Construction of asymmetric connectors of depth two

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Abstract

An (n, N)–connector of depth d is an acyclic digraph with n inputs and N outputs in which for any injective mapping of input vertices into output vertices there exist n vertex disjoint paths of length at most d joining each input to its corresponding output. In this paper we consider the problem of construction of sparse depth two connectors with $n \ll N$. We use posets of star products and their matching properties to construct such connectors. In particular this gives a simple explicit construction for connectors of size $O(N \log n / \log \log n)$.
Thus our earlier idea to use other posets than the family of subsets of a finite set was successful.

Keywords: connector, rearrangeable network, concentrator

1 Introduction

The study of connectors started from pioneering works [14], [15], [4], [3], in connection with practical problems in designing switching networks for telephone traffic. Later they were also studied as useful architectures for parallel machines (see [10] for a good survey).

An (n, N)– communication network is a directed acyclic graph with n distinguished vertices called inputs and N other distinguished vertices called outputs. All other vertices are called links. A route in a network is a directed path from an input to an output. The size of a network is the number of edges, and the depth is the length of the longest route in it.

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An (n, N, d)–connector, also called a rearrangeable network, is a network of depth d ($n \le N$), such that for every injective mapping of the set of input vertices into a set of output vertices there exist n vertex disjoint paths joining each input to its corresponding output.

Usually the size, in some approximate sense, corresponds to the cost and the depth corresponds to the delay of a communication network. Therefore for the networks intended for a certain communication task it is preferable to have small size and small depth.

Symmetric connectors, that is connectors with $n = N$ are well studied. Pippenger and Yao [12] obtained lower and upper bounds for the size of an (n, n, d)–connector: $\Omega(n^{1+1/d})$ and $\mathrm{O}(n^{1+1/d}(\log n)^{1/d})$, respectively. The best known explicit construction for odd depth $2i + 1$ has size $\mathrm{O}(n^{1+1/(i+1)})$ and is due to Pippenger [13]. Hwang and Richards [7] and Feldman, Friedman and Pippenger [6] gave explicit constructions for depth 2 connectors of size $\mathrm{O}(n^{5/3})$; the latter can be used for construction of connectors of depth $2i$ and size $\mathrm{O}(n^{1+2/(3i-1)})$.

For asymmetric connectors Oruc [10] gave constructions for depth $\Omega(\log_2 N + \log_2^2 n)$ of size $\mathrm{O}(N + n \log_2 n)$.

Explicit constructions for $(n, N, 2)$–connectors of size $O(N\sqrt{n})$ for $n \le \sqrt{N}$ (and prime power N) are given in [7] (see also [6]).

Baltz, Jäger and Srivastav [2] have shown by a probabilistic argument the existence of $(n, N, 2)$–connectors of size $\mathrm{O}(N)$, if $n \le N^{1/2-\varepsilon}$, $\varepsilon > 0$, and extended the construction of $\mathrm{O}(N\sqrt{n})$ size connectors to arbitrary $N \ge n^2$.

A construction of $(n, N, 2)$–connectors of size $(1+o(1))N \log_2 n$ for all N and $n = \mathrm{O}\big(N^{1/\sqrt{\log_2 N}}\big)$ is given in [1].

A challenging problem is to construct linear–sized $(n, N, 2)$–connectors (even for some restricted values of n and N).

In this paper we study the case when $n \ll N$. Such connectors of depth two are of particular interest in the design of sparse electronic switches. Also they may be useful as building blocks in multistage (or depth two) symmetric connectors. We improve the result of [1] giving simple constructions for $(n, N, 2)$–connectors of size $\mathrm{O}(N \log n/ \log\log n)$.

2 Basic definitions and results

For integers $a < b$ let us denote $[a, b] = \{a, a+1, \ldots, b\}$, and for $[1, b]$ we use the abbreviation $[b]$. Let also $S(k, q) \triangleq \{(x_1, \ldots, x_k) : \ x_i \in [0, q]\}$. We will also use for $q = \infty$ the notation $[0, \infty] \triangleq \{0\} \cup \mathbb{N}$ and $S(k, \infty)$.

Define now a partial ordering on elements of $S(k, q)$ as follows.
For $x, y \in S(k, q)$ we say that $x \le y$ if either $x_i = y_i$ or $x_i = 0$ for all $i = 1, \ldots, k$. Define also $r(x)$ = the number of nonzero coordinates of $x \in S(k, q)$ (note that $r(x)$ is usually called the Hamming weight of x).

Thus $S(k, q)$ is a partially ordered set ordered by $\le$ with the rank function $r(x)$ defined for each element $x \in S(k, q)$. In the literature $S(k, q)$ is usually called the product of stars (see e.g. [5]). By $S_r(k,q)$ we denote the elements of rank r, that is $S_r(k,q) = \{x \in S(k,q) : \ r(x) = r\}$. For short we will use the notation S_r, when k and q are specified. Thus $S(k,q) = S_0 \dot{\cup} S_1 \dot{\cup} \cdots \dot{\cup} S_k$, where $|S_i| = \binom{k}{i} q^i, \ i = 0, 1, \ \ldots, k$.

Given integers $1 \le l < r \le k$ and q (or $q = \infty$), the l-th shadow of $x \in S_r(k,q)$ is defined by $\partial_l x = \{y \in S_{r-l} : \ x \ge y\}$. Correspondingly for $X \subset S_r$, $\partial_l X = \{\partial_l x : \ x \in X\}$. For $l = 1$ we just write ∂X (and call it the shadow of X).

How small can be the shadow of a subset $X \subset S_r(k,q)$ with given size $|X| = m$? This problem was solved by Leeb [9] and the result was rediscovered later by several authors (see [5], Ch. 8).

For stating the result we have to introduce a linear order on $S(k,q)$. Define first $x(t) = \{i \in [k] : \ x_i = t\}, \ x \in S(k,q)$. Recall also the colexicographic order on the subsets of $[k]$. For $A, B \subset [k]$ we say $A \prec_{col} B$ iff $\max(A \setminus B) < \max(B \setminus A)$. Now for $x, y \in S(k,q)$ we define the linear ordering $\prec_L$ as follows:
$x \prec_L y$ iff $y(t) \prec_{col} x(t)$, where t is the smallest number such that $x(t) \ne y(t)$.
For a subset $X \subset S(k,q)$ let $\mathcal{C}(m, X)$ denote the set of the first m elements of X with respect to ordering $\prec_L$.

Theorem L [9] *Given integers $1 \le r \le k$, m and a subset $A \subset S_r(k,\infty)$ with $|A| = m$ we have*

$$\partial_l \mathcal{C}(m, S_r) \subseteq \mathcal{C}(|\partial_l A|, S_{r-1}). \tag{2.1}$$

In particular this implies that

$$|\partial_l A| \ge |\partial_l \mathcal{C}(m, S_r)|. \tag{2.2}$$

For our construction we use consequences of the theorem stated below.

Theorem 0 *For $1 \le l < r < k$, t and $A \subset S_r(k,\infty)$ with $|A| \le q^r \binom{r}{l}$ we have*

$$\frac{|\partial_l A|}{|A|} \ge \frac{\binom{r}{l}}{\binom{k-r+l}{l} q^l} \tag{2.3}$$

Proof By Theorem L we have

$$\begin{aligned} |\partial_l A| &\ge |\partial_l \mathcal{C}(|A|, S_r(k,\infty))| \\ &= |\partial_l \mathcal{C}(|A|, S_r(k,q))| \end{aligned}$$

since $|A| \le q^r \binom{r}{l}$. Assuming now that $A \subset S_r(k,q)$ we count in two different ways the number of pairs (a,b) with $a \in A, \ b \in \partial_l A$ and observe that

$$|A| \binom{r}{r-l} \le |\partial_l A| \binom{k-r+l}{l} q^l,$$

hence the result. $\square$

Corollary 1 *For $A \subset S_k(k,\infty)$ we have*

$$|\partial_l A| \ge |A| \ \textit{if} \ |A| \le \lfloor \frac{k}{l} \rfloor^k. \tag{2.4}$$

In particular

$$|\partial A| \geq |A| \text{ if } |A| \leq k^k. \tag{2.5}$$

Proof Theorem 0 with the inequality $\binom{k}{l} \geq \left(\frac{k}{l}\right)^l$ gives

$$\frac{|\partial_l A|}{|A|} \geq \frac{\binom{k}{l}}{\lfloor \frac{k}{l} \rfloor^l} \geq \frac{\left(\frac{k}{l}\right)^l}{\lfloor \frac{k}{l} \rfloor^l} \geq 1.$$

□

We note that one can give better upper bounds in (2.4) and (2.5) for the size of A which guarantee the condition $|\partial_l A| \geq |A|$. However, for our purposes these bounds are good enough.

3 The Construction

An (N, L, c)–concentrator is an (N, L)–network such that for every set of $t \leq c$ inputs there exist t disjoint routes containing these inputs. For concentrators of depth one (that is bipartite graphs) this is equivalent to the property that every $t \leq c$ input vertices have at least t neighbors, that is Hall's matching condition is satisfied for every set of $t \leq c$ input vertices.

One of standard approaches for construction of connectors is the concatenation of a connector with a concentrator. In particular, for depth two connectors this is equivalent to the following.

Suppose the vertex set $V = \mathcal{I} \cup \mathcal{L} \cup \mathcal{O}$ of a graph $G = (V, E)$ is partitioned into input vertices $\mathcal{I}$ with $|\mathcal{I}| = n$, link vertices $\mathcal{L}$ with $|\mathcal{L}| = L$ and output vertices $\mathcal{O}$ with $|\mathcal{O}| = N$.

C1: $\mathcal{I}$ and $\mathcal{L}$ form a depth one connector which clearly is a complete bipartite graph .

C2: $\mathcal{O}$ and $\mathcal{L}$ form an (N, L, n)-concentrator.

It is easy to see that G is an $(n, N, 2)$–connector. Later on we will call such connectors standard (n, N)–connectors.

Given n and N, let k be the minimum integer such that $n \leq k^k$. Let also $q > k$ be an integer. Define now the bipartite graph G_1 with the bipartition $(\mathcal{O}, \mathcal{L})$ by

$$\mathcal{O} \triangleq \mathcal{C}(N, S_k(k, q)), \quad \mathcal{L} \triangleq \partial\mathcal{C}(N, S_k(k, q)).$$

The edges are defined in a natural way: for $x \in \mathcal{O}$ and $y \in \mathcal{L}$ (x, y) is in the edge set iff $y \in \partial(x)$. By (2.5) for any subset $x \subset \mathcal{O}$ with $|X| \leq n$ we have $|\Gamma(X)| \geq |X|$. Hence G_1 is an (N, L, n)-concentrator.

Now we can construct a standard (n, N)-connector with $N \triangleq |\mathcal{O}|$ and $L \triangleq |\mathcal{L}|$ links. For ease of calculations let $N = q^k$ and hence $L = q^{k-1}k$. The size of the connector is $Ln + Nk \leq q^{k-1}k^k + q^k k$. Let also $q \geq k^k$. Then we get an (n, N)-connector of size

$$|E| \leq 2Nk = 2N\frac{\log n}{(1+o(1))\log\log n}. \tag{3.1}$$

Note that it is not necessary to take $N = q^k$. Taking any $N = \Omega(q^k)$ we get $\mathrm{O}(Nk)$ for the size of the corresponding connector.

A more general construction using l-shadows is as follows. Given $t > 2$ and $n \geq t^t$, let k be the minimum integer such that $n \leq t^k$. Suppose $k = tl + r$ where $0 \leq r < t$. Thus $t^{k-1} < n \leq t^k$. Let also for ease of calculations $N = q^k$ (in general, $N = \Omega(q^k)$) for some integer $q > \binom{k}{l}^{1/l}$ (in order to have $N > n$).

We construct now the standard connector with $\mathcal{O} \triangleq \mathcal{C}(N, S_k(k,q))$, $\mathcal{L} \triangleq \partial_l \mathcal{C}(N, S_k(k,q))$, $L \triangleq |\mathcal{L}|$ and $n \triangleq |\mathcal{I}| = \Theta(t^k)$.

Then the size of the connector

$$|E| = Ln + N\binom{k}{l} = q^{k-l}\binom{k}{l}\Theta(t^k) + q^k\binom{k}{l}.$$

Put now $q = t^t$ (or $q > t^t$). Then we have a $(\Theta(t^k), t^{tk})$-connector with $L = t^{t(k-l)}$. Taking into account that $\binom{k}{l} < (ke/l)^l$ we get

$$|E| \leq 2N\binom{k}{l} = 2Nn^{\frac{1}{t}(1+o(1))}.$$

Thus we have our main result

Theorem 1 *For all integers $t > 2$, $n \geq t^t$, $N = \Omega(n^t)$ the construction above gives $(n, N, 2)$-connectors of size $Nn^{\frac{1}{t}(1+o(1))}$. In particular, for all n and $N > N(n)$ this construction gives connectors of size $2N\log n/(1+o(1))\log\log n$.*

4 Concluding Remarks

We mentioned before (the result shown in [2]) that for $n \leq N^{1/2-\delta}$, $0 < \delta < 1/2$ there exist $(n, N, 2)$- connectors of size $\mathrm{O}(N)$. Actually the same probabilistic argument can be used to show the following

Proposition 1 *Given $n \geq 2$ and $N \geq N_0(n)$ there exist $(n, N, 2)$–connectors of size $2N(1+o(1))$ and this is asymptotically optimal.*

Proof Given n, L, N, let G_1 be a bipartite graph with bipartition $(\mathcal{L}, \mathcal{O})$, $|\mathcal{L}| = L$, $|\mathcal{O}| = N$. Assume also that each vertex of $\mathcal{O}$ has degree k. Let now p be the probability that G_1 is not an (N, L, n)-concentrator, i.e. the Hall's condition is not satisfied for some $k + 1 \leq i \leq n$.

Observe then that

$$\begin{aligned} p &= \sum_{i=k+1}^{n} \binom{N}{i}\binom{L}{i-1}\binom{i-1}{k}^{i} \Big/ \binom{L}{k}^{i} \\ &< \sum_{i=k+1}^{n} \frac{N^i L^{i-1}}{i!(i-1)!}\left(\frac{i-1}{L}\right)^{ki} \\ &< \sum_{i=k+1}^{n} N^i L^{i-1}\left(\frac{n}{L}\right)^{ki}. \end{aligned}$$

Thus given n, for some constant c we have

$$p < c\sum_{i=k+1}^{n} \frac{N^i}{L^{(k-1)i+1}}.$$

We put now

$$L = L^* = N^{\frac{n}{(k-1)n+1}+g(N)},$$

where $g(N) = o(1)$ and $g^{-1}(N) = o(\log N)$, say $g(N) = 1/\sqrt{\log N}$. In particular, taking $k = 2$ we have $L^* = N^{n/(n+1)+g(N)}$. Observe then that $p = o(1)$ as $N \to \infty$.
Hence for $N \geq N_0(n)$ there exist standard (n, N)–connectors of size

$$|E| \leq nL^* + 2N = (1 + o(1))2N.$$

Note also that in any $(n, N, 2)$-connector $(n \geq 2)$ at least $N - L$ output vertices have degree not less than two, and the number of edges between inputs and links is lower bounded by L. Hence the size of an $(n, N, 2)$-connector is lower bounded by $L + 2N - L = 2N$. $\square$

What can we say about $(n, N, 2)$–connectors with $n \geq N^{1/2}$?
One can easily observe that the lower bound $\Omega(N^{3/2})$ in [12] for the size of an $(N, N, 2)$–connector implies also a lower bound for $(n, N, 2)$.

To this end it is enough to apply the following known simple fact (see e.g. [6]). Let G_1 and G_2 be $(n_1, N, 2)$ and $(n_2, N, 2)$–connectors respectively and let $G_1 * G_2$ be the $(n_1+n_2, N, 2)$–network obtained by identifying the outputs of G_1 and G_2 by any one-to-one mapping. Then $G_1 * G_2$ is an $(n_1 + n_2, N, 2)$–connector. Clearly the size of the resulting connector equals to the sum of sizes of G_1 and G_2.

Suppose now there exists an (N^α, N)–connector G of size $\Omega(N^x)$ with $1/2 \leq \alpha \leq 1$. Then one can construct an $(N, N, 2)$–connector from G applying $(*)$ costruction with sufficiently many copies of G and then deleting all but N input vertices of the resulting network (connector). The constructed connector has size $\Omega(N^{1-\alpha}N^x)$. This with the lower bound $\Omega(N^{3/2})$ implies that G has size $\Omega(N^{1/2+\alpha})$.

Similarly (again in view of $(*)$ construction) the existence of linear-sized $(N^\delta, N, 2)$–connectors for any $0 < \delta < 1/2$ implies also existence of $(N^\alpha, N, 2)$– connectors of size $\mathrm{O}(N^{1/2+\alpha+\delta})$. This can be used to obtain upper bounds for the size of $(N^\alpha, N, 2)$–connectors. In particular one can show the existence of such connectors of size $\mathrm{O}(N^{\alpha+1/2}\log n)$. Thus we have

Proposition 2 *For the size of an (N^α, N)–connector with $1/2 \leq \alpha < 1$ we have lower and upper bounds: $\Omega(N^{\alpha+1/2})$ and $O(N^{\alpha+1/2}\log n)$ respectively.*

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