Note

A counterexample to Aharoni's

"Strongly maximal matching" conjecture

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It is conjectured (and proved for edge sets of graphs) in [1] that in every family $\mathcal{A}$ of finite sets a subfamily $\mathcal{B}$ of disjoint sets (called a "strongly maximal matching") exists, so that no replacement of k of them by more than k sets from $\mathcal{A}$ results again in a subfamily of disjoint sets.

As expected by Erdös (Introduction of [2]), the conjecture is false. A counterexample is $\mathcal{A}$, the family of those finite subsets of the set $\mathbb{N}$ of natural numbers, whose cardinality and smallest element (in canonical order) are equal.

In fact, suppose $\mathcal{A}$ contains a strongly maximal matching $\mathcal{B}$, then, by our definitions $\mathcal{B}$ is infinite, has an element $B = \{b_1 < b_2 < \cdots < b_t\}$ with $b_1 = t \geq 3$ and also an element $B' = \{b'_1 < b'_2 < \cdots < b'_{t'}\}$ with

(1) $t' = b'_1 \geq b_2 + b_3$.

By the disjointness property of $\mathcal{B}$

(2) $|B \cup B'| = t + t'$

and there exist disjoint $A_1, A_2, A_3 \in \mathcal{A}$:

(i) b_i is the minimal element of A_i and $|A_i| = b_i \quad (i = 1, 2, 3)$.

(ii) $A_1 \cup A_2 \cup A_3 \subset B \cup B'$.

The two sets $B, B' \in \mathcal{B}$ can be replaced by the three sets $A_1, A_2, A_3 \in \mathcal{A}$ without violating the disjointness property, but in violation of our supposition.

Remark: The conjecture remains open for families of sets of bounded sizes.

[1] R. Aharoni, "Infinite matching theory", Discrete Math. 95, 5–22, 1991.

[2] P. Erdös, "Problems and results in discrete mathematics", Discrete Math. 136, 53–73, 1994.

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