Note on the optimal structure of recovering set pairs in lattices: the sandglass conjecture

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Abstract

We present a conjecture concerning the optimal structure of a subset pair satisfying two dual requirements in a lattice that can be derived as the product of k finite length chains. The conjecture is proved for $k = 2$.

1 Introduction

On an Oberwolfach conference in 1989 the second author presented the following conjecture.

Conjecture: Let $\mathcal{A} = \{A_i\}_{i=1}^{M_1}$, $\mathcal{B} = \{B_i\}_{i=1}^{M_2}$ be two families of subsets of an n–set such that the following two conditions hold:

(i) $A_j \cup B_r = A_k \cup B_s \Rightarrow j = k$

(ii) $A_j \cap B_r = A_k \cap B_s \Rightarrow r = s$.

Then $M_1 M_2 \leq 2^n$.

$\blacksquare$

If true this upper bound is sharp as it is shown by the following simple construction. Fix an arbitrary $C \subseteq [n]$ and let $\mathcal{A} = \{A : C \subseteq A \subseteq [n]\}$ and $\mathcal{B} = \{B : B \subseteq C\}$. Then clearly for every $A \in \mathcal{A}$, $B \in \mathcal{B}$ we have $B \subseteq A$, i.e., $A \cup B = A$ and $A \cap B = B$ that assures the two conditions to be satisfied. On the other hand $|\mathcal{A}||\mathcal{B}| = 2^{n-|C|}2^{|C|} = 2^n$.

(The problem originally arose from the investigation of codes for so–called write–unidirectional memories. For a description of that model the interested reader is referred to [2].)

We will call (cf. the Definition below) a pair $(\mathcal{A}, \mathcal{B})$ satisfying conditions (i) and (ii) a recovering pair because condition (i) means that from the union of an A_i and a B_i we can always recover the A_i and condition (ii) is the dual statement meaning the recoverability of B_i from the intersection.

The above conjecture is still not proved or disproved. After the aforementioned meeting in Oberwolfach the first author asked what happens if we state the analogous question in a more general setting, namely, instead of the Boolean lattice we deal with products of chains. We are back to the original problem if all the chains have length two. In this note we deal with the "other end" of the problem, namely, when we have two chains of arbitrary finite length. We show that in this case a statement analogous to the above conjecture is true.

Before making the above mentioned generalization precise, let us make two remarks on the Boolean case.

Remark 1: The best upper bound for $|\mathcal{A}||\mathcal{B}|$ we know about is given by the following simple argument proposed by Gérard Cohen [1]. Let $t = \min_{A \in \mathcal{A}} |A|$. Then by condition (ii), a t–element subset $A \in \mathcal{A}$ of $[n]$ intersects every $B \in \mathcal{B}$ in a different subset implying $|\mathcal{B}| \leq 2^t$. On the other hand $|\mathcal{A}| \leq \sum_{i=t}^{n} \binom{n}{i}$ and thus $|\mathcal{A}||\mathcal{B}| \leq \sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$.

Note that this argument works for the relaxed problem when we have only one of the two conditions. (Because of symmetry it does not matter if it is condition (i) or (ii). Above we argued with condition (ii).)

Remark 2: If we drop one of the two conditions, say condition (i), then we can construct families $\mathcal{A}$ and $\mathcal{B}$ for which $|\mathcal{A}||\mathcal{B}| > 2^n$ as follows. Let $C_1, C_2, \ldots, C_{\frac{n}{2}}$ be disjoint subsets of $[n]$, each consisting of two elements except possibly the last that has three elements if n is odd. Now let $\mathcal{A} = \{A : A \cap C_i \neq \varnothing \ i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, $\mathcal{B} = \{B : B \cap C_i = \varnothing \text{ or } C_i \subseteq B \ i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. It is easy to check that B will always be recoverable from $A \cap B$ and

$$|\mathcal{A}||\mathcal{B}| = \begin{cases} 6^{n/2} & \text{if } n \text{ is even} \\ 6^{n-3/2}14 & \text{if } n \text{ is odd} \end{cases}.$$

2 The Sandglass Conjecture

Now we state our more general conjecture, a special case of which we are going to prove. We need two definitions.

Definition: Let us be given a lattice $\mathcal{L}$. Two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}$ form a *recovering pair* if for every $a, a', c, c' \in \mathcal{A}$ and $b, b', d, d' \in \mathcal{B}$ the following two conditions hold:

(i) $\max(a, b) = \max(a', b') \Rightarrow a' = a$

(ii) $\min(c, d) = \min(c', d') \Rightarrow d' = d$

Set $\mathcal{A}$ is called the *upper set* and $\mathcal{B}$ the *lower set* of the pair. We denote by $r(\mathcal{L})$ the maximum possible value of $|\mathcal{A}||\mathcal{B}|$ for a recovering pair of $\mathcal{L}$, i.e.,

$$r(\mathcal{L}) = \max_{\substack{\mathcal{A},\mathcal{B} \subseteq \mathcal{L} \\ (\mathcal{A},\mathcal{B}) \text{ is a recov. pair}}} |\mathcal{A}||\mathcal{B}|. \quad \otimes$$

The next definition gives a name to a natural configuration of two subsets of a lattice.

Definition: A pair $(\mathcal{A}, \mathcal{B})$ of subsets of a lattice $\mathcal{L}$ is said to form a *sandglass* if there exists an element c of $\mathcal{L}$ that satisfies $c \leq a$ for every $a \in \mathcal{A}$ and $c \geq b$ for every $b \in \mathcal{B}$. A sandglass is *full* if adding any new element to $\mathcal{A}$ or $\mathcal{B}$ the new pair will not be a sandglass any more. $\otimes$

Note that in a lattice we could equivalently define a sandglass by the property that $b \leq a$ holds for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. (For general partially ordered sets these two possible definitions would not coincide.)

It is clear that a sandglass always forms a recovering pair. Our conjecture is the following.

(Sandglass) Conjecture: Let $\mathcal{L}$ be a lattice that can be derived as the product of k finite length chains. Then there exists a (full) sandglass $(\mathcal{A}, \mathcal{B})$, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L}$ for which $|\mathcal{A}||\mathcal{B}| = r(\mathcal{L})$.

■

Remark 3: In fact, we do not have an example of any lattice where the analogous statement is not true. Still, we dare not to conjecture it to be true in general.

The Sandglass Conjecture is trivial for $k = 1$. We show that it holds for $k = 2$, too.

3 The case $k = 2$

Theorem: *Let $\mathcal{L}$ be a lattice obtained as the product of two finite length chains. Then $r(\mathcal{L})$ can be achieved by a sandglass.*

First we prove a few lemmas. Lemmas 1, 2 and 2' are valid for any lattice $\mathcal{L}$.

Lemma 1: *If $\mathcal{A}$ and $\mathcal{B}$ form a recovering pair with lower set $\mathcal{B}$ and $\exists a \in \mathcal{A}$, $b \in \mathcal{B}$ with $b \geq a$ then there exists a sandglass $(\mathcal{A}', \mathcal{B}')$ with $|\mathcal{A}'| \geq |\mathcal{A}|, |\mathcal{B}'| \geq |\mathcal{B}|$.*

Proof: It is clear from (i) in the definition of recovering pairs that

$$|\mathcal{A}| \leq \min_{b \in \mathcal{B}} |\{h \in \mathcal{L};\ h \geq b\}|$$

and similarly from (ii)

$$|\mathcal{B}| \leq \min_{a \in \mathcal{A}} |\{h \in \mathcal{L};\ h \leq a\}|.$$

If $\exists a \in \mathcal{A}$, $b \in \mathcal{B}$ with $b \geq a$ then consider the sandglass

$$\mathcal{A}' = \{h \in \mathcal{L};\ h \geq b\},\ \mathcal{B}' = \{h \in \mathcal{L};\ h \leq b\}.$$

Since $\{h \in \mathcal{L};\ h \leq a\} \subseteq \mathcal{B}'$ by $b \geq a$ we have $|\mathcal{A}'| \geq |\mathcal{A}|, |\mathcal{B}'| \geq |\mathcal{B}|$ by the above inequalities. This proves the lemma.

$\blacksquare$

We call a recovering pair $(\mathcal{A}, \mathcal{B})$ *canonical*, if for no $a \in \mathcal{A}$ and $b \in \mathcal{B}$ $b \geq a$ holds. Those pairs remain to be analysed.

For the next lemma we have to introduce some further concepts. Consider a recovering pair $(\mathcal{A}, \mathcal{B})$ with lower set $\mathcal{B}$. For each $a \in \mathcal{A}$ the *territory of* a is the set

$$\tau_{\mathcal{B}}(a) = \{\max(a, b) :\ b \in \mathcal{B}\}$$

and similarly the *territory of* b is

$$\omega_{\mathcal{A}}(b) = \{\min(a, b) :\ a \in \mathcal{A}\}.$$

Notice that the two conditions in the definition of recovering pairs is equivalent to

(i') $\tau_{\mathcal{B}}(a) \cap \tau_{\mathcal{B}}(a') = \varnothing$ if $a, a' \in \mathcal{A}$ and $a \neq a'$.

(ii') $\omega_{\mathcal{A}}(b) \cap \omega_{\mathcal{A}}(b') = \varnothing$ if $b, b' \in \mathcal{B}$ and $b \neq b'$.

The *peak of the territory* of an $a \in \mathcal{A}$ and a $b \in \mathcal{B}$ is defined by

$$t_{\mathcal{B}}(a) = \max_{c \in \tau_{\mathcal{B}}(a)} c \qquad \text{and} \qquad \mathrm{w}_{\mathcal{A}}(b) = \min_{d \in \omega_{\mathcal{A}}(b)} d$$

respectively.

Lemma 2: *If $\mathcal{A}$ and $\mathcal{B}$ form a recovering pair (with lower set $\mathcal{B}$) and $\exists a_0 \in \mathcal{A}$ with $t_{\mathcal{B}}(a_0) \in \tau_{\mathcal{B}}(a_0)$ then the set*

$$\mathcal{A}^+ = \{\mathcal{A}\backslash a_0\} \cup \{t_{\mathcal{B}}(a_0)\}$$

also forms a recovering pair with $\mathcal{B}$.

Proof: For $a \in \mathcal{A}$, $a \neq a_0$ the values $\min(a, b)$ and $\max(a, b)$ do not change if we substitute a_0 by $t_{\mathcal{B}}(a_0)$ in $\mathcal{A}$ (so obtaining $\mathcal{A}^+$).

By the definition of $t_{\mathcal{B}}(a_0)$, $\max(t_{\mathcal{B}}(a_0), b) = t_{\mathcal{B}}(a_0)$ for every $b \in \mathcal{B}$. Since $t_{\mathcal{B}}(a_0)$ was an element of $\tau_{\mathcal{B}}(a_0)$, it could not be contained in any other $\tau_{\mathcal{B}}(a)$ with $a \neq a_0$, and so (i') is satisfied for $\mathcal{A}^+$ and $\mathcal{B}$.

Since $t_{\mathcal{B}}(a_0) \geq b$ for any $b \in \mathcal{B}$, $\min(t_{\mathcal{B}}(a_0), b) = b$ for any $b \in \mathcal{B}$. It is obvious that $b \in \omega_{\mathcal{A}}(b')$ with $b' \neq b$ is impossible unless there exists an $a \in \mathcal{A}$ with $a \geq b$. But then $\min(a, b) = b$, too, i.e., $b \in \omega_{\mathcal{A}}(b) \cap \omega_{\mathcal{A}}(b')$ contradicting (ii'). So if (ii') was satisfied for $(\mathcal{A}, \mathcal{B})$ then it is so for $(\mathcal{A}^+, \mathcal{B})$. Thus $(\mathcal{A}^+, \mathcal{B})$ is a recovering pair.

$\blacksquare$

A similar argument proves

Lemma 2': *If $\mathcal{A}$ and $\mathcal{B}$ form a recovering pair (with lower set $\mathcal{B}$) and $\exists b_0 \in \mathcal{B}$ with $w_{\mathcal{A}}(b_0) \in \omega_{\mathcal{A}}(b_0)$ then $\mathcal{A}$ and $\mathcal{B}^- = \{\mathcal{B}\backslash b_0\} \cup \{w_{\mathcal{A}}(b_0)\}$ is also a recovering pair.*

$\blacksquare$

The following lemma will make use of the special structure of $\mathcal{L}$ in the Theorem.

Lemma 3: *If $\mathcal{L}$ is the product of two finite length chains then for any canonical recovering pair $(\mathcal{A}, \mathcal{B})$ (with lower set $\mathcal{B}$) containing an incomparable pair a, b, $a \in \mathcal{A}$, $b \in \mathcal{B}$, either there exists an element $a_0 \in \mathcal{A}$ with the properties $a_0 \neq t_{\mathcal{B}}(a_0)$, $t_{\mathcal{B}}(a_0) \in \tau_{\mathcal{B}}(a_0)$ or there exists a $b_0 \in \mathcal{B}$ with the properties $b_0 \neq w_{\mathcal{A}}(b_0)$, $w_{\mathcal{A}}(b_0) \in \omega_{\mathcal{A}}(b_0)$.*

Proof: Let the elements of $\mathcal{L}$ be denoted by (i, j) in the natural way, i.e., i is the corresponding element of the first and j is that of the second chain defining $\mathcal{L}$. Note that if two elements, (i, j) and (k, l), are incomparable, then either $i < k, j > l$ or $i > k, j < l$ holds.

Consider all those elements of $\mathcal{A}$ and $\mathcal{B}$ for which there are incomparable elements in the other set, i.e., define the set

$D = \{a \in \mathcal{A} :\ \exists b \in \mathcal{B},\ a$ and b are incomparable$\} \cup$
$\cup \{b \in \mathcal{B} :\ \exists a \in \mathcal{A},\ a$ and b are incomparable$\}$.

Now choose an element $(i, j) \in D$ for which the (possibly negative) value of $(i - j)$ is minimal within D. Denote it by (i_0, j_0). We claim that this element can take the role of a_0 or b_0 depending on whether it is in $\mathcal{A}$ of $\mathcal{B}$. Since (i_0, j_0) is in D, it is clearly not equal to the peak of its territory, so all we have to prove is that the peak of its territory is contained in its territory.

Assume $(i_0, j_0) \in \mathcal{A}$. Consider the elements of $\mathcal{B}$ that are incomparable with (i_0, j_0). Let (k, l) be an arbitrary one of them. By the choice of (i_0, j_0) we know that $i_0 - j_0 \leq k - l$. Since (i_0, j_0) and (k, l) are incomparable this implies $k > i_0$ and $l < j_0$, thus $\max((i_0, j_0), (k, l)) = (k, j_0)$. Since $(\mathcal{A}, \mathcal{B})$ is canonical this implies that every element of $\tau_{\mathcal{B}}((i_0, j_0))$ has the form $(., j_0)$. This means that $\tau_{\mathcal{B}}((i_0, j_0))$ is an ordered subset of $\mathcal{L}$. Thus it contains its maximum $t_{\mathcal{B}}((i_0, j_0))$.

Similarly, if $(i_0, j_0) \in \mathcal{B}$ then $\omega_{\mathcal{A}}((i_0, j_0))$ consists of elements of the form $(i_0, .)$ and so is an ordered subset of $\mathcal{L}$ therefore containing its minimum, $\mathrm{w}_{\mathcal{A}}((i_0, j_0))$. This completes the proof of the Lemma.

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Proof of the Theorem: By Lemma 1 it suffices to consider a canonical recovering pair $(\mathcal{A}, \mathcal{B})$. If it contains incomparable pairs (i.e., an $a \in \mathcal{A}$ and a $b \in \mathcal{B}$ that are incomparable), then by Lemmas 2, 2' and 3 we can modify these sets step by step in such a way that the cardinalities do not change and the modified sets form canonical recovering pairs while the number of incomparable pairs is strictly decreasing at each step. So this procedure ends with a canonical recovering pair $(\mathcal{A}', \mathcal{B}')$ where $|\mathcal{A}'| = |\mathcal{A}|$, $|\mathcal{B}'| = |\mathcal{B}|$ and every element of $\mathcal{A}'$ is comparable to every element of $\mathcal{B}'$. Then $(\mathcal{A}', \mathcal{B}')$ is a sandglass and so we are done.

■

4 References

[1] Cohen, G.: Problem 1 in: Bulletin of the Institute of Mathematics Academia Sinica, Vol. 16, No. 4, Dec. 1988, page 385.

[2] Simonyi, G.: On write–unidirectional memory codes, IEEE Trans. Inform. Theory, Vol. IT–35 (1989), No. 3, pp. 663–669.