].

Let M be a $k \times n$ matrix of the following form shown in Figure 1.

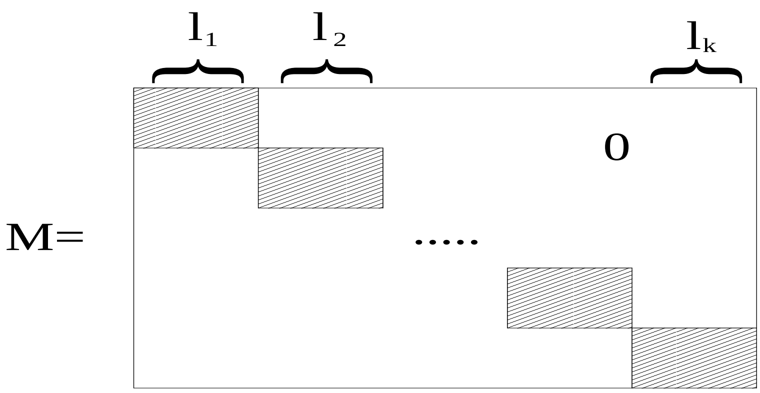

Fig. 1

Each shade of size $\ell_i \geq 1$ $(i = 1, \ldots, k)$, $\sum_{i=1}^{k} \ell_i = n$, depicts ℓ_i positive entries of the i–th row, and above the steps M has only zero entries.

We say that a $k \times n$ matrix is in the *positive step form* if it has the form of the matrix M (in Fig. 1) up to the permutations of the columns.

A nonzero vector of $\mathbb{R}^n$ is called positive (nonnegative) if all its coordinates are positive (nonnegative).

Lemma [1]. *A $k \times n$ matrix T can be transformed into a positive step form by elementary row operations if and only if the row space of T contains a positive vector.*

Let v be the sum of all vectors of $\mathcal{A}$. W.l.o.g. let $v = (a_1, \ldots, a_m, 0, \ldots, 0)$, where $a_1, \ldots, a_m > 0$ and $k \leq m \leq n$. In view of Lemma [1] there exists a generator matrix $G = [M \mid O]$ of the subspace $\mathcal{U}$, where M is a $k \times m$ matrix in the positive step form, and O is the $k \times (n-m)$ zero matrix.

In particular by elementary row operations and permutations of the columns G can be reduced to the following form

$$G' = [I_k \mid B \mid O]$$

where I_k is the $k \times k$ identity matrix and the $k \times m$ submatrix $[B]$ is in the positive step form. Let g be the first row of G'.

Let $\mathcal{B}$ be the set of those $(0,1)$–vectors of $\mathcal{U}$ that are generated without g, that is all vectors of $\mathcal{B}$ have zero in the first coordinate. Note that if $b \in \mathcal{B}$ and $(b+g) \in E(n)$ then b and $b+g$ form a chain.

This is clear because g is a nonnegative vector. Hence for any $b \in \mathcal{B}$ either b or $b+g$ is not in $\mathcal{A}$. This completes the proof since $|\mathcal{B}| \leq 2^{k-1}$.

$\square$

In this case there are many non–isomorphic maximum antichains. For example $\mathcal{A} = E(2,1)^{k-1} \times v$, for any $v \in E(n-2k+2)$.

References

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