ADVANCES IN MATHEMATICS 80, 137–151 (1990)

An Identity in Combinatorial Extremal Theory

R. AHLSWEDE AND Z. ZHANG

Universität Bielefeld, Fakultät für Mathematik,
Postfach 8640, 4800 Bielefeld 1, West Germany

Our main discovery is the following identity: For every family $\mathscr{A} \subset 2^{\Omega}$ of non-empty subsets of $\Omega = \{1, 2, ..., n\}$

$$\sum_{X \subset \Omega} \frac{W_{\mathscr{A}}(X)}{|X| \binom{n}{X}} \equiv 1,$$

where

$$W_{\mathscr{A}}(X) = \left| \bigcap_{X \supset A \in \mathscr{A}} A \right|.$$

It can be viewed as a sharpening of the famous LYM-inequality. We present also generalizations to other posets. The total impact for combinatorics remains to be explored. The identity seems to be particularly useful for uniqueness proofs in Sperner Theory. We also discuss a geometric consequence.

1. THE MAIN IDENTITY

We give first a slightly different formulation of our main identity in terms of concepts, which are needed later. For every $X \in \mathscr{P} = 2^{\Omega}$ and every $\mathscr{A} \subset \mathscr{P}$ we define

$$X_{\mathscr{A}} = \bigcap_{X \supset A \in \mathscr{A}} A \quad \text{and} \quad W_{\mathscr{A}}(X) = |X_{\mathscr{A}}|. \tag{1.1}$$

Using the functions

$$W_{\mathscr{A}}^{(i)} = \sum_{X \in \mathscr{P}_i} W_{\mathscr{A}}(X), \tag{1.2}$$

where $\mathscr{P}_i$ is the set of all i-element subsets of Ω, we can write

$$\sum_{X} \frac{W_{\mathscr{A}}(X)}{|X| \binom{n}{|X|}} = \sum_{i=1}^{n} \frac{W_{\mathscr{A}}^{(i)}}{i \binom{n}{i}}.$$

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THEOREM 1. *For every family $\mathscr{A}$ of non-empty subsets of $\Omega = \{1, 2, ..., n\}$*

$$\sum_{i=1}^{n} \frac{W_{\mathscr{A}}^{(i)}}{i\binom{n}{i}} = 1.$$

Proof. Note first that only the minimal elements in $\mathscr{A}$ determine $X_{\mathscr{A}}$ and therefore matter. We can assume therefore that $\mathscr{A}$ is an antichain.

Recall that in Lubell's proof of Sperner's Lemma all "saturated" chains which pass through members of $\mathscr{A}$, are counted:

$$\sum_{A \in \mathscr{A}} |A|!\,(n - |A|)!.$$

No chain is counted twice, because $\mathscr{A}$ is an antichain. Since there are $n!$ saturated chains in total, clearly

$$\sum_{A \in \mathscr{A}} |A|!\,(n - |A|)! \leqslant n!$$

or

$$\sum_{A \in \mathscr{A}} \frac{1}{\binom{n}{|A|}} \leqslant 1.$$

Our observation is that we can also count the saturated chains not passing through $\mathscr{A}$. The key idea is to assign to $\mathscr{A}$ the upset

$$\mathscr{U} = \{X \in \mathscr{P} : X \supset A \text{ for some } A \in \mathscr{A}\} \tag{1.3}$$

and to count saturated chains according to their exits in $\mathscr{U}$. For this we view $\mathscr{P}$ as a directed graph with an edge between vertices B, C exactly if $B \supset C$ and $|B \backslash C| = 1$.

Since $\varnothing \notin \mathscr{A}$, clearly $\varnothing \notin \mathscr{U}$. Therefore every saturated chain starting in $\Omega \in \mathscr{U}$ has a last set, say exit set, in $\mathscr{U}$. For every $U \in \mathscr{U}$ we call $e = (U, V)$ an exit edge, if $V \in \mathscr{P} \backslash \mathscr{U}$, and we denote the set of exit edges by $\mathscr{E}_{\mathscr{A}}(U)$. The number of saturated chains *leaving* $\mathscr{U}$ in U is then

$$(n - |U|)!\,|\mathscr{E}_{\mathscr{A}}(U)|\,(|U| - 1)!.$$

Therefore

$$\sum_{U \in \mathscr{U}} (n - |U|)!\,|\mathscr{E}_{\mathscr{A}}(U)|\,(|U| - 1)! = n! \tag{1.4}$$

and since $\mathscr{E}_{\mathscr{A}}(X) = \varnothing$ for $X \in \mathscr{P} - \mathscr{U}$, also

$$\sum_{X \in \mathscr{P}} \frac{|\mathscr{E}_{\mathscr{A}}(X)|}{|X|\binom{n}{|X|}} \equiv 1. \tag{1.5}$$

Now just verify that

$$|\mathscr{E}_{\mathscr{A}}(X)| = W_{\mathscr{A}}(X). \tag{1.6}$$

Remark. It is surprising that this identity has not been found earlier. For instance in [10] an effort is made to improve the LYM-inequality by adding some, but not all, missing terms.

The simple proof above was the result of an analysis of two somewhat more complicated proofs, which are reproduced in Section 6 for readers interested in "proof techniques." It extends to general posets.

The name LYM-inequality was introduced in the survey article [5] to honour the authors in [4], [2], and [3]. More recent books concerned with this subject are [6], [7], and [8].

2. Consequences for Families of Sets

For any antichain $\mathscr{A} \subset \mathscr{P}$ set $\mathscr{A}_i = \mathscr{A} \cap \mathscr{P}_i$. Then the LYM-inequality can be written in the form

$$\sum_{i=1}^{n} \frac{|\mathscr{A}_i|}{\binom{n}{i}} \leqslant 1. \tag{2.1}$$

The next trick is standard. For any function $c: \mathbb{N} \to \mathbb{R}_+$ one can rewrite the LYM-inequality as

$$\sum_{i=1}^{n} \frac{c(i)\,|\mathscr{A}_i|}{c(i)\binom{n}{i}} \leqslant 1,$$

which yields

$$\sum_{i=1}^{n} c(i)\,|\mathscr{A}_i| \leqslant \max_i c(i)\binom{n}{i}. \tag{2.2}$$

For the ease of reference we state the special case $c(i) = i$ for $i = 1, 2, ..., n$ as

Corollary 1. *For any antichain* $\mathscr{A} \subset \mathscr{P}$

$$\sum_{i=1}^{n} i|\mathscr{A}_i| \leqslant \max_{1 \leqslant i \leqslant n} i\binom{n}{i} = m\binom{n}{m}, \qquad \textit{where} \quad m = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n+1}{2} \right\rceil.$$

The bound is best possible.

A. *New Results for Generalized Antichains*

Theorem 1 together with (1.2) implies:

THEOREM 2. *For any family $\mathscr{A} \subset \mathscr{P}$*

$$\sum_{X \in \mathscr{P}} W_{\mathscr{A}}(X) \leqslant m \binom{n}{m}.$$

Obviously by (1.1) *this result implies Corollary* 1. *However, it covers also other cases of interest.*

DEFINITION. We call $\{\mathscr{A}(1), ..., \mathscr{A}(M)\}$ a cloud-antichain (CAC), if $\mathscr{A}(i) \subset \mathscr{P}$ for $1 \leqslant i \leqslant M$ and if for every $i \neq j$ any $A_i \in \mathscr{A}(i)$ and any $A_j \in \mathscr{A}(j)$ are incomparable.

In case $|\mathscr{A}(i)| = 1$ for $1 \leqslant i \leqslant M$ this reduces to the familiar concept of an antichain. From Theorem 2 we deduce an extension of Corollary 1.

THEOREM 3. *For the* CAC $\{\mathscr{A}(i)\colon 1 \leqslant i \leqslant M\}$ $\sum_{i=1}^{M} |\mathscr{A}(i)|\, |\bigcap_{A \in \mathscr{A}(i)} A| \leqslant m\binom{n}{m}$.

Proof. Let $\mathscr{M}(i)$ be the set of minimal elements in $\mathscr{A}(i)$ and let $\mathscr{M} = \bigcup_{i=1}^{M} \mathscr{M}(i)$. For $X \in \mathscr{A}(i)$ we have

$$\bigcap_{A \in \mathscr{A}(i)} A \subset \bigcap_{X \supset A \in \mathscr{M}(i)} A = \bigcap_{X \supset A \in \mathscr{M}} A \text{ (by incomparability)} = X_{\mathscr{M}}.$$

Thus $\sum_{i=1}^{M} |\mathscr{A}(i)|\, |\bigcap_{A \in \mathscr{A}(i)} A| \leqslant \sum_{i=1}^{M} \sum_{X \in \mathscr{A}(i)} W_{\mathscr{M}}(X) \leqslant \sum_{X \in \mathscr{P}} W_{\mathscr{M}}(X)$ and the result follows with Theorem 2.

B. *An Extension to Several Families*

THEOREM 4. *For k families $\mathscr{A}^1, ..., \mathscr{A}^k$ of non-empty subsets of Ω and $X \subset \Omega$ define*

$$W_{\mathscr{A}^1, ..., \mathscr{A}^k}(X) = \left| \bigcup_{j=1}^{k} X_{\mathscr{A}^j} \right|$$

and

$$W^{(i)}_{\mathscr{A}^1, ..., \mathscr{A}^k} = \sum_{X \in \mathscr{P}_i} W_{\mathscr{A}^1, ..., \mathscr{A}^k}(X).$$

Then

(i) $$\sum_{i=1}^{n} \frac{W^{(i)}_{\mathscr{A}^1, ..., \mathscr{A}^k}}{i\binom{n}{i}} \leqslant k$$

and

(ii) $$\sum_{X \in \mathscr{P}} W_{\mathscr{A}^1, \ldots, \mathscr{A}^k}(X) \leqslant \max_{0 \leqslant l \leqslant n-k} \sum_{r=l+1}^{l+k} r\binom{n}{r}.$$

Proof. Since by our definitions

$$W^{(i)}_{\mathscr{A}^1, \ldots, \mathscr{A}^k} \leqslant \sum_{j=1}^{k} W^{(i)}_{\mathscr{A}^j},$$

(i) immediately follows from Theorem 1.

To prove (ii), note first that from its definition

$$W^{(i)}_{\mathscr{A}^1, \ldots, \mathscr{A}^k} \leqslant i\binom{n}{i}$$

and therefore the numbers $\alpha_i = W^{(i)}_{\mathscr{A}^1, \ldots, \mathscr{A}^k} / i\binom{n}{i}$ satisfy $0 \leqslant \alpha_i \leqslant 1$ and by (i) $\sum_{i=1}^{n} \alpha_i \leqslant k$. Therefore

$$\sum_{i=1}^{n} \alpha_i i\binom{n}{i} \leqslant \max\left\{\sum_{i=1}^{n} \beta_i i\binom{n}{i} \,\middle|\, 0 \leqslant \beta_i \leqslant 1, \quad \sum_{i=1}^{k} \beta_i = 1\right\}$$

$$= \max_{0 \leqslant l \leqslant n-k} \sum_{r=l+1}^{l+k} r\binom{n}{r}$$

and since $\sum_{X \in \mathscr{P}} W_{\mathscr{A}^1, \ldots, \mathscr{A}^k}(X) = \sum_{i=1}^{n} W^{(i)}_{\mathscr{A}^1, \ldots, \mathscr{A}^k} = \sum_{i=1}^{n} \alpha_i i\binom{n}{i}$, the result (ii) follows.

3. Geometric Consequences

Theorem 5. *For an n-dimensional unit-cube $C^n = \{0,1\}^n$ with the usual $n\,2^{n-1}$ edges:*

(i) *a hyperplane cuts at most $m\binom{n}{m}$ edges and this bound is best possible;*

(ii) *it takes n hyperplanes with non-negative coefficients to cut all edges.*

Proof. (i) A hyperplane in n-dimensional space is determined by a vector $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ of coefficients for the linear equation $\lambda_0 = \sum_{t=1}^{n} \lambda_t x_t$.

Let us assume first that $\lambda_t \geqslant 0$ for $t = 1, 2, \ldots, n$. A vector $\mathbf{y} \in C^n$ is minimal, if $\sum_{t=1}^{n} \lambda_t y_t \geqslant \lambda_0$ and if replacement of 1 by 0 in *any* component of $\mathbf{y}$ results in a vector $\mathbf{y}'$, with $\sum_{t=1}^{n} \lambda_t y'_t < \lambda_0$. Let $\mathscr{V}(\boldsymbol{\lambda})$ be the set of minimal vectors. If an edge $(\mathbf{x}, \mathbf{x}')$ is cut by the hyperplane such that $\sum_{t=1}^{n} \lambda_t x_t \geqslant \lambda_0$ and $\sum_{t=1}^{n} \lambda_t x'_t < \lambda_0$, then $\mathbf{x}$ and $\mathbf{x}'$ differ in a component in which all vectors from $\mathscr{V}(\boldsymbol{\lambda})$ below $\mathbf{x}$ have a 1.

The vectors $\mathbf{x} \in C^n$ can be identified with the sets $X = \{t: x_t = 1\}$. Let $\mathscr{A}(\boldsymbol{\lambda})$ be the family of sets corresponding to $\mathscr{V}(\boldsymbol{\lambda})$. By the foregoing remarks the number of edges $\{(\mathbf{x}, \mathbf{x}'): \mathbf{x}' < \mathbf{x}\}$ with vertex $\mathbf{x}$ fixed, which are cut by the hyperplane, does not exceed $W_{\mathscr{A}(\boldsymbol{\lambda})}(X)$. Denoting the total number of edges cut by the hyperplane by $F(\boldsymbol{\lambda})$, we thus get

$$F(\boldsymbol{\lambda}) \leqslant \sum_{X \in \mathscr{P}} W_{\mathscr{A}(\boldsymbol{\lambda})}(X). \tag{3.1}$$

For hyperplanes with arbitrary coefficients $(\lambda_1, ..., \lambda_n)$ a coordinate transformation

$$T(x_t) = \begin{cases} x_t & \text{if } \lambda_t \geqslant 0 \\ 1 - x_t & \text{if } \lambda_t < 0 \end{cases}$$

leads to the case of non-negative coefficients just treated. By Theorem 2 therefore

$$\max_{\boldsymbol{\lambda}} F(\boldsymbol{\lambda}) \leqslant m \binom{n}{m}. \tag{3.2}$$

The case $\boldsymbol{\lambda} = (m, 1, ..., 1)$ shows that this bound is best possible.

(ii) For k hyperplanes with the vectors of coefficients $\boldsymbol{\lambda}^j$ $(j = 1, 2, ..., k)$ we define as before the set $\mathscr{A}(\boldsymbol{\lambda}^j)$ and put $\mathscr{A}^j = \mathscr{A}(\boldsymbol{\lambda}^j)$ in Theorem 4. The number $F(\boldsymbol{\lambda}^1, ..., \boldsymbol{\lambda}^k)$ of edges cut by these hyperplanes is bounded by $\sum_{X \in \mathscr{P}} W_{\mathscr{A}^1, ..., \mathscr{A}^k}(X)$. Since all edges shall be cut, by (ii) of Theorem 4

$$n\, 2^{n-1} \leqslant \max_{0 \leqslant l \leqslant n-k} \sum_{r=l+1}^{l+k} r \binom{n}{r}$$

and since $n\, 2^{n-1} = \sum_{r=1}^{n} r \binom{n}{r}$, necessarily $k \geqslant n$. That on the other hand n hyperplanes suffice, can be seen by the example $\boldsymbol{\lambda}^j = (j, 1, ..., 1)$ for $j = 1, 2, ..., n$ or also by the example

$$\lambda_t^j = \begin{cases} 1 & \text{for } j = t \\ 0 & \text{for } j \neq t, \end{cases}$$

$$\lambda_0^j = 1 \qquad \text{for } j, t = 1, 2, ..., n.$$

Remark. According to [9] it has been conjectured by S. Poljak that (even without any restrictions on the coefficients) it takes n hyperplanes to cut all edges of the n-dimensional unit cube. Note, however, that for $k<n$ the bound $\sum_{0\leqslant l\leqslant n-k}\sum_{r=l+1}^{l+k} r\binom{n}{r}$ is not an upper bound on the number of edges cut by k hyperplanes with arbitrary coefficients. Already in case $n=3$ and $k=2$ one can cut 10 edges, whereas $1\binom{3}{1}+2\binom{3}{2}=9$.

4. Uniqueness Proofs

We demonstrate now that simple uniqueness proofs can be given via our identity.

The reader is reminded of our convention

$$m=\left\lfloor\frac{n}{2}\right\rfloor+1. \tag{4.1}$$

We also use the elementary facts

$$\max_{1\leqslant l\leqslant m} l\binom{n}{l}=m\binom{n}{m} \tag{4.2}$$

$$l\binom{n}{l}<m\binom{n}{m}\begin{cases}\text{exactly} & \text{for } l\neq m, m-1, \text{ if } n \text{ is even}\\ & \text{for } l\neq m, \text{ if } n \text{ is odd.}\end{cases} \tag{4.3}$$

A. *Sperner's Case*

For an antichain $\mathscr{A}$ in $\mathscr{P}$ the identity says

$$1=\sum_{A\in\mathscr{A}}\frac{1}{\binom{n}{|A|}}+\sum_{X\notin\mathscr{A}}\frac{W_{\mathscr{A}}(X)}{|X|\binom{n}{|X|}}. \tag{4.4}$$

If $\mathscr{A}$ is maximal, then $|\mathscr{A}|=\binom{n}{\lfloor n/2\rfloor}$, and therefore by (4.4)

$$\mathscr{A}\subset\mathscr{P}_{\lfloor\frac{n}{2}\rfloor}\cup\mathscr{P}_{\lceil\frac{n}{2}\rceil}, \tag{4.5}$$

$$\sum_{X\notin\mathscr{A}}\frac{W_{\mathscr{A}}(X)}{|X|\binom{n}{|X|}}=0. \tag{4.6}$$

Obviously $\mathscr{A}=\mathscr{P}_k$, if $n=2k$. We show now that for $n=2k+1$ the assumption $\mathscr{A}\neq\mathscr{P}_k, \mathscr{P}_m$ violates (4.6).

For this note that $W_{\mathscr{A}}(X)=0$ for $X\in\mathscr{P}_m\backslash\mathscr{A}$ implies that in the graph defined on $\mathscr{P}_k\cup\mathscr{P}_m$ by containment X has no connections with $\mathscr{P}_k\backslash\mathscr{A}$. Since there are no connections between $\mathcal{A}_m = \mathcal{P}_m \cap \mathcal{A}$ and $\mathcal{A}_k = \mathcal{P}_k \cap \mathcal{A}$, we have two connected components $\mathcal{A}_m \cup (\mathcal{P}_k \backslash \mathcal{A})$ and $(\mathcal{P}_m \backslash \mathcal{A}) \cup \mathcal{A}_k$. However, $\mathcal{P}_m \cup \mathcal{P}_k$ is obviously connected.

B. *Uniqueness in Theorem* 2

Since for every family $\mathcal{A}$ with set of minimal elements $\mathcal{M}$ $W_{\mathcal{A}}(X) = W_{\mathcal{M}}(X)$, and since every antichain occurs as a set of minimal elements, it suffices to characterize those antichains $\mathcal{A}$ with

$$\sum_{X \in \mathcal{P}} W_{\mathcal{A}}(X) = m \binom{n}{m}. \tag{4.7}$$

THEOREM 2'. *Equality occurs in* (4.7) *for* $\mathcal{A} = \mathcal{P}_m$, *if* $n = 2k+1 = m+k$, *and for* every *antichain* $\mathcal{A} \subset \mathcal{P}_m \cup \mathcal{P}_k$, *which is "saturated" in* $\mathcal{P}_m$, *if* $n = 2k$.

Proof. It follows from (4.7) and our identity that

$$W_{\mathcal{A}}(X) = 0, \qquad \text{if } |X| \binom{n}{|X|} < m \binom{n}{m}. \tag{4.8}$$

For $n = 2k+1$ we have therefore $W_{\mathcal{A}}(X) = 0$, if $|X| \neq m$, and thus $\mathcal{A} = \mathcal{P}_m$.

On the other hand for $n = 2k$, $k\binom{n}{k} = m\binom{n}{m}$ and we can conclude only that $\mathcal{A} \subset \mathcal{P}_m \cup \mathcal{P}_k$. $\mathcal{A}$ is saturated in $\mathcal{P}_m$, because otherwise there is an antichain $\mathcal{A}' = \mathcal{A} \cup \{X\}$ with $X \in \mathcal{P}_m \backslash \mathcal{A}$ and $\sum_{Y \in \mathcal{P}} W_{\mathcal{A}'}(Y) = \sum_{Y \in \mathcal{P}} W_{\mathcal{A}}(Y) + |X|$, which contradicts the optimality of $\mathcal{A}$.

It remains to be seen that equality occurs in (4.7) for these antichains. Let $\mathcal{A}_k^+$ be the elements of $\mathcal{P}_m$, which are connected with an element of $\mathcal{A}_k$, and let $d(\mathcal{A}_k, X)$ count the number of connections of $\mathcal{A}_k$ with X. Then we have

$$\begin{aligned}
\sum_X W_{\mathcal{A}}(X) &= k|\mathcal{A}_k| + (k+1)\left(\binom{2k}{k+1} - |\mathcal{A}_k^+|\right) \\
&\quad + \sum_{X \in \mathcal{A}_k^+} (k+1) - d(\mathcal{A}_k, X) \\
&= k|\mathcal{A}_k| + (k+1)\binom{2k}{k+1} - (n-k)\,|\mathcal{A}_k| \\
&= (k+1)\binom{2k}{k+1}.
\end{aligned} \tag{4.9}$$

The notion of antichains $\mathcal{A} \subset \mathcal{P}_m \cup \mathcal{P}_k$, which are saturated in $\mathcal{P}_m$, but not in $\mathcal{P}_k$, is meaningful.

EXAMPLE. Let $(n, m, k) = (4, 3, 2)$ and

$$\mathscr{A}_m = \{\{1, 2, 3\}\}, \qquad \mathscr{A}_k = \{\{1, 4\}, \{2, 4\}\}.$$

The antichain $\mathscr{A} = \mathscr{A}_m \cup \mathscr{A}_k$ cannot be extended by $\{1, 2, 4\}$, $\{1, 3, 4\}$, or $\{2, 3, 4\}$, however, it can be extended by $\{3, 4\}$.

C. *Uniqueness in Corollary* 1

Since for an antichain $\mathscr{A}$, $\sum_{A \in \mathscr{A}} |A| \leqslant \sum_{X \in \mathscr{P}} W_{\mathscr{A}}(X)$, the equality

$$\sum_{A \in \mathscr{A}} |A| = m \binom{n}{m} \tag{4.10}$$

can occur only for antichains, which are contained in the class characterized in Theorem 2′. By (4.9) thus necessarily $\sum_{X \in \mathscr{A}_k^+} (k+1) = \sum_{X \in \mathscr{A}_k^+} d(\mathscr{A}_k, X)$ or, equivalently, $\mathscr{A}_k \cup \mathscr{A}_k^+$ is a connected component of $\mathscr{P}_k \cup \mathscr{P}_m$. This is possible only if $\mathscr{A}_k = \varnothing$ or $\mathscr{A}_k = \mathscr{P}_k$. We summarize this result.

COROLLARY 1′. *There is equality in* (4.10) *exactly for* $\mathscr{A} = \mathscr{P}_m$, *if* $n = 2k + 1$, *and for* $\mathscr{A} \in \{\mathscr{P}_m, \mathscr{P}_k\}$, *if* $n = 2k$.

D. *Announcement of Further Results*

In another paper we show that Bollobas' inequality [13, 6] can also be lifted to an identity and that this identity enables us to prove the uniqueness conjecture [14, Remark 1] of Griggs, Stahl, and Trotter.

5. AN IDENTITY FOR POSETS

Let now $\mathscr{P}$ be a finite partially ordered set. In case it doesn't have a unique maximal element, we introduce an element X_∞, which is above the original maximal elements. $\mathscr{P}$ can also be viewed as a directed graph with an edge $e = (X, Y)$ connecting X and Y exactly if $X > Y$ and there is no element between X and Y. The idea of using the concept of exit edges of an upset $\mathscr{U}$ for counting saturated chains applies to any such poset. The sets involved in such a count are:

$\mathscr{E}(U)$, the exit edges starting in U

$\overline{\mathscr{C}}(X)$, the saturated chains ending in X

$\underline{\mathscr{C}}(X)$, the saturated chains starting in X

$\mathscr{C}$, the set of all saturated chains in $\mathscr{P}$.

Then clearly

$$\sum_{U \in \mathscr{U}} \sum_{V:\,(U,V) \in \mathscr{E}(U)} |\mathscr{C}(U)|\,|\mathscr{C}(V)| = |\mathscr{C}|. \tag{5.1}$$

More generally we can incorporate the case of a weight function

$$F: \mathscr{C} \to \mathbb{R}. \tag{5.2}$$

With $\mathscr{C}(e)$ as the set of saturated chains passing through edge e we can express the identity in the following form.

THEOREM 6. $\sum_{U \in \mathscr{U}} \sum_{e \in \mathscr{E}(U)} \sum_{C \in \mathscr{C}(e)} F(C) = \sum_{C \in \mathscr{C}} F(C)$.

Sometimes the quantities in this identity can be fully or partially calculated. This was the case in (1.4), other examples are given below. To any $\mathscr{A} \subset \mathscr{P}$ we can assign as in (1.3) an upset $\mathscr{U}$ and the sets of exit edges $\mathscr{E}_{\mathscr{A}}(U)$, $U \in \mathscr{U}$. Thus Theorem 1 is an immediate consequence of Theorem 6. For an upset $\mathscr{U} \neq \{X_\infty\}$ there is still another way of using the idea of exit edges. Let $\mathscr{M}$ be the set of minimal elements in $\mathscr{U}$. We call an edge $e = (X, Y)$ free (relative to $\mathscr{U}$), if

$$X \in \mathscr{U} \setminus \mathscr{M} \quad \text{and} \quad Y \in \mathscr{M} \cup (\mathscr{P} \setminus \mathscr{U}).$$

Let $\mathscr{F}(Y)$ denote the set of free edges ending in Y.

THEOREM 7. $\sum_{Y \in \mathscr{M} \cup (\mathscr{P} \setminus \mathscr{U})} \sum_{e \in \mathscr{F}(Y)} \sum_{C \in \mathscr{C}(e)} F(C) = \sum_{C \in \mathscr{C}} F(C)$.

In the poset of subsets we get for $F \equiv 1$

$$\sum_{Y} |\mathscr{F}(Y)|\,(n - |Y| - 1)!\,|Y|! = n!$$

and

$$\mathscr{F}(Y) = \{X \in \mathscr{U} \setminus \mathscr{M} : X \supset Y,\ |X - Y| = 1\}.$$

Therefore

$$\sum_{Y} \frac{|\mathscr{F}(Y)|}{|Y| \binom{n}{|Y|}} = 1. \tag{5.3}$$

Actually, by looking at the upset $\mathscr{U}' = \mathscr{U} - \mathscr{M}$ we see that the only difference between the two identities is that we look at first entrances in the second and at last exits in the first. Other equivalent formulations can be obtained by looking at downsets rather than at upsets. *Moreover, we can give a general method for producing identities*: For any *blocking set* $\mathscr{B} \subset \mathscr{P}$, that is every $C \in \mathscr{C}$ meets an element $B \in \mathscr{B}$, we can count the saturated chains according to their *first entrance* (*or last exit*) in $\mathscr{B}$. Whether nice identities come up this way depends on the structures of $\mathscr{P}$ and $\mathscr{B}$. Insofar we have considered only blocks, which are upsets. For arbitrary $\mathscr{A} \subset \mathscr{P}$ we have set up identities via associated upsets $\mathscr{U} = \mathscr{U}_{\mathscr{A}}$. There are many "hull operations" which assign blocking sets to arbitrary $\mathscr{A}$ and which could have been used in place of $\mathscr{U}_{\mathscr{A}}$. We analyse now the identity in Theorem 6 in special cases.

EXAMPLE 1. THE α-REGULAR TREE $\mathscr{T}_{\alpha,n}$ OF DEPTH n. Here the rank function is the length of the path from the top to X. For any $\mathscr{A} \subset \mathscr{T}_{\alpha,n}$ define the upset $\mathscr{U} = \{X \in \mathscr{T}_{\alpha,n} : X \geqslant A \text{ for some } A \in \mathscr{A}\}$. By Theorem 6

$$\sum_{U \in \mathscr{U}} \sum_{e \in \mathscr{E}(U)} \alpha^{n-r(U)-1} = \alpha^n$$

or

$$\sum_{U \in \mathscr{U}} \frac{|\mathscr{E}(U)|}{\alpha \cdot \alpha^{r(U)}} = 1. \tag{5.3}$$

If $\mathscr{A}$ is an antichain, then we have $|\mathscr{E}(U)| = \alpha$ for $U \in \mathscr{A}$ and therefore

$$\sum_{U \in \mathscr{A}} \frac{1}{\alpha^{r(U)}} \leqslant 1, \tag{5.4}$$

which is Kraft's inequality [11]. If, in addition, $\mathscr{A}$ is blocking, then $\sum_{U \in \mathscr{A}} (1/\alpha^{r(U)}) = 1$.

EXAMPLE 2: CASCADE GRAPHS. It is a small exercise to show that Renyi's inequality [12] in his uniform flow theorem can be replaced by an identity.

EXAMPLE 3. SUBSPACES OVER $GF(q)$. Let $\mathscr{L}_n(q)$ denote the lattice of subspaces of a vector space of dimension n over the field $GF(q)$, ordered by inclusion. Then $\mathscr{L}_n(q)$ has a rank function $r(X) = \dim(X)$ and the number $N_k(\mathscr{L}_n(q))$ of subspaces of rank k is the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

Now for an $\mathscr{A}$ and the associated $\mathscr{U}$ in $\mathscr{L}_n(q)$

$$\sum_{U\in\mathscr{U}}\sum_{e\in\mathscr{E}(U)}\prod_{k=1}^{n-r(U)} N_1(\cdot\mathscr{L}_{n-r(U)+1-k}(q))\cdot\prod_{k=1}^{r(U)-1} N_1(\mathscr{L}_{r(U)-k}(q))$$
$$=\prod_{k=1}^{n} N_1(\mathscr{L}_{n-k+1}(q)),$$

that is

$$\sum_{U\in\mathscr{U}}|\mathscr{E}(U)|\prod_{k=1}^{n-r(U)}\frac{q^{n-r(U)+1-k}-1}{q-1}\prod_{k=1}^{r(U)-1}\frac{q^{r(U)-k}}{q-1}=\prod_{k=1}^{n}\frac{q^{n-k+1}-1}{q-1}$$

or equivalently

$$\sum_{U\in\mathscr{U}}|\mathscr{E}(U)|\frac{\prod_{k=1}^{r(U)-1}(q^{r(U)-k}-1)(q-1)}{\prod_{k=1}^{r(U)}(q^{n-k+1}-1)}=1$$

and finally

$$\sum_{U\in\mathscr{U}}|\mathscr{E}(U)|\frac{1}{\begin{bmatrix}n\\r(U)\end{bmatrix}_q\cdot\begin{bmatrix}r(U)\\1\end{bmatrix}_q}\equiv 1.$$

Since the calculation of $|\mathscr{E}(U)|$ is somewhat lengthy, we give only the result. For this let $\{A: U\supset A\in\mathscr{A}\}=\{A_1, ..., A_s\}$ and for $T\subset\{1, 2, ..., s\}$ denote by $A_{(T)}$ the subspace spanned by $\bigcup_{t\in T}A_t$, then by inclusion–exclusion

$$|\mathscr{E}(U)|=\sum_{t=0}^{s}(-1)^t\sum_{\substack{|T|=t\\T\subset\{1,2,...,s\}}}\frac{q^{n-r(A_{(T)})}-1}{q-1}.$$

6. Two Further Proofs of Theorem 1

In spite of the fact that the proof given in Section 1 is so perspicuous and short, we present here two proofs, which preceded it. We hope that this will be appreciated by readers interested in "proof techniques." The first proof is by induction on the ground set Ω and does not seem to have a parallel in Sperner Theory. The second proof resembles Sperner's pushing operations [1].

Proof I. Obviously $I_n(\mathscr{A})=\sum_{i=1}^{n}(W_{\mathscr{A}}^{(i)}/i\binom{n}{i})$ equals one for $n=1$. Assuming this for $|\Omega|<n$ we prove it for n. Clearly $W_{\mathscr{A}}^{(n)}=|T|$, if

For every $k \in \Omega = \{1, 2, ..., n\}$ and $\Omega \setminus \{k\}$ we define

$$\mathscr{A}_{(k)} = \{A \in \mathscr{A} : A \subset \Omega \setminus \{k\}\} \tag{6.1}$$

and for $X \subset \Omega \setminus \{k\}$

$$W_{\mathscr{A}(k)}(X, n-1) = \left| \bigcap_{X \subset A \in \mathscr{A}(k)} A \right|. \tag{6.2}$$

Then obviously for $X \subset \Omega - \{k\}$

$$W_{\mathscr{A}(k)}(X, n-1) = W_{\mathscr{A}}(X). \tag{6.3}$$

Since $\bigcap_{A \in \mathscr{A}(k)} A \supset T$, it suffices to consider only sets $X \supset T$. Therefore

$$I^*_{n-1}(\mathscr{A}_{(k)}) = \sum_{\Omega \setminus \{k\} \supset X \supset T} \frac{W_{\mathscr{A}(k)}(X, n-1)}{|X| \binom{n-1}{|X|}}$$

$$= \begin{cases} 1, & \text{if } \ T \subset \Omega \setminus \{k\} \\ 0, & \text{if } \ T \not\subset \Omega \setminus \{k\}! \end{cases}$$

and

$$\sum_{k=1}^{n} I^*_{n-1}(\mathscr{A}_{(k)}) = n - t. \tag{6.4}$$

On the other hand

$$\sum_{k=1}^{n} I^*_{n-1}(\mathscr{A}_{(k)}) = \sum_{k=1}^{n} \sum_{\Omega \setminus \{k\} \supset X} \frac{W_{\mathscr{A}(k)}(X, n-1)}{|X| \binom{n-1}{|X|}}$$

$$= \sum_{j=1}^{n-1} \sum_{|X|=j} \sum_{k: \Omega \setminus \{k\} \supset X} \frac{W_{\mathscr{A}(k)}(X, n-1)}{j \binom{n-1}{j}}$$

$$= \sum_{j=1}^{n-1} \sum_{|X|=j} \frac{(n-j)\, W_{\mathscr{A}}(X)}{j \binom{n-1}{jB}}.$$

This and (6.4) imply

$$\sum_{j=1}^{n-1} \sum_{|X|=j} \frac{(n-j)\, W_{\mathscr{A}}(X)}{nj \binom{n-1}{j}} = 1 - \frac{t}{n}$$

or

$$\sum_{j=1}^{n-1} \sum_{|X|=j} \frac{W_{\mathscr{A}}(t)}{j\binom{n}{j}} + \frac{t}{n} = 1.$$

Since $t/n = W_{\mathscr{A}}^{(n)}/n\binom{n}{n}$, we have established $I_n(\mathscr{A}) = 1$.

Proof II. We need a few definitions. For $\mathscr{T} \subset \mathscr{P}_r$ we set

$$\mathscr{T}^+ = \{X \in \mathscr{P}_{r+1} : X \supset Y \text{ for some } Y \in \mathscr{T}\}. \tag{6.5}$$

Also, for $A \in \mathscr{P}_r$ and $B \in \mathscr{P}_{r+1}$ we set

$$d(A, B) = \begin{cases} 1, & \text{if } A \subset B \\ 0, & \text{if } A \not\subset B. \end{cases} \tag{6.6}$$

For the given $\mathscr{A}$ let now l be the smallest number with $\mathscr{A}_l = \mathscr{A} \cap \mathscr{P}_l \neq \varnothing$. We introduce now an operation on families of subsets, which leaves I_n invariant. Repeated application of this operation leads finally to $\{\Omega\}$, for which I_n takes the value 1, because

$$W_{\{\Omega\}}(X) = \begin{cases} 0, & \text{if } X \neq \Omega \\ n, & \text{if } X = \Omega. \end{cases}$$

We define now the "*pushing up*" *transformation* S by

$$S(\mathscr{A}) = (\mathscr{A} \setminus \mathscr{A}_l) \cup \mathscr{A}_l^+ \tag{6.7}$$

and write $\mathscr{A}' = S(\mathscr{A})$.

We analyse now the effect of this transformation. Clearly, $Y_{\mathscr{A}} = Y_{\mathscr{A}'}$ for $|Y| \leqslant l-1$. Next we establish this identity for $|Y| \geqslant l+2$. It suffices to show that

$$\bigcap_{Y \supset A \in \mathscr{A}_l} A = \bigcap_{Y \supset B \in \mathscr{A}_l^+} B.$$

Since $|Y| \geqslant l+2$, for $A \subset Y$ there are $A \cup \{a\}$, $A \cup \{b\} \subset Y$ with intersection A. Therefore $\bigcap_{Y \supset A \in \mathscr{A}_l} A \supset \bigcap_{Y \supset B \in \mathscr{A}_l^+} B$ and the opposite inclusion is obvious. We calculate now the contributions of the set of cardinality l or $l+1$ to $I_n(\mathscr{A})$ and $I_n(\mathscr{A}')$. Clearly

$$W_{\mathscr{A}}^{(l)} = l|\mathscr{A}_l|, \quad W_{\mathscr{A}'}^{(l)} = 0 \tag{6.8}$$

$$W_{\mathscr{A}}^{(l+1)} = (l+1)\,|\mathscr{A}_{l+1} \setminus \mathscr{A}_l^+| + \sum_{Y \in \mathscr{A}_l^+} l+1-d(Y, \mathscr{A}_l), \tag{6.9}$$

where $d(Y, \mathscr{A}_l) = \sum_{A \in \mathscr{A}_l} d(Y, A)$, and

$$W_{\mathscr{A}}^{(l+1)} = (l+1)\,|\mathscr{A}_{l+1} \backslash \mathscr{A}_l^+| + (l+1)\,|\mathscr{A}_l^+|. \tag{6.10}$$

Now from these results

$$\begin{aligned} I_n(\mathscr{A}) - I_n(\mathscr{A}') &= \frac{|\mathscr{A}_l|}{\binom{n}{l}} - \sum_{Y \in \mathscr{A}_l^+} d(Y, \mathscr{A}_l) \frac{1}{(l+1)\binom{n}{l+1}} \\ &= \sum_{A \in \mathscr{A}_l} \left[\frac{1}{\binom{n}{l}} - \frac{d(\mathscr{A}_l^+, A)}{(l+1)\binom{n}{l+1}} \right] \\ &= \sum_{A \in \mathscr{A}_l} \left[\frac{1}{\binom{n}{l}} - \frac{n-l}{(l+1)\binom{n}{l+1}} \right] = 0. \end{aligned}$$

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