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ON CLOUD-ANTICHAINS AND RELATED CONFIGURATIONS

R. AHLSWEDE and Z. ZHANG
Universität Bielefeld, Fakultät für Mathematik, Postfach 8640, 4800 Bielefeld 1, FRG

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We introduce the concept of a cloud-antichain, which is a natural generalization of antichains in partially ordered sets, and solve some seemingly basic extremal problems for them.

Following the discovery of our identity (in [1]) we found the 'missing term', which converts Bollobás' inequality into an identity.

It immediately yields the uniqueness in the Sperner Theorem for unrelated chains of subsets, which are an example of a cloud-antichain.

Introduction

An antichain $\mathscr{A}$ in a poset $\mathscr{P}$ can be characterized as a subset without chains of length 1. This lead Erdős [7] to consider subsets without chains of length k. Instead of excluding certain chains one may exclude other configurations. This general point of view was taken in [5]. Here we give another generalisation of the concept of an antichain, which covers also cases which have been studied already (see [2]).

Definition 1. $(\mathscr{B}_i)_{i=1}^N$ is a *cloud-antichain* (CAC) of length N in $\mathscr{P}$, if

(a) $\mathscr{B}_i \subset \mathscr{P}$ for $i = 1, 2, \ldots, N$,

(b) for all pairs (i, j) of indices any members $B_i \in \mathscr{B}_i$ and $B_j \in \mathscr{B}_j$ are not comparable.

The sets $\mathscr{B}_1$ are also called clouds. They are obviously disjoint. Antichains are characterized by the property that all clouds have size one. This suggests the next notion.

Definition 2. $(\mathscr{B}_i)_{i=1}^N$ is a k-cloud-antichain (k-CAC), if $|\mathscr{B}_i| = k$ for $i = 1, 2, \ldots, N$.

In [2] the maximal length of k-CAC's with each $\mathscr{B}_i$ being a chain has been determined in case $\mathscr{P}$ is the poset of subsets of a finite set.

We prove uniqueness of the optimal configuration found in [2]. The proof is based on the approach of [1] to lift LYM-type inequalities to identities. Here it is

Bollobas' inequality [3–4], which we are able to improve to an identity. Another such example is an inequality in [5].

Next we consider incomparable varying length chains of subsets and determine their maximal length as well as their maximal weight under a canonical weight function. We also take a look at cloud-chains.

Definition 3. $(\mathscr{C}_i)_{i=1}^M$ is a cloud-chain (CC) of length M in $\mathscr{P}$, if
(a) $\mathscr{C}_i \subset \mathscr{P}$ for $i = 1, 2, \ldots, M$,
(b) for all members $C_i \in \mathscr{C}_i$ and $C_j \in \mathscr{C}_j$ we have $C_i < C_j$, if $i < j$.

Whereas CAC's are complex objects, it is easy to analyze CC's. In order to better understand CAC's we investigate here the case $N = 2$ and maximize the sizes of the clouds. We get a tight bound for $|\mathscr{B}_1|\,|\mathscr{B}_2|$. Amazingly, a neat proof can be based on an ancient inequality [6]. This encouraged us to consider other two family extremal problems, namely, mutually intersecting systems (MIS) $(\mathscr{A}, \mathscr{B})$ defined by the properties $\mathscr{A}, \mathscr{B} \subset 2^\Omega$, $\Omega = \{1, 2, \ldots, n\}$; $A \cap B \neq \emptyset$ for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and mutually comparable systems (MCS) $(\mathscr{A}, \mathscr{B})$, for which $A \supset B$ or $A \subset B$ holds for all $A \in \mathscr{A}$, $B \in \mathscr{B}$.

For the ranges of triples $(|\mathscr{A}|, |\mathscr{B}|, |\mathscr{A} \cap \mathscr{B}|)$ of cardinalities we establish asymptotically optimal results for MIS's and exact results for MCS's.

Part I: Identities

1. An identity behind Bollobás' inequality

First we generalize our basic identity [1] in the same way as Bollobás' inequality generalizes the LYM-inequality. For this we use an elementary result, which can be found with a short proof in [2].

Lemma 1. *For two sets $A, B \subset \Omega = \{1, 2, \ldots, n\}$ with $A \subset B$ exactly*

$$n!\binom{n - |B \setminus A|}{|A|}^{-1}$$

maximal chains in $\mathscr{P} = 2^\Omega$ the power set of Ω, meet $\{C: A \subset C \subset B\}$.

Theorem 1. *Suppose $A_1 \subset B_1, \ldots, A_l \subset B_l$ are subsets of $\{1, 2, \ldots, n\}$ such that $A_i \not\subset B_j$ for $i \neq j$, then*

$$\sum_{i=1}^{l}\binom{n - |B_i \setminus A_i|}{|A_i|}^{-1} + \sum_{X \in \mathscr{O}} \frac{W_A(X)}{|X|}\binom{n}{|X|}^{-1} = 1,$$

where

$$\mathscr{O} = \{Y: \nexists i: A_i \subset Y \subset B_i\} \quad \text{and} \quad W_{\mathscr{A}}(X) = \left|\bigcap_{A_i \subset X} A_i\right|.$$

Notice that the special case $A_i = B_i$ for $i = 1, \ldots, l$ is our identity of [1] and that a consequence is Bollobás' inequality

$$\sum_{i=1}^{l} \binom{n - |B_i \setminus A_i|}{|A_i|}^{-1} \leq 1. \tag{1.1}$$

Proof. We repeat the proof of our basic identity in [1] so that the meaning of occurring terms is clear.

For $\mathcal{U} = \{X \in \mathcal{P}: X \supset A \text{ for some } A \in \mathcal{A}\}$ the number of saturated chains leaving $\mathcal{U}$ at U is

$$(n - |U|)!\, W_A(U)(|U| - 1)!.$$

Since the sets $\{X: A_i \subset X \subset B_i\}$ are disjoint, we have therefore

$$\sum_{i=1}^{l} \sum_{A_i \subset U \subset B_i} (n - |U|)!\, |A_i|(|U| - 1)! + \sum_{U \in \mathcal{O}} (n - |U|)!\, W_{\mathcal{A}}(U)(|U| - 1)! = n!.$$

Furthermore, by Lemma 1

$$\sum_{A_i \subset U \subset B_i} (n - |U|)!\, |A_i|(U - 1)! = n! \binom{n - |B_i \setminus A_i|}{|A_i|}^{-1}$$

and the result follows. □

2. An identity related to Lemma 4.5 of [5]

In $\{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ let $(i, j) \leq (i', j')$ exactly if $i \leq i'$ and $j \leq j'$.

An auxiliary result in [5] is:

If $I \subset \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ is an antichain, then

$$\sum_{(i,j) \in I} \binom{m}{i}\binom{n}{j}\binom{m+n}{i+j}^{-1} \leq 1. \tag{2.1}$$

The authors of [5] express the opinion that this inequality is interesting in itself. Actually, it is the LYM-inequality for the poset defined. To see this just count the saturated chains through

$$(i_t, j_t) \in I = \{(i_1, j_1), \ldots, (i_T, j_T)\}.$$

Their number is

$$\binom{m + n - i_t - j_t}{m - i_t}\binom{i_t + j_t}{i_t}$$

and since the total number of saturated chains is $\binom{m+n}{m}$, we get

$$\sum_{t=1}^{T} \binom{m + n - i_t - j_t}{m - i_t}\binom{i_t + j_t}{i_t}\binom{m+n}{m}^{-1} \leq 1.$$

Now observe that the tth summand equals

$$\frac{(m+n-i_t-j_t)!\,(i_t+j_t)!}{(m+n)!}\cdot\frac{m!}{(m-i_t)!\,i_t!}\cdot\frac{n!}{(n-j_t)!\,j_t!},$$

which is just

$$\binom{m}{i_t}\binom{n}{j_t}\binom{m+n}{i_t+j_t}^{-1}.$$

This was to be shown.

Next we derive the following identity.

Theorem 2

$$\sum_{t=1}^{T}\left[\sum_{l>j_t}^{j_{t+1}-1}\frac{\binom{m}{i_t}i_t\binom{n}{l}}{(i_t+l)\binom{m+n}{i_t+l}}+\sum_{k>i_t}^{i_{t-1}-1}\frac{\binom{m}{k}j_t\binom{n}{j_t}}{(k+j_t)\binom{m+n}{k+j_t}}+\frac{\binom{m}{i_t}\binom{n}{j_t}(i_t+j_t)}{\binom{m+n}{i_t+j_t}(i_t+j_t)}\right]\equiv 1.$$

Proof. We follow again the approach of [1]. To

$$I=\{(i_t,j_t):1\leqslant t\leqslant T\}$$

we assign the upset

$$\mathcal{U}=\{(i,j):(i,j)\geqslant(i_t,j_t)\text{ for some }t\}$$

and we count again the saturated chains according to their exits from $\mathcal{U}$.

W.l.o.g. we assume $i_1\geqslant i_2\geqslant\cdots\geqslant i_T$. Then necessarily $j_1\leqslant j_2\leqslant\cdots\leqslant j_T$, because I is an antichain.

Therefore exits from $\mathcal{U}$ occur in three kinds of elements:

(a) (i_t,j) with $j_t<i\leqslant j_{t+1}-1$,

(b) (i,j_t) with $i_t<i\leqslant i_{t-1}-1$,

(c) (i_t,j_t).

Counting the chains accordingly and rearranging binomial coefficients gives the identity. □

Part II: Uniqueness via identities

3. Uniqueness for equality in Bollobás' inequality

Theorem 3. *Suppose that $A_1\subset B_1,\ldots,A_N\subset B_N$ are subsets of Ω such that $A_i\not\subset B_j$ for $i\neq j$, then*

$$\sum_{i=1}^{N}\binom{n-|B_i\backslash A_i|}{|A_i|}^{-1}=1$$

implies the existence of a $D \subset \Omega$ *and a* t *with*

$$\{A_i: 1 \leq i \leq N\} = \{A: A \subset D, |A| = t\} \quad \text{and} \quad B_i = A_i \cup \bar{D}.^1 \tag{3.1}$$

Proof. For a system reaching equality we know from Theorem 1 that

$$W_{\mathcal{A}}(X) = 0 \quad \text{for } X \in \mathcal{O}. \tag{3.2}$$

Let $t = \min_i |A_i|$ and without loss of generality $|A_1| = t$. For $y \in \bar{B}_1$ consider $A_1 \cup \{y\}$ and notice that $A_1 \cup \{y\} \in \mathcal{O}$. Therefore $W_{\mathcal{A}}(A_1 \cup \{y\} = 0)$ and all of the t-subsets of $A_1 \cup \{y\}$ with 1 element in $\bar{B}_1$ and $t-1$ elements in A_1 are in $\mathcal{A}$. We proceed now inductively. Suppose we know already that all t-subsets C with $|C \cap \bar{B}_1| = r$, $|C \cap A_1| = t - r$ are in $\mathcal{A}$, we show then that all t-subsets D with $|D \cap \bar{B}_1| = r + 1$, $|D \cap \bar{B}_1| = r + 1$, $|D \cap A_1| = t - r - 1$ are in $\mathcal{A}$.

For this notice that every $t+1$ subset E with $|E \cap \bar{B}_1| = r + 1$, $|E \cap A_1| = t - r$ contains by induction hypothesis at least $r + 1$ members of $\mathcal{A}$ (with t elements). Since $r + 1 \geq 2$, by the unrelatedness assumption necessarily $E \in \mathcal{O}$ and thus $W_A(E) = 0$.

Hence all t-subsets of E are in $\mathcal{A}$ and we have completed the induction: all t-subsets of $A_1 \cup \bar{B}_1$ are in $\mathcal{A}$. Let us call this set of subsets $\mathcal{A}'$.

Clearly,

$$|\mathcal{A}'| = \binom{n - |B_1 \backslash A_1|}{|A_1|}.$$

Furthermore, for every i by unrelatedness $B_i = A_i \cup C_i$ with $C_i \subset B_1 \backslash A_1$. Now any member A_j of $\mathcal{A}'$ could take the role of A_1 and therefore

$$B_j \backslash A_j = B_1 \backslash A_1 \quad \text{for } A_j \in \mathcal{A}'.$$

Thus

$$1 \geq \sum_{A_j \in \mathcal{A}'} \frac{1}{\binom{n - |B_i \backslash A_i|}{|A_i|}} = \frac{|\mathcal{A}'|}{\binom{n - |B_1 \backslash A_1|}{|A_1|}} = 1 \quad \text{and} \quad \mathcal{A} = \mathcal{A}'.$$

4. Uniqueness in the Sperner Theorem on unrelated chains of subsets by Griggs, Stahl and Trotter

A family of sets $\{A(i, j) \subset \{1, 2, \ldots, n\}: 1 \leq i \leq N, 0 \leq j \leq k\}$ satisfying

$$\text{for all } i, \quad A(i, 0) \subset A(i, 1) \subset \cdots \subset A(i, k), \tag{4.1}$$

$$\text{for all } i, i', j, j', \text{ with } i \neq i', \quad A(i, j) \not\subset A(i', j') \tag{4.2}$$

is called a collection of unrelated chains with $k + 1$ sets each. Denote the maximal value of N for such collections by $f_k(n)$.

[1] The bar denotes complementation in the ground set $\{1, 2, \ldots, n\}$. In later sections it also denotes complementation in the Minkowski sense for families of sets.

Theorem 2 of [2] states

$$f_k(n)=\binom{n-k}{\lfloor (n-k)/2\rfloor}. \tag{4.3}$$

The following collection achieves this optimum.

The sets $A(i,0)$ are the t-subsets of $\{k+1,\dots,n\}$ and for $j\geq 1$, $A(i,j)=A(i,0)\cup\{1,\dots,j\}$. Here t equals $\lfloor(n-k)/2\rfloor$ or $\lceil(n-k)/2\rceil$.

Notice that instead of speaking about a chain we could equivalently consider the family of sets between its first and last member. Thus for the above collection

$$A_i=A(i,0),\qquad B_i=A_i\cup\{1,\dots,k\}$$

and the A_i's are the $\lfloor(n-k)/2\rfloor$-subsets (resp. $\lceil(n-k)/2\rceil$)-subsets) of $\{k+1,\dots,n\}$. We call this a canonical collection.

Theorem 4. *A collection with $f_k(n)$ unrelated chains of length k is canonical.*

Proof. Let $\{A_i\colon 1\leq i\leq N\}$; $N=f_k(n)$; where $\mathcal{A}_i=\{A(i,0),\dots,A(i,k)\}$, be an optimal collection of unrelated chains. We begin as in [2]

$$\binom{n-|B_i\backslash A_i|}{|A_i|}\leq\binom{n-k}{|A_i|}\leq\binom{n-k}{\lfloor (n-k)/2\rfloor}. \tag{4.4}$$

Hence,

$$f_k(n)=\sum_{i=1}^{N}1\leq\sum_{i=1}^{N}\binom{n-k}{(n-k)/2}\binom{n-|B_i\backslash A_i|}{|A_i|}^{-1}\leq\binom{n-k}{(n-k)/2} \tag{4.5}$$

by Bollobás' inequality

$$\sum_{i=1}^{N}\binom{n-|B_i\backslash A_i|}{|A_i|}^{-1}\leq 1. \tag{4.6}$$

Since there must be equality everywhere in (4.5), we also have identity in (4.6) and in (4.4), that is, for $i=1,\dots,N$

$$|A_i|=\left\lfloor\frac{n-k}{2}\right\rfloor\quad\text{or}\quad\left\lceil\frac{n-k}{2}\right\rceil,\qquad |B_i\backslash A_i|=k \tag{4.7}$$

By Theorem 1 the collection is canonical.

Part III: Further special cloud-antichains and cloud-chains

5. On incomparable varying length chains of subsets

Suppose now that in a CAC $(\mathcal{B}_i)_{i=1}^N$ all clouds $\mathcal{B}_i$ are chains. What can be said about $\sum_{i=1}^N|\mathcal{B}_i|$?

A tight upper bound can be derived by the approach of [2]. There all chains are assumed to have $k+1$ elements. We allow chains of varying lengths and let $\beta(n)$ be the maximal number of elements in $\bigcup_{i=1}^{N} \mathcal{B}_i$. Using Bollobás' inequality one obtains as in [2]

$$\beta(n)=\sum_{i=1}^{N} |\mathcal{B}_i| \leq \max_{l}(l+1)\binom{n-l}{\lfloor (n-l)/2 \rfloor}. \tag{5.1}$$

The value

$$f(n, l) \stackrel{\text{def}}{=} (l+1)\binom{n-l}{\lfloor (n-l)/2 \rfloor}$$

is achieved for a canonical collection. It remains to be analyzed for which values of l we have $f(n, l)=\beta(n)$. By an elementary calculation we get

$$f(n, l) f(n, l+1)^{-1}=\frac{l+1}{l+2} \cdot \frac{n-l}{\lceil (n-l)/2 \rceil}. \tag{5.2}$$

We show next that

$$f(n, 1)>f(n, t) \quad \text{for } 2 \leq t \leq n \text{ and all } n \geq 5. \tag{5.3}$$

Notice that by (5.2)

$$f(n, 1) f(n, t)^{-1}=\prod_{l=1}^{t} \frac{l+1}{l+2} \frac{n-l}{\lceil (n-l)/2 \rceil}=\frac{2}{t+2} \prod_{l=1}^{t} \frac{n-l}{\lceil (n-l)/2 \rceil}.$$

For n odd therefore

$$f(n, 1) f(n, t)^{-1} \geq \frac{2}{t+2} 2^{\lceil t/2 \rceil} \cdot \frac{n-2}{\lceil (n-2)/2 \rceil} \geq \frac{3}{t+2} 2^{\lceil t/2 \rceil}>1.$$

Similarly for even $n \geq 6$

$$f(n, 1) f(n, t)^{-1} \geq \frac{2}{t+2} 2^{\lfloor t/2 \rfloor} \frac{n-1}{\lceil (n-1)/2 \rceil}>\frac{3}{t+2} 2^{\lfloor t/2 \rfloor} \geq 1$$

and (5.3) holds.

The optimal values in cases $n \leq 4$ are obtained by inspection.

$$f(1,1)=(1+1)\binom{1-1}{0}=2,$$

$$f(2,2)=(1+2)\binom{2-2}{0}=3,$$

$$f(3,1)=2 \cdot \binom{3-1}{1} \quad =4,$$

$$f(3,3)=4 \cdot 1 \quad =4,$$

$$f(4,0)=1 \cdot \binom{4}{2} \quad =6,$$

$$f(4,1)=2 \cdot \binom{4-1}{1} \quad =6,$$

$$f(4,2)=3 \cdot \binom{4-2}{1} \quad =6.$$

Thus for $n=3, 4$ there are more than one solution. Also, $l=1$ is only for the case $n=2$, where $f(2, 1)=2\binom{1}{0}=2<3$, not optimal. Furthermore, by Pascal's identity $f(n, 0)=f(n, 1)$ for n even, whereas $f(n, 1)>f(n, 0)$ for odd n.

We summarize our findings.

Theorem 5. *For $n \neq 2$*

$$\beta(n)=2\binom{n-2}{\lceil (n-2)/2\rceil}.$$

There are several non-isomorphic solutions for $n=3, 4$. There are two solutions for $n \geqslant 4$ even:

$$f(n, 0)=f(n, 1).$$

6. On incomparable chains with canonical weights

Next we study the function

$$R(n)=\max_{(\mathcal{B}_i)_{i\in I}} \sum_i \sum_{X\in\mathcal{B}_i} |X|.$$

Such questions involving $\operatorname{rang}(X)=|X|$ have been studied for ordinary antichains (cf. [9]).

For any $c:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ we can write Bollobás' inequality in the form

$$\sum_i \frac{c(|A_i|, |B_i|)}{c(|A_i|, |B_i|)\binom{n-|B_i\backslash A_i|}{|A_i|}} \leqslant 1, \tag{6.1}$$

where A_i is the minimal and B_i the maximal element in the chain $\mathcal{B}_i$. If we define now c by

$$c(l, l+\Delta)=(\Delta+1)\left(l+\frac{\Delta}{2}\right), \quad \text{then} \quad c(|A_i|, |B_i|)=\sum_{X\in\mathcal{B}_i} |X| \tag{6.2}$$

and from (6.1)

$$\sum_i \sum_{X\in\mathcal{B}_i} |X| \leqslant \max_{l+\Delta\leqslant n} c(l, l+\Delta)\binom{n-\Delta}{l}. \tag{6.3}$$

Therefore

$$R(n)\leqslant \max_{l+\Delta\leqslant n} (\Delta+1)\left(l+\frac{\Delta}{2}\right)\binom{n-\Delta}{l}. \tag{6.4}$$

On the other side the expression to the right can be achieved by the configurations described in Section 4 and we have equality in (6.4). The technical work starts now. What are optimal values for l, Δ?

Theorem 6.

$$R(n) = (2\lceil (n-1)/2 \rceil + 1)\binom{n-1}{\lceil (n-1)/2 \rceil}.$$

The proof is based on three properties of the function

$$g(n, \Delta) \triangleq \max_{l:\Delta+l\leqslant n} (\Delta+1)\left(l+\frac{\Delta}{2}\right)\binom{n-\Delta}{l}. \tag{6.5}$$

Lemma 2.

(a) $g(n, 0) = \binom{n}{\lceil n/2 \rceil} \cdot \lceil n/2 \rceil,$

(b) $g(n, 1) = \binom{n-1}{\lceil (n-1)/2 \rceil} \cdot (2\lceil (n-1)/2 \rceil + 1),$

(c) $g(n, \Delta+1) \leqslant g(n, \Delta)$ *for* $\Delta \geqslant 1$.

Proof. Since

$$\binom{n-1}{l}(2l+1) = \binom{n-1}{l}2l + \binom{n-1}{l}$$
$$\leqslant \binom{n-1}{\lceil (n-1/2) \rceil}2\lceil (n-1)/2 \rceil + \binom{n-1}{\lceil (n-1)/2 \rceil},$$

(b) follows and (a) is obvious.

For the proof of (c) set now

$$s = \left\lceil \frac{n-\Delta}{2} \right\rceil. \tag{6.6}$$

Then

$$g(n, \Delta) = \binom{n-\Delta}{s}\left[(\Delta+1)\left(s+\frac{\Delta}{2}\right)\right]$$

and by Pascal's identity

$$g(n, \Delta) = \left[\binom{n-(\Delta+1)}{s} + \binom{n-(\Delta+1)}{s-1}\right]\left[(\Delta+1)\left(s+\frac{\Delta}{2}\right)\right]. \tag{6.7}$$

Case: $n - \Delta$ *even.*

$$g(n, \Delta+1) = \binom{n-(\Delta+1)}{s}\left[(\Delta+2)\left(s+\frac{\Delta+1}{2}\right)\right].$$

Notice that $2s - 1 = n - \Delta - 1$ and thus $(n-(\Delta+1)/s) = (n-(\Delta+1)/s-1)$. Therefore

$$g(n, \Delta) = 2\binom{n-(\Delta+1)}{s}\left[(\Delta+1)\left(s+\frac{\Delta}{2}\right)\right].$$

It suffices to show that

$$2(\Delta+1)\left(s+\frac{\Delta}{2}\right) \geq (\Delta+2)\left(s+\frac{\Delta+1}{2}\right)$$

or (equivalently) that

$$\Delta s+\frac{\Delta^2}{2} \geq \frac{\Delta}{2}+1. \tag{6.8}$$

The case $\Delta=1$, $s=0$, that is $n=1$, does not occur with $\Delta+1$.

Otherwise (6.8) holds for all $\Delta \geq 1$ and $s \geq 0$.

Case: $n-\Delta$ odd.

Now we have

$$\left\lceil \frac{n-(\Delta+1)}{2} \right\rceil = s-1$$

and

$$g(n, \Delta+1) = \binom{n-(\Delta+1)}{s-1}\left[(\Delta+2)\left((s-1)+\frac{\Delta+1}{2}\right)\right].$$

Using (6.7) we get

$$g(n, \Delta)-g(n, \Delta+1) = \binom{n-(\Delta+1)}{s}\left[(\Delta+1)\left(s+\frac{\Delta}{2}\right)\right] - \binom{n-(\Delta+1)}{s-1}[s-1].$$

Since

$$(\Delta+1)\left(s+\frac{\Delta}{2}\right) \geq 2s \geq 2(s-1),$$

it suffices to show that

$$\binom{n-(\Delta+1)}{s} \geq \binom{n-(\Delta+1)}{s-1}.$$

By our assumptions this amounts to showing that

$$\binom{2k}{k+1} \geq \frac{1}{2}\binom{2k}{k} \quad \text{or that} \quad \frac{k}{k+1} \geq \tfrac{1}{2} \quad \text{for } k \geq 1.$$

The Theorem states that $g(n, 1)$ is optimal. By (c) in Lemma 1 the only competitor left is $g(n, 0)$. Now one readily verifies that

$$g(2m+1, 0) = g(2m+1, 1), \qquad g(2m, 0) < g(2m, 1). \tag{6.9}$$

At any rate, $g(n, 1)$ cannot be defeated. □

7. On cloud-chains

We call a cloud-chain (CC) $(\mathscr{C}_i)_{i=1}^M$ a k-CC, if

$$|\mathscr{C}_i| = k \quad \text{for } i=1, 2, \ldots, M. \tag{7.1}$$

Let $M(n, k)$ be the maximal M for which a k-CC $(\mathscr{C}_i)_{i=1}^M$ exists in $2^{\{1,2,\dots,n\}}$. This quantity may be termed the maximal length of cloud chains with parameters k and n.

Theorem 7. *For* $n \in \mathbb{N}$ *and* $1 \leqslant k \leqslant 2^n$

$$M(n,k)=\begin{cases} \left\lfloor \dfrac{n+1}{l} \right\rfloor & \text{if } 2^{l-1}<k<2^l, \\ \max\left(2\left\lfloor \dfrac{n+1}{2l+1}\right\rfloor, 2\left\lfloor \dfrac{n+1-l}{2l+1}\right\rfloor+1\right) & \text{if } k=2^l. \end{cases}$$

Proof. We can associate with a CC $(\mathscr{C}_i)_{i=1}^M$ an ordinary chain $(C_i)_{i=1}^M$ defined by

$$C_i = \bigcup_{C \in \mathscr{C}_i} C. \tag{7.2}$$

Notice that

$$C_i \subset C \quad \text{for all } C \in \mathscr{C}_{i+1}. \tag{7.3}$$

C_i may be the largest element in $\mathscr{C}_i$ and it also can be the smallest element in $\mathscr{C}_{i+1}$. For a k-CC necessarily

$$|C_{i+1}| - |C_i| \geqslant \lceil \log_2 k \rceil \tag{7.4}$$

and therefore

$$M \leqslant \left\lfloor \frac{n+1}{\lceil \log_2 k \rceil} \right\rfloor. \tag{7.5}$$

In case $2^{l-1} < k < 2^l$ we have $\lceil \log_2 k \rceil = l$ and equality can be achieved in (7.5) with the following CC: Start with $\mathscr{C}_1$ as set of all proper subsets of an l-element set C_1, then define $\mathscr{C}_2 = \{C: C_1 \subset C \subsetneqq C_2\}$, where C_2 extends C_1 by l-elements, etc.

In case $k = 2^l$, C_1 has to be member of $\mathscr{C}_1$ and an $l+1$-element extension of C_1 is needed now for C_2. Here C_2 can be in $\mathscr{C}_3$ and thus an l-element extension of C_2 suffices now again for C_3. This procedure keeps alternating. Therefore

$$M(n, 2^l) = \max\left(\max_{t(2l+1) \leqslant n+1} 2t \ , \ \max_{t(2l+1)+l \leqslant n+1} 2t+1 \right) \tag{7.6}$$

and the result follows. □

Part IV: Basic extremal configurations involving two families

8. Cloud-antichains of length 2

Several two family extremal problems have been studied in the literature, however, to our knowledge not the following seemingly basic problem.

In case $N=2$ the CAC consists of two families, say $\mathscr{A}$ and $\mathscr{B}$, of subsets of $\mathscr{P}=2^{\Omega}$, with the property

$$\text{all } A \in \mathscr{A} \text{ and } B \in \mathscr{B} \text{ are incomparable.} \tag{8.1}$$

What can be said about the cardinalities of $\mathscr{A}$ and $\mathscr{B}$?

Theorem 8. *For $\Omega = \{1, 2, \ldots, n\}$ a* CAC $\{\mathscr{A}, \mathscr{B})$ *in $\mathscr{P}=2^{\Omega}$ satisfies*

(i) $|\mathscr{A}| \cdot |\mathscr{B}| \leq 2^{2n-4}$,
(ii) $\min\{|\mathscr{A}|, |\mathscr{B}|\} \leq 2^{n-2}$

and these bounds are tight.

Proof. Obviously (i) implies (ii).
Equalities occur for instance if

$$\mathscr{A}=\{X \in \mathscr{P}: 1 \in X, 2 \notin X\} \quad \text{and} \quad \mathscr{B}=\{x \in \mathscr{P}: 1 \notin X, 2 \in X\}.$$

There are other such configurations as for example for n even

$$\mathscr{A}'=\{X \in \mathscr{P}: 1 \in X, |X-\{1\}| \leq \lfloor (n-1)/2 \rfloor\},$$
$$\mathscr{B}'=\{x \in \mathscr{P}: 1 \notin X, |X| \geq \lceil (n-1)/2 \rceil\}.$$

Now we come to the heart of the matter. Define

$$\mathscr{A} \vee \mathscr{B}=\{A \cup B: A \in \mathscr{A}, B \in \mathscr{B}\} \tag{8.2}$$

and analogously $\mathscr{A} \wedge \mathscr{B}$.

A special case of the inequality in [6] says

$$|\mathscr{A}| \cdot |\mathscr{B}| \leq |\mathscr{A} \vee \mathscr{B}| \cdot |\mathscr{A} \wedge \mathscr{B}|. \tag{8.3}$$

The key observation now is that all four sets are disjoint. For $\mathscr{A}$ and $\mathscr{B}$ this follows from the incomparability. If now $A' \in (\mathscr{A} \vee \mathscr{B}) \cap \mathscr{A}$, then $A'=A \cup B$, which would imply that contrary to the assumption $B \subset A'$. Similarly $A' \in (\mathscr{A} \wedge \mathscr{B}) \cap \mathscr{A}$ implies $A'=A \cap B$ and thus $A' \subset B$, finally, $A' \in (\mathscr{A} \vee \mathscr{B}) \cap (\mathscr{A} \wedge \mathscr{B})$ yields $A \cup B = A'' \cap B''$ and thus $B \subset A''$.

Now, since $|\mathscr{A}|+|\mathscr{B}|+|\mathscr{A} \vee \mathscr{B}|+|\mathscr{A} \wedge \mathscr{B}| \leq 2^n$, by the inequality of the geometric and arithmetic means $|\mathscr{A}| \cdot |\mathscr{B}| \cdot |\mathscr{A} \vee \mathscr{B}| \cdot |\mathscr{A} \wedge \mathscr{B}| \leq (2^n/4)^4$, which, together with (8.3) implies the result. □

9. The asymptotic growth of mutually intersecting systems (MIS)

Recall the definition of an MIS in the Introduction. We speak of an MIS $(\mathscr{A}, \mathscr{B})$ as an (n, α)-MIS, if

$$|\mathscr{A} \cap \mathscr{B}| \leq \alpha \cdot 2^n; \quad \mathscr{A}, \mathscr{B} \subset 2^{\Omega}. \tag{9.1}$$

We determine here the optimal rate

$$\rho(\alpha)=\overline{\sup_{n}}\sup_{(\mathcal{A},\mathcal{B})(n,\alpha)\text{-MIS}}\frac{|\mathcal{A}|\,|\mathcal{B}|}{2^{2n}}. \tag{9.2}$$

Theorem 9. *With the abbreviation* $\gamma=\sqrt{1-3\alpha}$

$$\rho(\alpha)=\begin{cases}\frac{2\alpha}{3}+\frac{1}{18}+\frac{\gamma}{18}+\left(\frac{1}{9}+\frac{\gamma}{9}-\frac{2\alpha}{3}\right)\sqrt{\frac{5}{36}-\frac{\alpha}{3}-\frac{\gamma}{9}} & \text{for } 0\le\alpha\le\frac{1}{4},\\ \frac{1}{4} & \text{for } \frac{1}{4}\le\alpha\le 1.\end{cases}$$

In particular for $\alpha=0$, $\rho(0)=4/27$.

Proof. With the help of an ancient inequality we derive first an upper bound on $\rho(\alpha)$. This bound is described as the solution of an analytic optimisation problem. Next this problem will be solved. The parameters, for which the optimum is assumed, can be determined. Finally, using properties of these parameters we construct a sequence of MIS's, which approach in rate $\rho(\alpha)$.

The upper bound. Assume that $\mathcal{A}$ and $\mathcal{B}$ are a non-extendable (n,α)-MIS. Then for the set

$$\mathcal{C}=\{C: C\cap X\neq\emptyset \text{ for all } X\in\mathcal{A}\cup\mathcal{B}\} \tag{9.3}$$

necessarily

$$\mathcal{C}\supset\mathcal{A}\cap\mathcal{B}. \tag{9.4}$$

Furthermore, for the set

$$\bar{\mathcal{C}}=\{\bar{C}: C\in\mathcal{C}\} \tag{9.5}$$

we have

$$\bar{\mathcal{C}}\cap(\mathcal{A}\cup\mathcal{B})=\emptyset, \tag{9.6}$$

because otherwise by definition of $\mathcal{C}$ for some $\bar{C}$ $C\cap\bar{C}\neq\emptyset$, a contradiction.

Define now

$$\mathcal{A}^*=\mathcal{A}\setminus\mathcal{C},\qquad \mathcal{B}^*=\mathcal{B}\setminus C. \tag{9.7}$$

We derive a bound on $|\mathcal{A}|\cdot|\mathcal{B}|$ via the inequality

$$|\mathcal{A}^*|\cdot|\mathcal{B}^*|\le|\mathcal{A}^*\vee\mathcal{B}^*|\cdot|\mathcal{A}^*\wedge\mathcal{B}^*|. \tag{9.8}$$

Since $\mathcal{A}\vee\mathcal{B}\subset\mathcal{C}$ we have a fortiori

$$\mathcal{A}^*\vee\mathcal{B}^*\subset\mathcal{C}. \tag{9.9}$$

We show that also

$$\mathcal{A}^*\wedge\mathcal{B}^*\subset\bar{\mathcal{C}}. \tag{9.10}$$

Assume to the contrary that for some $A^* \in \mathscr{A}^*$ and $B^* \in \mathscr{B}^*$ we have $\overline{A^* \cap B^*} \notin \mathscr{C}$. Then by definition of $\mathscr{C}$ there exists an $X \in \mathscr{A} \cup \mathscr{B}$ (say $X \in \mathscr{A}$) with $\overline{A^* \cap B^*} \cap X = \emptyset$ and thus $X \subset A^* \cap B^*$. Now, since by definition of $\mathscr{B}^*$ we have $\mathscr{B}^* \notin \mathscr{C}$, there exists a $Y \in \mathscr{B}$ with $Y \cap B^* = \emptyset$. Therefore also $X \cap Y = \emptyset$, which contradicts the fact that $(\mathscr{A}, \mathscr{B})$ is an intersecting system.

From (9.8)–(9.10) we conclude that

$$|\mathscr{A}^*| \cdot |\mathscr{B}^*| \leq |\mathscr{C}| \cdot |\bar{\mathscr{C}}| = |\mathscr{C}|^2. \tag{9.11}$$

Now define

$$\mathscr{C}_1 = \mathscr{A} \cap \mathscr{C}, \qquad \mathscr{C}_2 = \mathscr{B} \cap \mathscr{C}.$$

By (9.7), $\mathscr{A} = \mathscr{A}^* \dot{\cup} \mathscr{C}_1$, $\mathscr{B} = \mathscr{B}^* \dot{\cup} \mathscr{C}_2$ and by (9.4), $\mathscr{A} \cap \mathscr{B} = \mathscr{C}_1 \cap \mathscr{C}_2$. Therefore

$$|\mathscr{A}| \cdot |\mathscr{B}| \leq (|\mathscr{A}^*| + |\mathscr{C}_1|)(|\mathscr{B}^*| + |\mathscr{C}_2|), \tag{9.12}$$

$$|\mathscr{C}_1| + |\mathscr{C}_2| \leq |\mathscr{C}| + \alpha \cdot 2^n. \tag{9.13}$$

Furthermore, the sets $\mathscr{A}^*$, $\mathscr{B}^*$, $\mathscr{C}$, and $\bar{\mathscr{C}}$ are disjoint. This is clear for $\mathscr{A}^*$, $\mathscr{B}^*$, and $\mathscr{C}$ by the definitions and follows for $\bar{\mathscr{C}}$ from (9.6) and the fact $\mathscr{A} \cup \mathscr{B} = \mathscr{A}^* \cup \mathscr{B}^* \cup \mathscr{C}$. We thus have

$$2|\mathscr{C}| + |\mathscr{A}^*| + |\mathscr{B}^*| \leq 2^n. \tag{9.14}$$

The inequalities (9.11)–(9.14) give after normalisation by 2^n

$$\rho(\alpha) \leq \beta(\alpha) \triangleq \sup_{(*)} (a + c_1)(b + c_2), \tag{9.15}$$

where the supremum is subject to the constraints

$$(*) \ ab \leq c^2, \qquad c_1 + c_2 = c + \alpha, \qquad a + b + 2c = 1,$$

$$\alpha \leq c_1, c_2, c \quad \text{and} \quad \text{all numbers are nonnegative.}$$

Here we have replaced $a + b + 2c \leq 1$ by the equality, because increasing c until equality is reached just increases $(a + c_1)(b + c_2)$ without violating the other constraints.

The optimisation. We distinguish two cases.

Case: $a + \alpha \leq b + c$.

The case $a + \alpha > b + c$ and symmetrically the case $a + c < b + \alpha$ are treated later. Therefore we can assume $a + c \geq b + \alpha$ and that there exist c_1, c_2 with

$$a + c_1 = b + c_2 = \frac{a + b + c + \alpha}{2} = \frac{1 - c + \alpha}{2}. \tag{9.16}$$

This shows that $c = \sqrt{ab}$ is an optimal choice within this case. Furthermore, if $a > b$ (resp. $a \leq b$) then by $\sqrt{a} + \sqrt{b} = 1$, $\sqrt{ab}$ decreases if a increases (resp. decreases) and we have reduced our case to $a + \alpha = b + c$ (resp. $a + c = b + \alpha$).

In any case we are, by symmetry, in the next case.

Case: $a+\alpha \geq b+c$.

From $ab = c^2 - \varepsilon$, $\varepsilon \geq 0$, and $a+b+2c=1$ we derive that a and b are the two roots of $(1-2c-x)\,x = c^2-\varepsilon$. W.l.o.g. $a \geq b$ and thus

$$a = \tfrac{1}{2} - c + \sqrt{\tfrac{1}{4} - c + \varepsilon}, \qquad b = \tfrac{1}{2} - c - \sqrt{\tfrac{1}{4} - c + \varepsilon}. \tag{9.17}$$

By our assumption and the arithmetic-geometric mean inequality it is clear that a best choice is $c_1 = \alpha$, $c_2 = c$. We get

$$\begin{aligned}(a+c_1)(b+c_2) &= (\tfrac{1}{2} - c + \sqrt{\tfrac{1}{4} - c + \varepsilon} + \alpha)(\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c + \varepsilon}) \\ &= c - \varepsilon + (c-\alpha)\sqrt{\tfrac{1}{4} - c + \varepsilon} - \tfrac{1}{2}(c-\alpha) \triangleq f(\varepsilon),\end{aligned}$$

and

$$f'(\varepsilon) = -1 + \frac{c-\alpha}{2\sqrt{\tfrac{1}{4} - c + \varepsilon}} \leq 0,$$

because by assumption $a - b = 2\sqrt{\tfrac{1}{4} - c + \varepsilon} \geq c - \alpha$.

Therefore the choice $\varepsilon(c) = \max(0, c - \tfrac{1}{4})$ is optimal. We are left with the maximisation in c of

$$g(c) \triangleq c - \varepsilon(c) + (c-\alpha)\sqrt{\tfrac{1}{4} - c + \varepsilon(c)} - \tfrac{1}{2}(c-\alpha).$$

Define

$$g_1(c) = \begin{cases} g(c) & \text{for } 0 \leq c \leq \tfrac{1}{4}, \\ 0 & \text{for } \tfrac{1}{4} < c \leq 1, \end{cases}$$

$$g_2(c) = \begin{cases} g(c) & \text{for } \tfrac{1}{4} < c \leq 1, \\ 0 & \text{for } 0 \leq c \leq \tfrac{1}{4}. \end{cases}$$

Thus

$$g(c) = g_1(c) + g_2(c). \tag{9.18}$$

Now for $c \leq \tfrac{1}{4}$

$$g_1'(c) = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c} - \frac{c-\alpha}{2\sqrt{\tfrac{1}{4} - c}} = 0$$

yields

$$1 = \frac{c - \alpha - \tfrac{1}{2} + 2c}{\sqrt{\tfrac{1}{4} - c}} = \frac{(3c - 2 - \tfrac{1}{2})^2}{\tfrac{1}{4} - c}$$

or

$$9c^2 - (6\alpha + 2)c + \alpha^2 + \alpha = 0.$$

We get

$$c = \tfrac{1}{18}(6\alpha + 2) \pm \tfrac{1}{18}\sqrt{(6\alpha+2)^2 - 36(\alpha^2 + \alpha)} = \tfrac{1}{3}\alpha + \tfrac{1}{9} \pm \tfrac{1}{9}\sqrt{1 - 3\alpha}$$

and for

$$c = \tfrac{1}{3}\alpha + \tfrac{1}{9} + \tfrac{1}{9}\sqrt{1 - 3\alpha} \tag{9.19}$$

it can be shown that $g_1''(c)\leqslant 0$. Thus with $\gamma=\sqrt{1-3\alpha}$

$$\max_{0\leqslant c\leqslant\frac{1}{4}} g_1(c)=\tfrac{2}{3}\alpha+\tfrac{1}{18}+\tfrac{1}{18}\gamma+(\tfrac{1}{9}+\tfrac{1}{9}\gamma-\tfrac{2}{3}\alpha)\sqrt{\tfrac{5}{36}-\tfrac{1}{3}\alpha-\tfrac{1}{9}\gamma}\triangleq\beta^*(\alpha). \quad (9.20)$$

Now for $\frac{1}{4}<c$, $g_2(c)=\frac{1}{4}-\frac{1}{2}(c-\alpha)$ and for $\alpha\geqslant\frac{1}{4}$ therefore $\max_{\frac{1}{4}\leqslant c} g_2(c)=\frac{1}{4}$. For $\alpha<\frac{1}{4}$, $\max_{\frac{1}{4}\leqslant c} g_2(c)=\frac{1}{8}+\frac{1}{2}\alpha$.

In summary

$$\beta(\alpha)=\begin{cases}\max(\beta^*(\alpha),\frac{1}{8}+\frac{1}{2}\alpha) & \text{for } \alpha<\frac{1}{4},\\ \frac{1}{4} & \text{for } \alpha\geqslant\frac{1}{4}.\end{cases} \quad (9.21)$$

Actually, $\frac{1}{8}+\frac{1}{2}\alpha\leqslant\beta^*(\alpha)$, because for $\alpha=\frac{1}{4}$ the quantities are equal to $\frac{1}{4}$, for $\alpha=0$ $\beta^*(0)=\frac{4}{27}>\frac{1}{8}$ and $\beta^{*\prime}(\alpha)<\frac{1}{2}$ for all $\alpha\leqslant\frac{1}{4}$.

Construction of asymptotically optimal MIS. In the preceding analysis we encountered the following facts: For $\alpha\geqslant\frac{1}{4}$, $c=c_1=c_2=\alpha$, $ab=(c-\frac{1}{2})^2$, and $a=b=\frac{1}{2}-\alpha$ are optimal and for $\alpha\leqslant\frac{1}{4}$ this is the case for $c_1=\alpha$, $c_2=c=\frac{1}{3}\alpha+\frac{1}{9}+\frac{1}{9}\sqrt{1-3\alpha}$ and $ab=c^2$. By our definition of an MIS in (9.1) in case $\alpha\geqslant\frac{1}{4}$ it suffices to give a configuration for $\alpha=\frac{1}{4}$ achieving the value $\frac{1}{4}$. In this case and also in case $\alpha\leqslant\frac{1}{4}$ the constructions are obtained by proper adjustments of the parameters in the following basic scheme. Let $n=k\cdot l$ and partition $\Omega=\{1,2,\dots,n\}$ into the sets $P_i=\{k(i-1)+1,\dots,ki\}$; $i=1,2,\dots,l$. Consider

$$\mathscr{E}=\{E\subset\Omega\colon E\cap P_i=P_i \text{ for at least one } i\},$$
$$\mathscr{F}=\{F\subset\Omega\colon F\cap P_i\neq\emptyset \text{ for all } i=1,2,\dots,l\}.$$

Clearly, $(\mathscr{E},\mathscr{F})$ is an MIS. It has the parameters

$$|\mathscr{E}|=2^{kl}-(2^k-1)^l, \quad (9.22)$$
$$|\mathscr{F}|=(2^k-1)^l, \quad (9.23)$$
$$|\mathscr{E}\cap\mathscr{F}|=(2^k-1)^l-(2^k-2)^l. \quad (9.24)$$

Case: $\alpha=\frac{1}{4}$.

Define $\mathscr{A}=\mathscr{E}$, $\mathscr{B}=\mathscr{F}$, $\mathscr{C}=\mathscr{A}\cap\mathscr{B}$, $\mathscr{A}^*=\mathscr{A}\setminus\mathscr{C}$, $\mathscr{B}^*=\mathscr{B}\setminus\mathscr{C}$. Thus

$$|\mathscr{A}|\cdot|\mathscr{B}|=2^{2kl}\left[1-\left(1-\frac{1}{2k}\right)^l\right]\left[1-\frac{1}{2k}\right]^l$$

and with the choice $l=\mu 2^k$ we have for any $\nu>0$ and $k\geqslant k(\nu,\mu)$

$$|\mathscr{A}|\cdot|\mathscr{B}|\geqslant 2^{2n}[(1-e^{-\mu})e^{-\mu}-\nu],$$
$$|\mathscr{A}\cap\mathscr{B}|\leqslant 2^n[e^{-\mu}-e^{-2\mu}+\nu].$$

The result follows with the choice $e^{-\mu}=\frac{1}{2}$.

A shifting procedure. One could have defined a (n,α)-MIS by requiring equality in (9.1). For the rate function ρ' corresponding to this notion $\rho'(\alpha)\leqslant\rho(\alpha)$. We show first that $\rho'(\alpha)=\rho(\alpha)$, for $\alpha \geqslant \frac{1}{4}$ by shifting elements in the previous MIS towards the common part. This technique is genuinely used for our proof of $\rho(\alpha)\geqslant\beta(\alpha)$ for $\alpha\leqslant\frac{1}{4}$. In the construction above $|\mathcal{A}|\,2^{-n}\to\frac{1}{2}$, $|\mathcal{B}|\,2^{-n}\to\frac{1}{2}$, $|\mathcal{A}\cap\mathcal{B}|\,2^{-n}\to\frac{1}{4}$.

Consider now any $B\in \mathrm{MIN}\,\mathcal{B}^*=\{X\in\mathcal{B}: \nexists X'\in\mathcal{B}^*, X\supsetneqq X'\}$ and observe that

$$\bar{B}\cap B'\neq\emptyset \quad \text{for all } B'\in\mathcal{B}^*\setminus\{B\}. \tag{9.25}$$

Define now $\mathcal{B}^{**}=\mathcal{B}^*-\{B,\bar{B}\}$ and similarly for $A\in\mathrm{MIN}\,\mathcal{A}^*\mathcal{A}^{**}=\mathcal{A}^*\setminus\{A,\bar{A}\}$. Finally set $\mathcal{C}^{**}=\mathcal{C}\cup\{\bar{A},\bar{B}\}$. These definitions are possible, because $\mathcal{A}^*$ and $\mathcal{B}^*$ are closed under complementation! Now (9.25) and its analogue for $\mathcal{A}$ imply that $(\mathcal{A}^{**}\cup\mathcal{C}^{**}, \mathcal{B}^{**}\cup\mathcal{C}^{**})$ is an MIS. Its common part has increased by 2 elements. Iteration of such shifts of elements gives for any $k<\frac{1}{2}|\mathcal{A}^*|$ an MIS $(\mathcal{A}^{(k)},\mathcal{B}^{(k)})$ with

$$|\mathcal{A}^{(k)}|=|\mathcal{A}|,\ |\mathcal{B}^{(k)}|=|\mathcal{B}|$$

and

$$|\mathcal{A}^{(k)}|\cap\mathcal{B}^{(k)}|=|\mathcal{A}\cap\mathcal{B}|+2k\sim 2^n\alpha,$$

if $2k\sim(\alpha-\frac{1}{4})2^n$. This is possible because $\alpha\leqslant\frac{1}{2}$ and $|\mathcal{A}^*|\sim\frac{1}{2}2^n$.

Case: $\alpha\leqslant\frac{1}{4}$.

In the previous construction we have

$$|\mathcal{A}^*|=|\mathcal{A}\setminus\mathcal{A}\cap\mathcal{B}|\sim 2^n(1-\mathrm{e}^{-\mu})^2,$$
$$|\mathcal{B}^*|=|\mathcal{B}\setminus\mathcal{A}\cap\mathcal{B}|\sim 2^n(\mathrm{e}^{-\mu})^2,$$
$$|\mathcal{A}\cap\mathcal{B}|\sim 2^n(1-\mathrm{e}^{-\mu})\mathrm{e}^{-\mu}$$

and therefore with $\mathcal{C}=\mathcal{A}\cap\mathcal{B}$, $|\mathcal{A}^*|\,|\mathcal{B}^*|\sim|\mathcal{C}|^2$.

From the proof of the converse we know that $ab=c^2$, $c_1=\alpha$, $c_2=c$ are optimal. We have to choose $\mathcal{C}_1,\mathcal{C}_2\subset\mathcal{C}$ with

$$\mathcal{C}_2=\mathcal{C},\qquad \mathcal{C}_1\subset\mathcal{C},\qquad |\mathcal{C}_1|\sim\alpha\cdot 2^n \tag{9.26}$$

and can define

$$\mathcal{A}^\alpha=\mathcal{A}^*\cup\mathcal{C}_1,\qquad \mathcal{B}^\alpha=\mathcal{B}^*\cup\mathcal{C}_2. \tag{9.27}$$

Now

$$|\mathcal{A}^\alpha|\,|\mathcal{B}^\alpha|\,2^{-2n}\sim((1-\mathrm{e}^{-\mu})^2+\alpha)\mathrm{e}^{-\mu}$$

and since

$$\max_\mu\,[(1-\mathrm{e}^{-\mu})^2+\alpha]\mathrm{e}^{-\mu}=\beta(\alpha)$$

the proof is complete. □

Remark. A closer look at our proof shows that it yields more than we stated in Theorem 9. If we call a triple (r_1, r_2, α) achievable in case there exist for all large n MIS's $(\mathcal{A},\mathcal{B})$ with

$$\frac{|\mathcal{A}|}{2^n}\geqslant r_1,\qquad \frac{|\mathcal{B}|}{2^n}\geqslant r_2,\quad\text{and}\quad \frac{|\mathcal{A}\cap\mathcal{B}|}{2^n}\leqslant\alpha,$$

then we can actually get these triples by the specifications

$$r_1 = a + c_1, \quad r_2 = b + c_2, \quad ab \leqslant c^2, \quad c_1 + c_2 = c + \alpha,$$
$$a + b + 2c = 1, \quad \text{and} \quad \alpha \leqslant c_1, c_2, c.$$

10. Mutually comparable systems (MCS)

$(\mathscr{A}, \mathscr{B})$; $\emptyset \neq \mathscr{A}, \mathscr{B} \subset 2^{\Omega}$; is an MCS, if all members $A \in \mathscr{A}$ and $B \in \mathscr{B}$ are comparable, that is, $A \supset B$ or $A \subset B$ holds.

We write $A \supset\subset B$, if A and B are comparable and also $\mathscr{A} \supset\subset \mathscr{B}$, if $(\mathscr{A}, \mathscr{B})$ is an MCS.

Our first observation is:

$$X \supset\subset A_1, X \supset\subset A_2 \;\Rightarrow\; X \supset\subset A_1 \cap A_2, X \supset\subset A_1 \cup A_2. \tag{10.1}$$

We verify this now.

If $X \subset A_1$, $X \subset A_2$, then $X \subset A_1 \cap A_2 \subset A_1 \cup A_2$;
if $X \supset A_1$, $X \supset A_2$, then $X \supset A_1 \cup A_2 \supset A_1 \cap A_2$; and
if $X \supset A_1$, $X \subset A_2$, then $A_1 \cap A_2 = A_1 \subset X \subset A_2 = A_1 \cup A_2$.

This implies that an MCS $(\mathscr{A}, \mathscr{B})$ is transformed into an MCS, if we replace $\mathscr{A}$ (resp. $\mathscr{B}$) by $\tilde{\mathscr{A}}$ (resp. $\tilde{\mathscr{B}}$), the smallest set containing $\mathscr{A}$, which is closed under unions and intersections. Furthermore, also $(\mathscr{A} \cup \{\emptyset, \Omega\}, \mathscr{B} \cup \{\emptyset, \Omega\})$ is again an MCS. Whereas for MIS we presented asymptotically optimal results, we give now exact answers for the corresponding problems for MCS.

We assign to $(\mathscr{A}, \mathscr{B})$ the sets

$$\mathscr{C} = \mathscr{A} \cap \mathscr{B}, \quad \mathscr{A}^* = \mathscr{A} \setminus \mathscr{C}, \quad \mathscr{B}^* = \mathscr{B} \setminus \mathscr{C} \tag{10.2}$$

and the parameters (a, b, c), where

$$a = |\mathscr{A}|, \quad b = |\mathscr{B}|, \quad c = |\mathscr{C}|. \tag{10.3}$$

Let $\mathscr{T}(n)$ be the set of triples achievable by MCS's. It suffices to characterize the set $\mathscr{E}(n)$ of extremal triples in $\mathscr{T}(n)$ because all other triples can be obtained by omitting elements from $\mathscr{A}$, $\mathscr{B}$ or $\mathscr{C}$. Notice that $\mathscr{C}$ is always a chain and therefore only the values $c = 0, 1, \ldots, n+1$ can occur.

Notice also that extremal triples can be achieved only for MCS's $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$. The following result is the basic tool for our analysis of such systems

Lemma 3. *For every* MCS $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$ *with* $\mathscr{A}' = \tilde{\mathscr{A}} \setminus \{\emptyset, \Omega\}$, $\mathscr{B}' = \tilde{\mathscr{B}} \setminus \{\emptyset, \Omega\}$ *there is a non-empty* $T \subset \Omega$ *with*

(i) $T \subset X$ *for all* $X \in \mathscr{A}'$ *or* $T \subset X$ *for all* $X \in \mathscr{B}'$,
(ii) $\{T\} \supset\subset \tilde{\mathscr{A}} \cup \tilde{\mathscr{B}}$.

Proof. Let Z have minimal cardinality among the members of $\mathscr{A}' \cup \mathscr{B}'$ and w.l.o.g. let $Z \in \mathscr{B}'$. The minimality of Z and the relation $\{Z\} \supset\subset \mathscr{A}'$ imply

$$Z \subset A \quad \text{for all } A \in \mathscr{A}'. \tag{10.4}$$

Therefore the set

$$T = \bigcap_{A \in \mathscr{A}'} A \tag{10.5}$$

is not empty and satisfies (i).

Furthermore, by our definitions $T \in \mathscr{A}'$. Therefore $\{T\} \supset\subset \tilde{\mathscr{B}}$ and (ii) follows with (i).

The Lemma says that for a $T_1 \neq \emptyset$ only some members of $\mathscr{B}$ (resp. $\mathscr{A}$) are contained in T_1 and the others as well as all members of $\mathscr{A}$ (resp. $\mathscr{B}$) contain T_1.

Applying the Lemma next to $\Omega_1 = \Omega \setminus T_1$, $\mathscr{A}_1 = \{A \setminus T_1 : A \in \mathscr{A}\}$, $B_1 = \{B \setminus T_1 : B \in \mathscr{B}\}$ we find a $T_2 \subset \Omega \setminus T$; $T_2 \neq \emptyset$. Reiteration by a most n steps of this construction leads to a sequence of sets

$$S_1 = T_1, \qquad S_2 = T_1 \cup T_2, \qquad S_3 = T_1 \cup T_2 \cup T_3, \ldots$$

and a sequence of families of subsets of Ω

$$\mathscr{D}_0, \mathscr{D}_1, \ldots$$

such that for $j = 0, 1, 2, \ldots$ and $S_0 = \emptyset$

$$S_j \subset D \subset S_{j+1} \quad \text{for all } D \in \mathscr{D}_j \tag{10.6}$$

and

$$\mathscr{D}_j \subset \mathscr{A} \quad \text{or} \quad \mathscr{D}_j \subset \mathscr{B}, \qquad \bigcup \mathscr{D}_j = \mathscr{A} \cup \mathscr{B}. \tag{10.7}$$

The S_j may belong to both sets $\mathscr{A}$ and $\mathscr{B}$, to one of them, or to none of them.

The last case can be ignored, because it is not extremal. A $\mathscr{D}_j$ may be empty, but then again for an extremal configuration always $|S_{j+1}| = |S_j| + 1$.

Otherwise in extremal cases by (10.6) and (10.7) the $\mathscr{D}_j$'s must be 'intervals'. In an ad hoc terminology $\mathscr{X} \subset 2^{\Omega}$ is an (i, j)-interval, if

$$\mathscr{X} = \{X : Y \subsetneq X \subsetneq Z\} \quad \text{and} \quad |Y| = i, \qquad |Z| = j. \tag{10.8}$$

Similarly, $[i, j)$-intervals, $(i, j]$-intervals and $[i, j]$-intervals are defined. The notation reflects whether Y or Z or both are in $\mathscr{X}$. Correspondingly we speak of open, half-open or closed intervals.

We call $\mathscr{X}$ an $i - j$-chain, if

$$\mathscr{X} = \{\{x_1, \ldots, x_i\}, \{x_1, \ldots, x_{i+1}\}, \ldots, \{x_1, \ldots, x_j\}\}. \tag{10.9}$$

We summarize now the foregoing explanations.

An extremal $(\mathscr{A}, \mathscr{B})$ is of the following form: There is a sequence of intervals (of the 4 types described above) and chains with increasing indices such that neither chains nor intervals occur in immediate succession. The chains are contained in $\mathscr{C}$ and the intervals are contained in $\mathscr{A}^*$ or $\mathscr{B}^*$. At this point calculations are necessary to establish the following result.

Theorem 10. *The extremal set of parameters for* MCS *is*

$$\begin{aligned}\mathscr{E}(n) = \{(a, b, c)&: a = 2^\alpha - 2 + c,\ b = 2^\beta - 2 + c,\ c = n + 3 - \alpha - \beta; \\ &\text{where } 1 \leq \alpha, \beta \leq \alpha + \beta \leq n\} \\ &\cup \{(a, b, c): a = 2^\alpha - 2 + c,\ b = 2,\ c = n + 2 - \alpha; \text{ where } 1 \leq \alpha \leq n\} \\ &\cup \{(a, b, c): a = c,\ b = 2^\beta - 2 + c,\ c = n + 2 - \beta; \text{ where } 1 \leq \beta \leq n\}.\end{aligned}$$

Proof. (1) In the configuration described above we can assume that all intervals are open, because otherwise we can shift the 'boundary points' to the neighbouring chain and (a, b, c) is transformed into $(a, b + 1, c + 2)$, if the interval is in $\mathscr{A}^*$.

(2) No two intervals are in $\mathscr{A}^*$ or in $\mathscr{B}^*$, because

$$2^{l_1 + l_2} - 2 > 2^{l_1} - 2 + 2^{l_2} - 2$$

says that replacement of two intervals by one is an improvement.

(3) We are left with two possible situations:

(a) 0–i-chain→(i, j)-interval→j–k-chain→(k, l)-interval→l–n-chain, where $0 \leq i < j \leq k < l \leq n$. Here

$$c = i + 1 + k - j + 1 + n - l + 1,$$

$$a = 2^{j-i} - 2 + c, \qquad b = 2^{l-k} - 2 + c$$

and with the correspondence $\alpha = j - i$, $\beta = l - k$ the result follows.

(b) 0–i-chain→(i, j)-interval→j–n-chain. Here

$$c = i + 1 + n - j + 1, \qquad a = 2^{j-i} - 2 + c, \qquad b = c$$

(or symmetrically $a = c$, $b = 2^{j-i} - 2 + c$), $0 \leq i < j \leq n$. For $\alpha = j - i$ we get finally

$$a = 2^\alpha - 2 + c, \qquad b = c, \qquad c = n + 2 - \alpha; \quad 1 \leq \alpha \leq n. \qquad \square$$

Notice that for $\alpha = 1$ $\mathscr{A}$ and $\mathscr{B}$ equal the 0–n-chain, and for $\alpha = n$ $\mathscr{A} = 2^\Omega$ and $\mathscr{B} = \{\emptyset, \Omega\}$.

Remarks. (1) If (a, b, c) is extremal, then $(a^*, b^*, c) = (a - c, b - c, c)$ is not necessarily extremal. However, the set $\mathscr{E}^*(n) = \{(a^*, b^*, c)$: extremal triples achievable by MCS) can also easily be determined by using not only open intervals.

(2) An analysis of MCS with more than two sets can be given along these lines.

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