ON EXTREMAL SET PARTITIONS IN CARTESIAN PRODUCT SPACES

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1. Introduction

Consider $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a finite set and $\mathcal{E}$ is a system of subsets of $\mathcal{V}$. For the cartesian products $\mathcal{V}^n = \prod_1^n \mathcal{V}$ and $\mathcal{E}^n = \prod_1^n \mathcal{E}$ let $\pi(n)$ denote the minimal size of a partition of $\mathcal{V}^n$ into sets which are elements of $\mathcal{E}^n$, if a partition exists at all, otherwise $\pi(n)$ is not defined. This is obviously exactly the case if it is so for $n=1$.

Whereas the packing number $p(n)$, that is the maximal size of a system of disjoint sets from $\mathcal{E}^n$, and the covering number $c(n)$, that is the minimal number of sets from $\mathcal{E}^n$ to cover $\mathcal{V}^n$, have been studied in the literature, this seems to be not the case for the partition number $\pi(n)$.

Obviously, $c(n) \leq \pi(n) \leq p(n)$, if $c(n)$ and $\pi(n)$ are well–defined. The quantity $\lim_{n\to\infty} \frac{1}{n} \log p(n)$ is Shannon's zero error capacity ([4]). Whereas it is known only in very few cases (see [5]), for $\lim_{n\to\infty} \frac{1}{n} \log c(n)$ a nice formula exists (see [6], [7]).

The difficulties in analyzing $\pi(n)$ are similar to those for $p(n)$. For the case of graphs with edge set $\mathcal{E}$ including all loops we prove that $\pi(n) = \pi(1)^n$ (Theorem 3). This result is derived from the corresponding result for complete graphs (Theorem 2) with the help of Gallai's Lemma in matching theory [9]. More general results concern products of hypergraphs with non–identical factors. Another interesting quantity is $\mu(n)$, the maximal size of a partition of $\mathcal{V}^n$ into sets who are elements of $\mathcal{E}^n$ (Again only hypergraphs $(\mathcal{V},\mathcal{E})$ with a partition are considered). We call μ also the maximal partition number. It behaves more like the packing number (see example 5). Clearly $\pi(n) \leq \mu(n) \leq p(n)$. It seems to us that an understanding of these partition problems would be a significant contribution to an understanding of the basic and seemingly simple notion of Cartesian products. Another partition problem was formulated in [1]. Among the contributions to this problem we refer to [1], [2], and [3].

2. Products of complete graphs: First results

For a complete graph $\mathcal{C} = \{\mathcal{V},\mathcal{E}\}$ let $\mathcal{E}^* = \mathcal{E} \cup \big\{\{v\} : v \in \mathcal{V}\big\}$ and define the hypergraph $\mathbb{C}^n = \{\mathcal{V}^n, \mathcal{E}^n\}$, where $\mathcal{V}^n = \prod_1^n \mathcal{V}$ and $\mathcal{E}^n = \prod_1^n \mathcal{E}^*$.

We study the partition number $\pi(n)$ first for $\mathbb{C}^n$ and in later sections extend our results to hypergraphs, which are products of arbitrary graphs including all loops, however, again.

First we introduce now the map $\sigma : \mathcal{E}^n \to \{0,1\}^n$, where

$$s^n = \sigma(E^n) = (\log|E_1|, \ldots, \log|E_n|). \tag{2.1}$$

As weight of E^n , in short $w(E^n)$, we choose the Hamming weight $w_H(s^n) = \sum_{t=1}^n s_t$. Notice that the cardinality $|E^n|$ equals $2^{w(E^n)}$.

Instead of partitions we consider more generally a packing $\mathcal{P}$ of $\mathbb{C}^n$. We set

$$\mathcal{P}_i = \left\{E^n \in \mathcal{P} : w(E^n) = i\right\}, P_i = |\mathcal{P}_i| \tag{2.2}$$

and call $\{P_i\}_{i=0}^n$ the weight distribution of $\mathcal{P}$.
With $\mathcal{P}$ we associate the set of shadows $\mathcal{Q} \subset \mathcal{Z}^n$, defined by

$$\mathcal{Q} = \{E^n \in \mathcal{E}^n : E^n \subset F^n \text{ for some } F^n \in \mathcal{P}\}, \tag{2.3}$$

and its level sets

$$\mathcal{Q}_i = \{E^n \in \mathcal{Q} : w(E^n) = i\}, 0 \leq i \leq n. \tag{2.4}$$

It is convenient to write $Q_i = |\mathcal{Q}_i|$.
$\{Q_i\}_{i=0}^n$ is the weight distribution of $\mathcal{Q} = \text{shad}(\mathcal{P})$.

We establish first simple connections between these weight distributions.

Lemma 1. *For a packing $\mathcal{P}$ of $\mathbb{C}^n$*

$$\sum_{i=k}^n 2^{i-k} \binom{i}{k} P_i = Q_k. \tag{2.5}$$

Proof. Consider any edge E^n with weight $w(E^n) = i \geq k$. There are exactly $2^{i-k}\binom{i}{k}$ edges in $\mathcal{E}^n$ contained in it, which have weight k . Therefore we have always

$$\sum_{i=k}^n 2^{i-k} \binom{i}{k} P_i \geq Q_k. \tag{2.6}$$

Lemma 2. *For a packing $\mathcal{P}$ of $\mathbb{C}^n$*

$$|\mathcal{P}| = \sum_{i=0}^n P_i = \sum_{k=0}^n (-1)^k Q_k. \tag{2.7}$$

Proof. An edge $E^n \in \mathcal{P}_i$ contributes to $\sum_{k=0}^n (-1)^k Q_k$ the amount $\sum_{k=0}^i (-1)^k 2^{1-k}\binom{i}{k} = (2-1)^i = 1$.

Lemma 3. *For a packing* $\mathcal{P}$ *of* $\mathbb{C}^n$

$$P_0 = \sum_{k=0}^{n} (-1)^k 2^k Q_k \tag{2.8}$$

and if in addition $\mathcal{P}$ *is a partition and* $S = |\mathcal{V}|$ *is odd, then*

$$\sum_{k=0}^{n} (-1)^k 2^k Q_k - 1 \geq 0. \tag{2.9}$$

Proof. An edge $E^n \in \mathcal{P}_i$ contributes to $\sum_{k=0}^{n}(-1)^k 2^k Q_k$ the amount $\sum_{k=0}^{i}(-1)^k 2^k 2^{i-k}\binom{i}{k} = 2^i(1-1)^i$, which equals 1 , if $i = 0$, and equals 0 , otherwise. Therefore (2.8) holds.

Furthermore, if S is odd, then so is S^n and there must be an edge in the partition of odd size, that is, $P_0 \geq 1$ or, equivalently, by (2.8), (2.9) must hold.

Remark 1: The last two Lemmas can be derived more systematically from Lemma 1 by Möbius Inversion. Here this machinery can be avoided, but we need it for the more abstract setting of [11].

3. Products of complete graphs: The main results

We shall exploit now Lemma 3 by applying it to classes of subhypergraphs, which we now define. For any $I \subset \{1, 2, \ldots, n\}$ and any specification $(v_j)_{j \in I^c}$, where $v_j \in \mathcal{V}_j$, we set

$$\mathbb{C}^n\big(I, (v_j)_{j\in I^c}\big) = \left(\prod_{i=1}^{n} \mathcal{U}_i,\ \prod_{i=1}^{n} \mathcal{F}_i\right) = \big(\mathcal{U}^n, \mathcal{F}^n\big), \tag{3.1}$$

where

$$\mathcal{U}_i = \begin{cases} \mathcal{V}_i \\ \{v_i\} \end{cases} \quad \text{and} \quad \mathcal{F}_i = \begin{cases} \mathcal{E}_i & \text{for } i \in I \\ \{v_i\} & \text{for } i \in I^c. \end{cases} \tag{3.2}$$

Clearly, for a partition $\mathcal{P}$ of $\mathbb{C}^n$ and $\mathcal{Q} = \text{shad}\mathcal{P}$ the set $\mathcal{Q}\big(I, (v_j)_{j\in I^c}\big) = \mathcal{Q} \cap \mathcal{F}^n$ is a downset and the map

$$\psi : \mathcal{F}^n \to \prod_{i\in I} \mathcal{E}_i,\ \psi\left(\prod_{i=1}^{n} E_i\right) = \prod_{i\in I} E_i \tag{3.3}$$

is a bijection.

Write $\widetilde{\mathcal{Q}} = \psi(\mathcal{Q} \cap \mathcal{F}^n)$ and let $\widetilde{\mathcal{Q}}_i$ count the members of $\widetilde{\mathcal{Q}}$ of weight i . Since $\widetilde{\mathcal{Q}}$ is a downset in $\prod_{i\in I} \mathcal{E}_i$ and its maximal elements form a partition of $\prod_{i\in I} \mathcal{V}_i$, we know that $\widetilde{\mathcal{Q}}_0 = S^m$. This fact and Lemma 3 yield

$$S^m + \sum_{k=1}^{m} (-1)^k 2^k \widetilde{\mathcal{Q}}_k - 1 \geq 0. \tag{3.4}$$

This is the key in the proof of the following important result.

Theorem 1. *For a partition* $\mathcal{P}$ *of* $\mathbb{C}^n = (\mathcal{V}^n, \mathcal{E}^n)$ *with* $\mathcal{V}^n = \prod_{i=1}^n \mathcal{V}_i$, $|\mathcal{V}_i| = S$ *for* $i = 1, 2, \ldots, n$ *the weight distribution* $(Q_k)_{k=0}^n$ *of* $Q = shad\mathcal{P}$ *satisfies for* $1 \leq m \leq n$

$$\binom{n}{m} S^n + \sum_{k=1}^{m} (-1)^k \binom{n-k}{m-k} 2^k Q_k - \binom{n}{m} S^{n-m} \geq 0. \tag{3.5}$$

Proof. The map ψ preserves inclusions and weights. The total number of pairs $(I, (v_j)_{j\in I^c})$ with $|I| = m$ equals $\binom{n}{m} S^{n-m}$. Finally, each $E^n \in \mathcal{Q}$ with $w(E^n) = k$ is contained in exactly $\binom{n-k}{m-k}$ sets of the form $\mathcal{Q}\big(I, (v_j)_{j\in I^c}\big)$ and thus for the sets of weight k

$$\binom{n-k}{m-k} Q_k = \sum_{\big(I, (v_j)_{j\in I^c}\big), |I|=m} \left| \mathcal{Q}_k\big(I, (v_j)_{j\in I^c}\big) \right| . \tag{3.6}$$

We have one equation of the form (3.4) for each pair $\big(I, (v_j)_{j\in I_c}\big)$. Summation of their left hand sides gives therefore

$$\binom{n}{m} S^{n-m} \cdot S^m + \sum_{k=1}^{m} (-1)^k 2^k \binom{n-k}{m-k} Q_k - \binom{n}{m} S^{n-m} \geq 0$$

and hence (3.5).

Now comes the harvest.

Theorem 2. *For a partition* $\mathcal{P}$ *of* $\mathbb{C}^n$

$$|\mathcal{P}| \geq \left\lceil \frac{S}{2} \right\rceil^n .$$

Proof. Since $|E^n| \leq 2^n$, obviously $|\mathcal{P}| \geq \frac{S^n}{2^n}$ and for $S = 2\alpha$ even, the result obviously holds. Let now $S = 2\alpha + 1$.

Summing the left hand side expressions in (3.5) for $m = 1, 2, \dots, n$ results in

$$\sum_{m=1}^{n} \binom{n}{m} S^n + \sum_{m=1}^{n} \sum_{k=1}^{m} (-1)^k \binom{n-k}{m-k} 2^k Q_k - \sum_{m=1}^{n} \binom{n}{m} S^{n-m} \geq 0$$

or in

$$(2^n - 1)S^n + \sum_{k=1}^{n} (-1)^k 2^k Q_k \sum_{m=k}^{n} \binom{n-k}{m-k} - \big[(S+1)^n - S^n\big] \geq 0.$$

This is equivalent to

$$2^n \cdot \big[S^n + \sum_{k=1}^{n} (-1)^k Q_k\big] - (S+1)^n \geq 0.$$

Since $Q_0 = S^n$ we conclude with Lemma 2

$$P \geq (S+1)^n \cdot 2^{-n} = \left\lceil \frac{S}{2} \right\rceil^n , \quad \text{if } S \text{ is odd.}$$

4. Non–identical factors: a generalization

We consider now hypergraphs $\mathbb{C}^n$ with vertex sets $\mathcal{V}^n = \prod_{t=1}^{n} \mathcal{V}_t$ and edge sets $\mathcal{E}^n = \prod_{t=1}^{n} \mathcal{E}_t$, where the $\mathcal{V}_t$'s are finite sets of not necessarily equal cardinalities S_t . The factors $\mathcal{E}_t$ are such that $(\mathcal{V}_t, \mathcal{E}_t)$ is a complete graph with all loops included. We shall write with positive integers α_t

$$|\mathcal{V}_t| = 2\alpha_t + \varepsilon_t,\ \varepsilon_t \in \{0, 1\}. \tag{4.1}$$

Inspection shows that the sizes of factors do not affect the proofs of Lemmas 1 and 2. Also (2.8) in Lemma 2 holds and since $P_0 \geq 1$, if $\varepsilon_t = 1$ for $t = 1, 2, \dots, n$ we can generalize (2.9) to

$$\sum_{k=0}^{n} (-1)^k 2^k Q_k - \prod_{k=1}^{n} \varepsilon_k \geq 0. \tag{4.2}$$

Theorem 1 in Section 3 generalizes to

Theorem 1'. *For a partition* $\mathcal{P}$ *of* $\mathbb{C}'^n$

$$\binom{n}{m}\prod_{i=1}^{n} S_i + \sum_{k=1}^{m}(-1)^k\binom{n-k}{m-k}2^k Q_k - \sum_{I:|I|=m}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j \geq 0. \tag{4.3}$$

Sketch of proof. Replace in the proof of Theorem 1 S^m by $\prod_{i\in I} S_i$ and inequality (3.4) by

$$\prod_{i\in I} S_i + \sum_{k=1}^{n}(-1)^k 2^k \widetilde{Q}_k - \prod_{i\in I}\varepsilon_i \geq 0. \tag{4.4}$$

Theorem 2'. *For a partition* $\mathcal{P}$ *of* $\mathbb{C}'^n$

$$|\mathcal{P}| \geq \prod_{i=1}^{n}\left\lceil \frac{S_i}{2}\right\rceil. \tag{4.5}$$

Proof. Summing the left hand side expressions in (4.3) for $m = 1, 2, \dots, n$ results in

$$\begin{aligned}
0 &\leq \sum_{m=1}^{n}\binom{n}{m}\prod_{i=1}^{n} S_i + \sum_{m=1}^{n}\sum_{k=1}^{m}\binom{n-k}{m-k}(-1)^k 2^k Q_k - \sum_{m=1}^{n}\sum_{I:|I|=m}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j \\
&= (2^n-1)\prod_{i=1}^{n} S_i + \sum_{k=1}^{n}(-1)^k 2^k Q_k \sum_{m=k}^{n}\binom{n-k}{m-k} - \sum_{\phi\neq I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_i \\
&= 2^n\left[\prod_{i=1}^{n} S_i + \sum_{k=1}^{n}(-1)^k Q_k\right] - \sum_{I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j \quad \text{or}
\end{aligned}$$

$$|\mathcal{P}| \geq 2^{-n}\sum_{I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j. \tag{4.6}$$

We evaluate the right hand side expression by introducing $J = \{\ell : 1 \leq \ell \leq n\ ,\ \varepsilon_\ell = 1\}$ and $I^* = J \setminus I$. Then

$$\sum_{I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j = \sum_{I\subset J}\prod_{j\in I^*} S_j \cdot \prod_{j\in J^c} S_j$$

$$= \prod_{j\in J}(S_j+1)\cdot\prod_{j\in J^c} S_j = \prod_{j=1}^{n}(S_j+\varepsilon_j) \quad \text{and (4.5) follows.}$$

Corollary.

The partition number $\pi(\mathbb{C}'^n)$ equals $\prod_{j=1}^n \left\lceil \frac{S_j}{2} \right\rceil$.

Proof. The partition number of $(\mathcal{V}_j, \mathcal{E}_j)$ is $\left\lceil \frac{S_j}{2} \right\rceil$. Take a product of optimal partitions for the factors. This construction gives the lower bound in Theorem 2'.

5. Products of general graphs

We assume now that the factors $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ $(t = 1, 2, \dots, n)$ are arbitrary finite graphs with all loops included.

Obviously, we have for the partition number

$$\pi(\mathcal{G}_t) = |\mathcal{V}_t| - \nu(\mathcal{G}_t), \tag{5.1}$$

where $\nu(\mathcal{G}_t)$ is the matching number of $\mathcal{G}_t$.

Theorem 3. *For the hypergraph product* $\mathcal{H}^n = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$

$$\pi(\mathcal{H}^n) = \prod_{t=1}^n \pi(\mathcal{G}_t). \tag{5.2}$$

Here only the inequality

$$\pi(\mathcal{H}^n) \geq \prod_{t=1}^n \pi(\mathcal{G}_t) \tag{5.3}$$

is non–trivial. We make use of a well–known result from matching theory.

Gallai's Lemma. *([9] or [10], page 89)*

If a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is connected and for all* $v \in \mathcal{V}$ $\nu(\mathcal{G} - v) = \nu(\mathcal{G})$, *then* $\mathcal{G}$ *is factor–critical, that is, for all* $v \in \mathcal{V}$ $\mathcal{G} - v$ *has a perfect matching.*

Proof of (5.3).

For every $t \in \{1, 2, \ldots, n\}$ we modify $\mathcal{G}_t$ as follows: we remove any vertex $v \in \mathcal{V}_t$ with $\nu(\mathcal{G}_t - v) < \nu(\mathcal{G}_t)$ and reiterate this until we obtain a graph $\mathcal{G}_t^*$ with $\nu(\mathcal{G}_t^* - v) = \nu(\mathcal{G}_t^*)$ for all $v \in \mathcal{V}_t^*$.

Notice that (5.1) insures that

$$\pi(\mathcal{G}_t) = \pi(\mathcal{G}_t^*). \tag{5.4}$$

Denote the set of connected components of $\mathcal{G}_t^*$ by $\big\{\mathcal{G}_t^{*(j)}\big\}_{j \in J_t}$. Clearly,

$$\pi(\mathcal{G}_t^*) = \sum_{j \in J_t} \pi\big(\mathcal{G}_t^{*(j)}\big). \tag{5.5}$$

Moreover, by Gallai's Lemma each component $\mathcal{G}_t^{*(j)}$ has a vertex set $\mathcal{V}_t^{*(j)}$ of odd size and

$$\nu\big(\mathcal{G}_t^{*(j)}\big) = \big(|\mathcal{V}_t^{*(j)}| - 1\big)2^{-1} \triangleq \alpha_t^j, \ \text{ say.}$$

Thus,

$$\pi(\mathcal{G}_t^*) = \sum_j (\alpha_t^j + 1). \tag{5.6}$$

Now realize that for $\mathcal{H}^{*n} = \prod_1^n \mathcal{G}_t^*$

$$\pi(\mathcal{H}^n) \geq \pi(\mathcal{H}^{*n}), \tag{5.7}$$

because the modifications described above transform a partition of $\mathcal{H}^n$ into a partition of $\mathcal{H}^{*n}$ with not more parts.

Finally, we have for the product $\mathbb{C}^n$ of complete graphs with vertex sets $\mathcal{V}_t^{*(j)}$ by Theorem 2' that

$$\pi\big(\mathcal{G}_1^{*(j_1)} \times \cdots \times \mathcal{G}_n^{*(j_n)}\big) \geq \pi(\mathbb{C}^n) = (\alpha_1^{j_1} + 1) \ldots (\alpha_n^{j_n} + 1). \tag{5.8}$$

Therefore,

$$\begin{aligned}
\pi(\mathcal{H}^{*n}) &= \sum_{j_1 \in J_1, \ldots, j_n \in J_n} \pi\big(\mathcal{G}_1^{*(j_1)} \times \cdots \times \mathcal{G}_n^{*(j_n)}\big) \\
&\geq \sum_{(j_1, \ldots, j_n)} (\alpha_1^{j_1} + 1) \ldots (\alpha_n^{j_n} + 1) = \prod_{t=1}^n \sum_{j \in J_t} (\alpha_t^j + 1) \\
&= \prod_{t=1}^n \pi(\mathcal{G}_t^*) = \prod_{t=1}^n \pi(\mathcal{G}_t).
\end{aligned}$$

This and (5.7) imply (5.3).

6. Examples for deviation from multiplicative behaviour

We give now first two examples of product hypergraphs $\mathcal{H} \times \mathcal{H}'$ for which the partition number π is not multiplicative in the factors. They are due to K.U. Koschnick.

Example 1.

$\mathcal{V}_1 = \{0,1,2,\dots,6\}$, $\mathcal{E}_1 = \big\{E \subseteq \mathcal{V}_1 : |E| \in \{1,4\}\big\}$. Clearly, $\pi(\mathcal{H}_1) = 4$ and the partition

$$\begin{aligned}&\big\{\{i\} \times \{0,1,2,3\} : i = 0,1,2\big\} \cup \big\{\{i\} \times \{3,4,5,6\} : i = 4,5,6\big\}\\ &\cup \big\{\{0,1,2,3\} \times \{j\} : j = 4,5,6\big\} \cup \big\{\{3,4,5,6\} \times \{j\} : j = \{0,1,2\}\big\}\\ &\cup \big\{\{3\} \times \{3\}\big\}\end{aligned}$$

has 13 members. Therefore

$$\pi(\mathcal{H}_1 \times \mathcal{H}_1) \leq 13 < \pi(\mathcal{H}_1)\pi(\mathcal{H}_1) = 16. \tag{6.1}$$

Whereas this example seems to be the smallest possible, one can also do better with non–identical factors:

$\mathcal{H}_1 \times \mathcal{H}_1'$, where $\mathcal{V}_1' = \{0,1,2,3,4\}$ and $\mathcal{E}_1' = \big\{E \subset \mathcal{V}_1' : |E| \in \{1,3\}\big\}$. Here by a similar construction $\pi(\mathcal{H}_1 \times \mathcal{H}_1') \leq 11$, whereas $\pi(\mathcal{H}_1) \cdot \pi(\mathcal{H}_1') = 4 \cdot 3 = 12$.

Example 2.

Since π is multiplicative for graphs one may wonder whether it is multiplicative if one factor is a graph.

Consider $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{0,1,\dots,4\}$ and $\mathcal{E} = \big\{\{i, i{+}1 \mod 5\} : i = 0,1,\dots,4\big\} \cup \{i : 0 \leq i \leq 4\}$, that is, the pentagon with all loops.

Define $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ with $\mathcal{V}' = \{1,2,\dots,14\}$ and $\mathcal{E}' = \big\{E \subset \mathcal{V}' : |E| \in \{1,9\}\big\}$.

Notice that $\pi(G) = 3$, $\pi(\mathcal{H}') = 7$ and that the following construction insures $\pi(G \times \mathcal{H}') \leq 20 < 21 = \pi(G) \cdot \pi(\mathcal{H}')$:

$$\begin{aligned}&\big\{\{i\} \times \{j+k \mod 14 : 0 \leq k \leq 8\} : (i,j) \in \{(0,0),(1,3),(2,6),(3,9),(4,12)\}\big\}\\ &\cup \big\{\{1,2\} \times \{j\} : j = 0,1,2\big\} \cup \big\{\{2,3\} \times \{j\} : j = 2,3,5\big\}\\ &\cup \big\{\{3,4\} \times \{j\} : j = 6,7,8\big\} \cup \big\{\{4,0\} \times \{j\} : j = 9,10,11\big\}\\ &\cup \big\{\{0,1\} \times \{j\} : j = 12,13,14\big\}\end{aligned}$$

is a set of $5 + 5 \cdot 3 = 20$ edges partitioning $\mathcal{V} \times \mathcal{V}'$.

For the orientation of the reader we add three examples, which demonstrate that also the covering number c , the packing number p and the maximal partition number μ are not multiplicative in the factors.

Example 3. $\mathcal{V}_3 = \{0, 1, 2\}$, $\mathcal{E}_3 = \{E \subseteq \mathcal{V} : |E| = 2\}$

We have

$$3 = c(\mathcal{H}_3 \times \mathcal{H}_3) \neq c(\mathcal{H}_3) \cdot c(\mathcal{H}_3) = 4, \tag{6.2}$$

because $\mathcal{C}\big\{\{0,1\} \times \{0,1\}, \{0,2\} \times \{0,2\}, \{1,2\} \times \{1,2\}\big\}$ covers $\mathcal{V}_3 \times \mathcal{V}_3$ and there is no covering with 2 edges.

This is the smallest example in terms of the number of vertices.

Remark 2. Quite generally, even in case of non–identical factors $\mathcal{H}_t = (\mathcal{V}_t, \mathcal{E}_t)$, $t \in \mathbb{N}$, with $\max_t |\mathcal{E}_t| < \infty$ the asymptotic behaviour of $c(n)$ is known ([7]):

$\lim\limits_{n\to\infty} \frac{1}{n}\left(\log c(n) - \sum\limits_{t=1}^{n} \log\left(\max\limits_{q\in \mathrm{Prob}(\mathcal{E}_t)} \min\limits_{v\in\mathcal{E}_t} \sum\limits_{E\in\mathcal{E}_t} 1_E(v) q_E\right)^{-1}\right) = 0$, where $\mathrm{Prob}(\mathcal{E}_t)$ is the set of all probability distributions on $\mathcal{E}$, q_E is the probability of E under q and 1_E is the indicator function of the set E .

Example 4.

$\mathcal{V}_4 = \{0, 1, 2, 3, 4\}$, $\mathcal{E}_4 = \big\{\{x, x+1 \mod 5\} : x \in \mathcal{V}_4\big\}$. Here we have

$$5 = p(\mathcal{H}_4 \times \mathcal{H}_4) \neq p(\mathcal{H}_4) p(\mathcal{H}_4) = 4. \tag{6.3}$$

It was shown in [4] that this is the smallest example in the previous sense. Notice that it is bigger than the previous one.

Example 5. In order to avoid heavy notation we write $\mathcal{H}_5 = (\mathcal{V}_5, \mathcal{E}_5)$ simply without an index as $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. It is constituted by the 5 vertex sets

$$\mathcal{W}_i = \{x_{ij} : j = 1, 2, \dots, m\}, 3 \leq m (i = 0, 1, 2, \dots, 4)$$

and the 6 edge sets

$$\mathcal{G}_i = \big\{(x_{ij}, x_{i+1 \mod 5, j}) : j = 1, 2, \dots, m\big\} (i = 0, 1, 2, \dots, 4)$$

and the 5 edges $\mathcal{W}_i (i = 1, 2, \dots, 4)$. Thus

$$\mathcal{V} = \bigcup_{i=0}^{4} \mathcal{W}_i, \mathcal{E} = \{\mathcal{W}_1, \dots, \mathcal{W}_4\} \cup (\bigcup_{i=0}^{4} \mathcal{G}_i).$$

A look at the pentagon with vertex set $\{x_{01}, x_{11}, x_{21}, x_{31}, x_{41}\}$ shows that a partition of $\mathcal{H}$ must contain at least one of the edges $\mathcal{W}_i$ as a member. On the other hand the vertices $\mathcal{V} \smallsetminus \mathcal{W}_i$ have a maximal partition of size $2m$. Therefore we have shown that $\mu(\mathcal{H}) = 2m + 1$. We shall next consider $\mu(\mathcal{H} \times \mathcal{H})$. For this we introduce the superedges

$$\mathcal{G}_i^* = \mathcal{W}_i \cup \mathcal{W}_{i+1 \mod 5} (i = 0, 1, \dots, 4)$$

in $\mathcal{H}$ and the superedges $\mathcal{G}_i^* \times \mathcal{G}_{i'}^* (i, i' = 0, 1, \dots, 4)$ in $\mathcal{H} \times \mathcal{H}$. Whereas $\mathcal{G}_i^*$ can be partitioned into m edges, they can be partitioned into m^2 edges.

Now first of all we divide $\mathcal{V} \times \mathcal{V}$ into 25 parts $\{\mathcal{W}_i \times \mathcal{W}_{i'} : i, i' = 0, 1, \dots, 4\}$. Then we pack 5 superedges (as in Shannon's construction) into $\mathcal{V} \times \mathcal{V}$. They cover 20 parts and the remaining 5 parts are packed with 5 edges of type $\mathcal{W}_i \times \mathcal{W}_{i'}$. Finally we partition the 5 superedges into the edges of $\mathcal{H} \times \mathcal{H}$. Thus we obtain a desired partition with $5 + 5m^2$ edges. Notice that $\mu(\mathcal{H} \times \mathcal{H}) \geq 5 + 5m^2 > (2m + 1)^2 = \mu(\mathcal{H})^2$ for $m \geq 3$. The smallest example in this class has 15 vertices.

Remark 3. The construction is based on the pentagon. Its vertices are replaced by sets of vertices $\mathcal{W}_i$ with a numbering. The vertices with the same number in the $\mathcal{W}_i$'s form a pentagon. Thus we obtained $m = |\mathcal{W}_i|$ many pentagons. Then we added the $\mathcal{W}_i'$ as further edges. Finally we used the superedges to mimic the original small edges. We can make this construction starting with any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. If it has the property $p(\mathcal{H})^2 < p(\mathcal{H} \times \mathcal{H})$, then for m large enough our construction gives an associated hypergraph for which μ is not multiplicative.

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