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Edge Isoperimetric Theorems for Integer Point Arrays

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Abstract—We consider subsets of the n-dimensional grid with the Manhattan metrics, (i.e., the Cartesian product of chains of lengths $k_1, \dots, k_n$) and study those of them which have maximal number of induced edges of the grid, and those which are separable from their complement by the least number of edges. The first problem was considered for $k_1 = \cdots = k_n$ by Bollobás and Leader [1]. Here we extend their result to arbitrary $k_1, \dots, k_n$, and give also a simpler proof based on a new approach. For the second problem, [1] offers only an inequality. We show that our approach to the first problem also gives a solution for the second problem, if all $k_i = \infty$. If all k_i's are finite, we present an exact solution for $n = 2$.

Keywords—Discrete isoperimetric properties, $\mathcal{E}$-order, Lexicographic order, Manhattan metric.

1. INTRODUCTION

For nonnegative integers $k_1, k_2, \dots, k_n$, set

$$V^n = \{\mathbf{x} = (x_1, \dots, x_n) : 0 \le x_i \le k_i,\ x_i\text{'s are integers}\}.$$

Consider the grid graph M^n with the vertex set V^n, two vertices $\mathbf{x}, \mathbf{y}$ of which are joined by an edge iff $\rho(\mathbf{x}, \mathbf{y}) = 1$, where ρ is the Manhattan metric, $\rho(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$. Clearly, the graph M^n may be considered as the Cartesian product of chains of lengths $k_1, \dots, k_n$.

For $A \subseteq V^n$ and $\mathbf{x}, \mathbf{y} \in V^n$, $\rho(\mathbf{x}, \mathbf{y}) = 1$, we say that the edge $(\mathbf{x}, \mathbf{y})$ is an **inner edge** of the set A, if $\mathbf{x}, \mathbf{y} \in A$. Otherwise, if one of $\mathbf{x}, \mathbf{y}$ is in A and the other is not in A, the edge $(\mathbf{x}, \mathbf{y})$ is called a **boundary edge** of the set A. Denote by $E(A)$ (resp., $R(A)$), the collection of all inner (resp., boundary) edges of A.

Now let m be an integer. Consider all the m-element subsets of V^n and the following two extremal problems:

PROBLEM 1. *Find a set A with maximal possible value of $|E(A)|$.*

PROBLEM 2. *Find a set A with minimal possible value of $|R(A)|$.*

Similar problems may be considered with respect to any graph G. Notice that if G is regular of degree d, then

$$2 \cdot |E(A)| + |R(A)| = d \cdot |A|. \tag{1}$$

Thus, in this case, Problems 1 and 2 are equivalent in the sense that a solution of one of these problems is at the same time a solution of the other.

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In the binary case (i.e., when $k_1 = k_2 = \cdots = k_n = 1$), Problem 1 was first solved by Harper [2], and for arbitrary finite k_i's under the Hamming metric by Lindsey [3]. They proved that for each m, the set of the first m vertices of V^n in the **lexicographic order**, gives a solution for Problem 1 (and also for Problem 2). Here, by the lexicographic order $\mathcal{L}$, we mean the order induced by the following relation: a vector $\mathbf{x} \in V^n$ precedes $\mathbf{y} \in V^n$ if $x_i < y_i$ for some i with $x_1 = y_1, \ldots, x_{i-1} = y_{i-1}$. For the Hamming metrics, it is natural to assume that all k_i's are finite, as otherwise if, say, k_i is infinite, then the set $\{(0, \ldots, 0, x_i, 0, \ldots, 0),\ 0 \le x_i \le m-1\}$ gives a solution, since it contains an inner edge between any pair of its vertices.

In the nonbinary case under the Manhattan metric, the graph M^n is not regular, and so the equivalence of Problems 1 and 2 is not insured. It turned out, however, that if all k_i's are infinite, these problems have a common solution. It is interesting that in the "bounded" case, i.e., when all k_i's are finite, Problem 2 has no nested structure of solutions, while Problem 1 always has it, and so in this case, our problems are not equivalent.

Problem 1 was solved first by Bollobás and Leader [1] for $k_1 = \cdots = k_n$. In the next section, we present a simpler proof, which works for arbitrary k_i's. It turned out that the solution we give works either for the "infinite" case or for the "bounded" one.

Section 3 of our paper is devoted to Problem 2 in the "infinite" case, i.e., when $k_i = \infty$, $i = 1, \ldots, n$. For the "bounded" version, we are able to give an exact solution for the two-dimensional case only. It turned out that there exist only two sets, "suspicious" to optimality, and when m grows, the solution structure switches ones from one set to another. The study of such switches is of particular interest, since, if a problem has no nested structure of solutions, the present techniques, as a rule, cannot be applied for solving it. Some other examples of dealing successfully with "jumping" solutions one can find in [4], where there exist many switches, and in [5], with only one switch. Finally, in [2], one can find an edge isoperimetric inequality for Problem 2, from which an exact solution for some particular values of m follows for $n \ge 3$.

2. SOLUTION OF PROBLEM 1

Denote $V^{n,\infty} = \{(x_1, \ldots, x_n) : x_i \ge 0,\ 1 \le i \le n\}$. We introduce an order $\mathcal{E}$ on $V^{n,\infty}$ and prove that, for any m, the set induced by the initial segment of length m in $\mathcal{E}$ gives a solution of Problem 1.

Notice that $\mathcal{E}$ induces also some order on the set V^n. Denote by $I_\mathcal{E}(m) \subseteq V^n$ the initial segment of length m in this induced order. Throughout this section, we assume that $1 \le k_1 \le \cdots \le k_n$.

For $\mathbf{x} = (x_1, \ldots, x_n) \in V^n$, denote $]\mathbf{x}[\, = \max_i x_i$ and let $\tilde{\mathbf{x}}$ be the vector obtained from $\mathbf{x}$ by replacing all entries not equal to $]\mathbf{x}[$ by 0. The order $\mathcal{E}$ is defined inductively. For $\mathbf{x}, \mathbf{y} \in V^n$, we say $\mathbf{x} >_\mathcal{E} \mathbf{y}$ iff

(i) $]\mathbf{x}[\, > \,]\mathbf{y}[$, or
(ii) $]\mathbf{x}[\, = \,]\mathbf{y}[$ and $\tilde{\mathbf{x}} >_\mathcal{L} \tilde{\mathbf{y}}$, or
(iii) $]\mathbf{x}[\, = \,]\mathbf{y}[\, = t > 1$, $\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$, and $\mathbf{x}' >_\mathcal{E} \mathbf{y}'$,

where $\mathbf{x}', \mathbf{y}'$ are obtained from $\mathbf{x}, \mathbf{y}$, respectively, by deleting all entries with $x_i = y_i = t$.

Therefore, we first order lexicographically all vectors with binary entries, and, in the binary case, our order $\mathcal{E}$ is just the lexicographic order. As an example, we list the vertices of V^3 for $k_1 = k_2 = k_3 = 2$ in increasing order of $\mathcal{E}$: 000 001 010 011 100 101 110 111 002 012 102 112 020 021 120 121 022 122 200 201 210 211 202 212 220 221 222.

LEMMA 1. *Let $\mathbf{x} >_\mathcal{E} \mathbf{y}$ and $x_i = y_i$. Then $\mathbf{x}' >_\mathcal{E} \mathbf{y}'$, where $\mathbf{x}', \mathbf{y}'$ are obtained from $\mathbf{x}, \mathbf{y}$, respectively, by deleting the i^{th} entry.*

PROOF. We apply induction on n and follow the definition of the order $\mathcal{E}$. For $n = 1$, the lemma is obviously true, so let $n \ge 2$. If $]\mathbf{x}[\, > \,]\mathbf{y}[$ holds, then $]\mathbf{x}'[\, > \,]\mathbf{y}'[$, and we are done. If $]\mathbf{x}[\, = \,]\mathbf{y}[$ and $\tilde{\mathbf{x}} >_\mathcal{L} \tilde{\mathbf{y}}$, then either $]\mathbf{x}[\, = \,]\mathbf{x}'[\, > \,]\mathbf{y}'[$ or $\tilde{\mathbf{x}'} >_\mathcal{L} \tilde{\mathbf{y}'}$, by the definition of the lexicographic order, and so $\mathbf{x}' >_\mathcal{E} \mathbf{y}'$. Finally, let $]\mathbf{x}[\, = \,]\mathbf{y}[$ and $\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$. If $x_i = \,]\mathbf{x}[$, then $\mathbf{x}' >_\mathcal{E} \mathbf{y}'$ by (iii) in the definition of $\mathcal{E}$. If $x_i \neq]\mathbf{x}[$, then delete all the entries of $\mathbf{x}, \mathbf{y}$ which are equal to $]\mathbf{x}[$. We get vectors $\mathbf{x}'', \mathbf{y}''$ of smaller length with $\mathbf{x}'' >_{\mathcal{E}} \mathbf{y}''$, and the lemma follows by induction.

We introduce $V_i^n(j) = \{(x_1, \dots, x_n) \in V^n : x_i = j\}$, $i = 1, \dots, n$, $j = 0, \dots, k_i$ and $N(x)$ as the position number of $x \in V^n$ in the order $\mathcal{E}$. We also define

$$N(A) = \sum_{x \in A} N(x), A_i(j) = A \cap V_i^n(j),$$

and introduce the compression operator $C_i(A)$, which replaces the part $A_i(j)$ of A by the collection of the first $|A_i(j)|$ elements of $V_i^n(j)$ in order $\mathcal{E}$ simultaneously for each $j = 1, \dots, k_i$. Clearly, $N(C_i A) \leq N(A)$ by Lemma 1.

For $\mathbf{x} = (x_1, \dots, x_n) \in V^{n,\infty}$, we denote by $\underline{\mathbf{x}}$ the projection of $\mathbf{x}$ on the set V^n, i.e., the vector, whose i^{th} entry equals $\underline{x_i} = \min\{x_i, k_i\}$, $1 \leq i \leq n$.

Since our proof technique works for $n \geq 3$ only, we consider the case $n = 2$ separately.

LEMMA 2. *Let $A \subseteq V^2$ and $C_i(A) = A$ for $i = 1, 2$. Then $|E(I_{\mathcal{E}}(|A|))| \geq |E(A)|$.*

PROOF. One has

$$|E(A)| = 2|A| - (|A_1(0)| + |A_2(0)|). \tag{2}$$

Let $\mathbf{x} = (x_1, x_2) \notin A$ and $x_1, x_2 > 0$. We call $\mathbf{x}$ corner vector if $(x_1 - 1, x_2) \in A$ and $(x_1, x_2 - 1) \in A$.

Assume first that $|A_1(0)| \leq |A_2(0)|$. Consider the vector $\mathbf{y} = (y_1, y_2) \in A$ with $y_1 = |A_2(0)|$ and y_2 maximal possible and replace it with some corner vector. It is clear that this replacement decreases the function N. So, if there are corner vectors, then using such replacements one can transform the set A to $I_{\mathcal{E}}(m)$.

Consider the case where there is no corner vector. If now $|A_1(0)| = k_2$, then $A = I_{\mathcal{E}}(m)$ and we are done. Otherwise, if $|A_1(0)| < k_2$, replace the set $A_1(|A_2(0)|)$ by the set $\{(x_1, x_2) : 0 \leq x_1 \leq |B_2(0)| - 1,\ x_2 = |B_1(0)| + 1\}$. One gets a set B with $|E(B)| = |E(A)|$, but $N(D) < N(B)$. Clearly, there exists at least one corner vector for the set B, and we apply the replacements above to the set B. The proof in case $|A_1(0)| > |A_2(0)|$ is similar.

THEOREM 1. *$|E(I_{\mathcal{E}}(|A|))| \geq |E(A)|$ for any $A \subseteq V^n$.*

PROOF. Assume that $|A| = m$ and $A \neq I_{\mathcal{E}}(m)$. We use induction on n. For $n = 1, 2$, the inequality is true. Let us proceed with the inductive step for $n \geq 3$. Since for any set A one has

$$|E(A)| \leq \sum_{j=0}^{k_i} |E(A_i(j))| + \sum_{j=1}^{k_i} \min\{|A_i(j)|, |A_i(j-1)|\}, \tag{3}$$

then, using the induction hypothesis, it follows that $|E(C_i A)| \geq |E(A)|$. Lemma 1 implies that $N(A)$ cannot increase after the transformation C_i. Clearly, $N(A)$ strictly decreases, if the transformation C_i is nontrivial. Therefore, after a finite number of applications of C_i with $i = 1, 2, \dots, n, 1, 2, \dots$, one gets a stable set B for which $C_i B = B$ holds for $i = 1, 2, \dots, n$. Notice that for a stable set, the conditions $(x_1, \dots, x_n) \in B$ and $x_i > 0$ imply $(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_n) \in B$.

We proceed with more operations, which decrease the function N and transform a stable set B into $I_{\mathcal{E}}(|B|)$ without decreasing E. Denote by $\mathbf{x}$ the greatest vector of B in the order $\mathcal{E}$, and by $\mathbf{y}$ the least vector in order $\mathcal{E}$ which is not in B. Then $\mathbf{x} >_{\mathcal{E}} \mathbf{y}$. If now $x_i = y_i$ for some i, then $\mathbf{y} \in B$ follows from Lemma 1 and $C_i B = B$.

Assume that $]\mathbf{x}[= t >]\mathbf{y}[> 0$ and show that $T = \{\mathbf{z} \in V^n :]\mathbf{z}[= t - 1\} \subseteq B$. Clearly, $(0, \dots, 0, t) \in B$, hence, $(\underline{t-1}, \dots, \underline{t-1}, 0, \underline{t-1}) \in B$. Therefore, one has only to prove that $P \subseteq B$ and $Q \subseteq B$, where

$$P = \{(\underline{t-1}, \dots, \underline{t-1}, p, \underline{t-1}) : 1 \leq p \leq \underline{t-1}\}, \quad Q = \{(\underline{t-1}, \dots, \underline{t-1}, q) : 0 \leq q \leq t-1\}.$$

Since $|E(B \setminus \{\mathbf{x}\})| \geq |E(B)| - n$ and for any $\mathbf{z} \in P \setminus B$, one has $|E(B \cup \{\mathbf{z}\})| = |E(B)| + n$, and since $N(\mathbf{z}) < N(\mathbf{x})$, after replacing $\mathbf{x}$ by $\mathbf{z}$, we either transform the set B to $I_{\mathcal{E}}(m)$ or $P \subseteq B$.

As to the set Q, if $\mathbf{t} = (\underline{t-1}, \ldots, \underline{t-1}, 0) \in B$, then we apply the arguments above. So let $\mathbf{t} \notin B$. Without loss of generality, one may assume that there exists a j for which $x_j < \underline{t-1}$ holds, since otherwise $(\underline{t-1}, \ldots, \underline{t-1}) \in B$ and we are done. Consider the set $T = \{(z_1, \ldots, z_n) : z_i = x_i \text{ for } i \neq j,\ 0 \leq z_j \leq x_j\}$ and replace it with the set $S = \{(\underline{t-1}, \ldots, \underline{t-1}, s) : 0 \leq s \leq x_j\}$. Then

$$|E((B \setminus T) \cup S)| = |E(B)|,$$

but the function N decreases. Now we have $(\underline{t-1}, \ldots, \underline{t-1}, 0) \in B$, and either $B = I_{\mathcal{E}}(m)$ or $(\underline{t-1}, \ldots, \underline{t-1}) \in B$, and so $T \subseteq B$ holds.

Now let $]\mathbf{x}[\ = \ ]\mathbf{y}[\ = t > 1$. If $x_i = y_i = t$ for some i, then by similar reasoning to above $\mathbf{y} \in B$. So, we may assume that for some subscript i the following holds: $x_i = t$, $y_i < t$, and either $i = 1$ or $x_j < t$, $y_j < t$ for $1 \leq j < i$. Notice that if $x_j > y_j$ for some $j \neq i$, then $\mathbf{y} \in B$. Indeed, consider the vector $\mathbf{z}$ obtained from $\mathbf{x}$ by replacing x_j by y_j. One has $\mathbf{x} >_{\mathcal{E}} \mathbf{z} >_{\mathcal{E}} \mathbf{y}$ and $\mathbf{z} \in B$ implies $\mathbf{y} \in B$. Hence, $x_j < y_j$ for $j \neq i$, and so $y_j \neq 0$ for $j \neq i$. If now $y_i \neq 0$, then $|E(B \cup \{\mathbf{y}\})| = |E(B)| + n$, and we may replace the vector $\mathbf{x}$ by $\mathbf{y}$ without decreasing E, but with decreasing N.

Finally, if $y_i = 0$, then the two following cases are possible. In the first case, assume $x_j = 0$ for all $j \neq i$. Then clearly one could replace $\mathbf{x}$ by $\mathbf{y}$ without increasing E. Otherwise, if $x_j \neq 0$ for some $j \neq i$, then similarly to the above consider the sets

$$T = \{(z_1, \ldots, z_n) : z_s = x_s \text{ for } s \neq j, \text{ and } 0 \leq z_j \leq x_j\},$$
$$S = \{(z_1, \ldots, z_n) : z_s = y_s \text{ for } s \neq i, \text{ and } 0 \leq z_i \leq x_j\}.$$

Since $T \subseteq B$ and $S \cap B = \emptyset$, one may replace T by S without decreasing E, but with decreasing N.

In order to complete the proof of the whole theorem, we have to consider the case $t = 1$. In this case, $\mathbf{x}$ and $\mathbf{y}$ are binary vectors, and one may assume that $\mathbf{y}$ is the binary coordinatewise negation of $\mathbf{x}$, since otherwise $\mathbf{y} \in B$ as above. If there exists a vector $\mathbf{z}$ with $\mathbf{x} >_{\mathcal{L}} \mathbf{z} >_{\mathcal{L}} \mathbf{y}$, then $\mathbf{y} \in B$, since $x_i = z_i$ and $z_j = y_j$ for some i, j. Therefore, one has to consider only the case $\mathbf{x} = (1, 0, \ldots, 0)$, $\mathbf{y} = (0, 1, \ldots, 1)$. But in this case, replacement of $\mathbf{x}$ by $\mathbf{y}$ strictly increases the number of inner edges, which completes the proof.

3. SOLUTION OF PROBLEM 2

Consider first the case when all k_i are infinite, i.e., $V^n = V^{n,\infty}$. We will show that any initial segment in the order $\mathcal{E}$ gives a solution.

THEOREM 2. *$|R(I_{\mathcal{E}}(|A|))| \leq |R(A)|$ for any $A \subseteq V^n$.*

PROOF. The proof is very similar to the proof of Theorem 1. We go along the lines of this proof and discuss only the differences. So, assume that $|A| = m$ and $A \neq I_{\mathcal{E}}(m)$. We use induction on n. For $n = 1$, the Theorem is obviously true. For $n = 2$ instead of (2), we have

$$|R(A)| = |A_1(0)| + |A_2(0)|, \tag{4}$$

and so we have to maximize the same quantity as in (2) again, which proves this case.

Let us proceed with the induction step for $n \geq 3$. Instead of (3), one has

$$|R(A)| \geq \sum_{j=0}^{k_i} |R(A_i(j))| + \sum_{j \geq 1} \Big|\, |A_i(j)| - |A_i(j-1)| \,\Big|, \tag{5}$$

and hence, by the induction hypothesis, it follows that $|R(C_i A)| \leq |R(A)|$. Therefore, we may restrict ourselves to consider only a stable set B.

Denote again by $\mathbf{x}$ the greatest vector of B in the order $\mathcal{E}$, and by $\mathbf{y}$ the least vector in order $\mathcal{E}$ which is not in B. Then $\mathbf{x} >_{\mathcal{E}} \mathbf{y}$. We may assume $x_i \neq y_i$ for $1 \leq i \leq n$.

Assume that $]\mathbf{x}[= t >]\mathbf{y}[> 0$ and show that $T = \{\mathbf{z} \in V^n :]\mathbf{z}[= t-1\} \subseteq B$. Clearly, $(0, \dots, 0, t) \in B$, hence, $(t-1, \dots, t-1, 0, t-1) \in B$. Therefore, one has to prove only that $P \subseteq B$ and $Q \subseteq B$ where

$$P = \{(t-1, \dots, t-1, p, t-1) : 1 \leq p \leq t-1\}, \; Q = \{(t-1, \dots, t-1, q) : 0 \leq q \leq t-1\}.$$

Since $|R(B \setminus \{\mathbf{x}\})| \leq |R(B)| + n$ and for any $\mathbf{z} \in P \setminus B$ one has $|R(B \cup \{\mathbf{z}\})| = |R(B)| - n$, and since $N(\mathbf{z}) < N(\mathbf{x})$, after replacing $\mathbf{x}$ by $\mathbf{z}$, we either transform the set B into $I_{\mathcal{E}}(m)$ or $P \subseteq B$.

As to the set Q, if $\mathbf{t} = (t-1, \dots, t-1, 0) \in B$, then one can apply the arguments from above. Let $\mathbf{t} \notin B$. Notice that there exists a j for which $x_j < t-1$ holds. Consider the set $T = \{(z_1, \dots, z_n) : z_i = x_i \text{ for } i \neq j, \, 0 \leq z_j \leq x_j\}$ and replace it by the set $S = \{(t-1, \dots, t-1, s) : 0 \leq s \leq x_j\}$. Then

$$|R((B \setminus T) \cup S)| = |R(B)|,$$

but the function N decreases. Now we have $(t-1, \dots, t-1, 0) \in B$, and either $B = I_{\mathcal{E}}(m)$ or $(t-1, \dots, t-1) \in B$. Thus $T \subseteq B$ holds.

Now let $]\mathbf{x}[=]\mathbf{y}[= t > 1$. Then for some i, one has $x_i = t$, $y_i < t$, and either $i = 1$ or $x_j, y_j < t$ for $j < i$. There is no loss of generality to assume that $x_j < y_j$ for $j \neq i$, and so $y_j \neq 0$ for $j \neq i$. If now $y_i \neq 0$, then $|R(B \cup \{\mathbf{y}\})| = |R(B)| - n$, and we may replace the vector $\mathbf{x}$ by $\mathbf{y}$ without increasing R, but with decreasing N.

Finally, if $y_i = 0$, then similarly (see the proof of Theorem 1) we replace the vector $\mathbf{x}$ by the vector $\mathbf{y}$ or the set T by the set S without increasing R, but with decreasing N.

In the last case $t = 1$, the proof is quite similar to the proof of Theorem 1).

Consider now the "bounded" 2-dimensional version of this problem, i.e., let $k_1, k_2 < \infty$ and $k_1 \leq k_2$. Let A be an optimal m-element subset. We may restrict our attention considering the case $m \leq k_1 k_2/2$ only, because the number of boundary edges of the set and its complement are the same.

THEOREM 3.

(i) *If* $m \leq \lfloor \sqrt{k_1/2} \rfloor$, *then* $|R(I_{\mathcal{E}}(m))| \leq |R(A)|$ *for any* $A \subseteq V^2$;
(ii) *if* $\lfloor \sqrt{k_1/2} \rfloor \leq m \leq k_1 k_2/2$, *then* $|R(I_{\mathcal{L}}(m) \leq |R(A)|$ *for any* $A \subseteq V^2$.

PROOF. Without loss of generality, we may assume that A is stable, i.e., $C_i(A) = A$ for $i = 1, 2$. Denote by l_1 (respectively, by l_2), the number of vectors of A of the form $(0, x)$ (respectively, $(x, 0)$). Then the two following cases are possible:

CASE 1. $l_1 < k_1$ and $l_2 < k_2$. Here, the number of boundary edges for such a set A equals simply $l_1 + l_2$. It is clear that A is inside an $l_1 \times l_2$ rectangular area, and so if $m = q^2 + p$, then $|R(A)| \geq 2q$, if $p = 0$, or $|R(A)| \geq 2q + 1$, if $p > 0$, i.e., the square is an optimal solution.

CASE 2. $l_1 = k_1$ or $l_2 = k_2$. Assume first that only one of these inequalities holds. Then $|R(A)| \geq \min\{k_1, k_2\} = k_1$, and clearly, $I_{\mathcal{L}}(m)$ has exactly k_1 boundary edges.

Now let $l_1 = k_1$ and $l_2 = k_2$ hold. Then $|R(A)| = k_1 + k_2 - (r + c)$, where r and c are, respectively, the numbers of completely filled rows and columns of the grid V^n in the set A. One has $k_1 + k_2 - r - c > k_1$, because otherwise, if $r + c > k_2$, then $|A| \geq k_1 r + k_2 c - rc > k_1 k_2/2$, which contradicts our assumptions.

Therefore, the solution of our problem is either $I_{\mathcal{E}}(m)$ or $I_{\mathcal{L}}(m)$. Notice that the number of boundary edges for the first set is an increasing function of m, while for the second set, it increases first, and then jumps between k_1 and $k_1 + 1$. Hence, as m increases until some m_0, there may exist two solutions, among which is $I_{\mathcal{E}}(m)$, and for $m_0 < m \leq k_1 k_2/2$, the set $I_{\mathcal{L}}(m)$ is better.

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