Every channel with time structure has a capacity sequence

Rudolf Ahlswede
Department of Mathematics
University of Bielefeld
PO Box 100131
33501 Bielefeld
Germany
Email: ahlswede@math.uni-bielefeld.de

I. The Concept

On August 25th 2008 we lectured at the workshop in Budapest which honored I. Csiszár in the year of his 70th birthday, on the result that for every AVC under maximal error probability pessimistic capacity and optimistic capacity are equal. This strongly motivated us to think again about performance criteria and we came back to what we called already a long time ago [1] a (weak) capacity function (now sequence!). But this time we were bold enough to conjecture the theorem below. Its proof was done in hours. In the light of this striking observation we omit now the word "weak" which came from the connection with the weak converse and make the following

Definition 1: For a channel with time structure $\mathcal{K} = (W^n)_{n=1}^{\infty}$ $C : \mathbb{N} \to \mathbb{R}_+$ is a capacity sequence, if for maximal code size $M(n,\lambda)$, where n is the block length or time and λ is the permitted error probability, and the corresponding rate $R(n,\lambda) = \frac{1}{n}\log M(n,\lambda)$

$$\inf_{\lambda>0} \underline{\lim}_{n\to\infty} \left(R(n,\lambda) - C(n)\right) \geq 0 \tag{1}$$

$$\inf_{\lambda>0} \overline{\lim}_{n\to\infty} \left(R(n,\lambda) - C(n)\right) \leq 0. \tag{2}$$

Recall that the pessimistic capacity is $\underline{C} = \inf_{\lambda>0} \underline{\lim}_{n\to\infty} R(n,\lambda)$ and the optimistic capacity is $\overline{C} = \inf_{\lambda>0} \overline{\lim}_{n\to\infty} R(n,\lambda)$.

II. The Existence Result and Its Proof

Theorem 1: Every channel with time structure has a capacity sequence, if $\overline{C} > \infty$.

Moreover, if $(C, C, C, \dots)$ is a capacity sequence, then $C = \underline{C} = \overline{C}$.

Proof. We use only that $R(n,\lambda)$ is not decreasing in λ.

Let $(\delta_l)_{l=1}^{\infty}$ be a null-sequence and let $(\lambda_l)_{l=1}^{\infty}$ be a strictly decreasing null-sequence in $(0,1)$ such that

$$\underline{C} + \delta_l \geq \underline{\lim}_{n\to\infty} R(n,\lambda_l) \geq \underline{C} \tag{3}$$

$$\overline{C} + \delta_l \geq \overline{\lim}_{n\to\infty} R(n,\lambda_l) \geq \overline{C}. \tag{4}$$

Moreover, let $(n_l)_{l=1}^{\infty}$ be a monotone increasing sequence of natural numbers such that for all $n \geq n_l$

$$\overline{C} + \delta_l \geq R(n,\lambda_l) \geq \underline{C} - \delta_l. \tag{5}$$

For $d_l = \lceil \frac{\overline{C} - \underline{C}}{\delta_l} \rceil$ define

$$\begin{aligned} A_l(i) &= \{n : n_l \leq n, \overline{C} - (i-1)\delta_l > R(n,\lambda_l) \geq \overline{C} - i\delta_l\}, \\ &\quad \text{if } 2 \leq i \leq d_l - 1 \\ A_l(1) &= \{n : n_l \leq n, \overline{C} + \delta_l > R(n,\lambda_l) \geq \overline{C} - \delta_l\}, \\ A_l(d_l) &= \{n : n_l \leq n, \overline{C} - (d_l-1)\delta_l > R(n,\lambda_l) \geq \underline{C} - \delta_l\}. \end{aligned}$$

Define by using lower end points

$$C(n) = \begin{cases} \overline{C} - i\delta_l & \text{for } n \in A_l(i), n < n_{l+1} \\ & \quad \text{and } 1 \leq i \leq d_l - 1 \\ \underline{C} - \delta_l & \text{for } n \in A_l(d_l), n < n_{l+1}. \end{cases}$$

Now for any $\lambda \in (0,1)$ and $\lambda_l < \lambda$

$$R(n,\lambda) - C(n) \geq R(n,\lambda_{l+j}) - C(n) \geq 0$$

for $n_{l+j} \leq n < n_{l+j+1}$ and $j = 0,1,2,\dots$ and thus

$$\underline{\lim}_{n\to\infty} R(n,\lambda) - C(n) \geq 0$$

and (1) follows.

Finally, for any $\lambda < \lambda_l$ by monotonicity of $R(n,\lambda)$

$$\overline{\lim}_{n\to\infty} R(n,\lambda) - C(n) \leq \overline{\lim}_{n\to\infty} R(n,\lambda_l) - C(n) \leq 2\delta_l$$

and

$$\inf_{\lambda\in(0,1)} \overline{\lim}_{n\to\infty} R(n,\lambda) - C(n) \leq \lim_{l\to\infty} 2\delta_l = 0$$

and (2) holds. □

Example: The channel with $\frac{1}{n}\log M(n,\lambda) = \begin{cases} \lambda n & \text{for even } n \\ 0 & \text{for odd } n \end{cases}$ has $\underline{C} = 0$, $\overline{C} = \infty$ and no capacity sequence.

For a discussion of other performance criteria we refer to [2].

References

[1] R. Ahlswede, Beiträge zur Shannonschen Informationstheorie im Fall nichtstationärer Kanäle, Z. Wahrscheinlichkeitstheorie und verw. Geb. 10, 1–42, 1968. (Dipl. Thesis Nichtstationäre Kanäle, Göttingen 1963.)

[2] R. Ahlswede, On concepts of performance parameters for channels, General Theory of Information Transfer and Combinatorics, Lecture Notes in Computer Science, Vol. 4123, Springer Verlag, 639-663, 2006.