3.5 EIGHT PROBLEMS IN INFORMATION THEORY

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1. Multiuser Information Theory.

Problem 1: So far, the capacity regions of multiway channels have been characterized in only a few cases. The main difficulty consists of finding appropriate methods for *single-letterization.*

For complex channels this seems to be a hopeless task. We therefore suggest settling for somewhat less, that is, a description of the capacity region as the limit of information quantities depending on vector-valued random variables such that the speed of convergence in terms of the number of components can be bounded from above. There ought to be a way to do this.

Problem 2: There are nonprobabilistic channels that have never been considered in a multiuser situation. We suggest doing this for the permuting channels, which have been studied in [1].

Problem 3: One of the very challenging problems has been to determine the capacity region of the broadcast channel (Cover, 1972).

The following simpler problem encounters some of the typical difficulties. Suppose that V is a finite set, then a family $\{ E_{ij} : 1 \le i \le I , 1 \le j \le J \}$ of subsets of V is ε-good if for $A_i = \bigcup_j E_{ij}$ and $B_j = \bigcup_i E_{ij}$

(i) $| A_i \cap E_{i'j} | \le \varepsilon | E_{i'j} |$ for all j and all $i' \ne i$;

(ii) $| B_j \cap E_{ij'} | \le \varepsilon | E_{ij'} |$ for all i and all $j' \ne j$.

Derive bounds on I and J in terms of $| V |$ and ε .

Problem 4: Whereas there is an extensive literature on coding schemes for multiway channels with feedback, it seems that there is no theory for multisources in case of feedback. Such a theory should include various search problems such as group testing.

Problem 5: In [2], we studied several source coding problems involving decompositions of $n \times n$ arrays into as few as possible partial transversals such that each transversal has distinct symbols as entries. It is therefore of interest to know the possible lengths of such transversals. In particular we have the following:

Conjecture: Suppose that in an $n \times n$ array no symbol occurs more than n times as an entry. Then there exists a partial transversal of length $n-1$ with distinct symbols. The example $\left(\begin{smallmatrix} ab \\ ba \end{smallmatrix}\right)$ shows that one cannot always expect a transversal of length n.

2. Noiseless Coding for Multiple Purposes.

Consider a Bernoulli source $X^n = (X_1, \dots, X_n)$. Suppose that there are n persons and that person t is interested in the outcome of X_t ($1 \leq t \leq n$). A multiple purpose encoding (or program) shall be a sequence $f = (f_1(X^n), f_2(X^n), \dots)$ of 0-1 valued functions f_i.

Person t requests sequentially the values of $f_1, f_2, \dots,$ and stops as soon as he has identified the value of X_t. Let $l(f, t)$ denote the expected number of requests of person t for program f. We are interested in the quantity $L(n) = \min_f \max_{1 \leq t \leq n} l(f, t)$. The choice $f_i(X^n) = X_i$ $(1 \leq i \leq n)$ gives $l(f, t) = t$ for $1 \leq t \leq n$. Since $\frac{1}{n} \sum_{t=1}^{n} l(f, t) = \frac{n+1}{2}$ one should do better.

Problem 6: What is the asymptotic growth of $L(n)$? There are obvious generalizations of this problem.

3. Correlation Inequalities.

Correlation inequalities play a role in statistical physics, reliability theory, and so on. A systematic study was made in [3]. Instead of the Boolean operations $\vee$, $\wedge$ usually occurring in those inequalities, one can consider any two operations $\phi, \psi : S \times S \to S$, where S is a finite set. Further progress depends on the solution of the following.

Problem 7: For two maps $\phi_S : S \times S \to S$ and $\phi_T : T \times T \to T$, define the product

$$\phi_{ST} : (S \times T) \times (S \times T) \to S \times T$$

by

$$\phi_{ST}\ ((s_1, t_1), (s_2, t_2)) = (\phi_S(s_1, s_2), \phi_T(t_1, t_2))$$

$$\text{for all} \quad s_1, s_2 \in S,\ t_1, t_2 \in T .$$

Also ϕ associates to $A, B \subset S$ a new set in the Minkowski sense $\phi\ (A, B) \triangleq \{\ \phi\ (a, b) : a \in A,\ b \in B\ \}$. The pair (ϕ, ψ) is called *expansive*, if $|A|\ \ |B| \leq |\phi(A, B)|\ \ |\psi(A, B)|$ for all $A, B \subset S$.

Conjecture ([3]). If (ϕ_S, ψ_S) and (ϕ_T, ψ_T) are expansive, then the pair of products (ϕ_{ST}, ψ_{ST}) is also expansive.

4. Random Selection and Equidistribution.

Existence proofs by random selection are very popular in combinatorics, information theory, complexity theory and so on. We wonder whether they can be replaced by deterministic procedures, which have certain equidistribution properties. Our ideas are not yet precise. We came across the following number theoretical problem, which does not seem to fit into the classical theory of equidistribution.

Problem 8: Consider, for instance, the sets $A_n \triangleq \{\ \sum_{i=1}^{n} \varepsilon_i 5^i :$ $\varepsilon_i \in \{0, 1\}\ \}$. Do the sets $A_n(m) \triangleq \{\ k \in A_n : k \equiv m \bmod 2^n\ \}$ satisfy for all $0 \leq m \leq 2^n-1$ $|A_n(m)|\ 2^{-n} = O(1)$ (or at least $|A_n(m)|\ 2^{-n} = 2^{o(n)}$) ?

REFERENCES

[1] R. Ahlswede and A. Kaspi, "Optimal Coding Strategies for Certain Permuting Channels," submitted to *IEEE Trans. Inf. Theory.*

[2] R. Ahlswede, "Coloring Hypergraphs: A New Approach to Multiuser Source Coding," Part I, *J. Combinatorics, Inf. Syst. Sci.* 4, No. 1, pp. 76-115 (1979); Part II, *ibid.* 5, No. 3, pp. 220-268 (1980).

[3] R. Ahlswede and D.E. Daykin, "Inequalities for a Pair of Maps $S \times S \to S$ with S a Finite Set," *Math. Z.* 165, pp. 267-289 (1979).