THE DIAMETRIC THEOREM IN HAMMING SPACES — OPTIMAL ANTICODES

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Abstract

For a Hamming space $(\mathcal{X}_\alpha^n, d_H)$, the set of n–length words over the alphabet $\mathcal{X}_\alpha = \{0, 1, \dots, \alpha-1\}$ endowed with the distance d_H, which for two words $x^n = (x_1, \dots, x_n)$, $y^n = (y_1, \dots, y_n) \in \mathcal{X}_\alpha^n$ counts the number of different components, we determine the maximal cardinality of subsets with a prescribed diameter d or, in another language, anticodes with distance d. We refer to the result as Diametric Theorem.

In a sense anticodes are dual to codes, which have a prescribed *lower* bound on the pairwise distance. It is a hopeless task to determine their maximal sizes exactly.

We find it remarkable that the Diametric Theorem (for arbitrary α) can be derived from our recent Complete Intersection Theorem, which can be viewed as a Diametric Theorem (for $\alpha = 2$) in the restricted case, where all n–length words considered have exactly k ones.

1. Previous results, conjectures, and the new theorem

This paper is another demonstration of the power of the methods of [3]. We stick to the earlier notation as far as possible and first repeat it. Then we state the complete Intersection Theorem in its historical connections, because this enables us to put the new result in proper perspective. Here we need some more terminology for the formulation of known results and conjectures for the diametric problem in Hamming space or related intersection problems. Finally we state the new Diametric Theorem.

$\mathbb{N}$ denotes the set of positive integers and for $i, j \in \mathbb{N}$, $i < j$, the set $\{i, i+1, \dots, j\}$ is abbreviated as $[i, j]$. Moreover, for $[1, j]$ we also write $[j]$. For $k, n \in \mathbb{N}$, $k \le n$, we set

$$2^{[n]} = \{F : F \subset [1, n]\} \text{ and } \binom{[n]}{k} = \{F \in 2^{[n]} : |F| = k\}.$$

A system of sets $\mathcal{A} \subset 2^{[n]}$ is called t–intersecting , if

$$|A_1 \cap A_2| \ge t \text{ for all } A_1, A_2 \in \mathcal{A}, \tag{1.1}$$

and $I(n, t)$ denotes the set of all such systems.

Moreover, we define $I(n, k, t) = \left\{\mathcal{A} \in I(n, t) : \mathcal{A} \subset \binom{[n]}{k}\right\}$. The investigation of the function $M(n, k, t) = \max\limits_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}|$, $1 \le t \le k \le n$, and the structure of maximal systems was initiated by Erdös, Ko, and Rado

Theorem EKR [5]. *For $1 \le t \le k$ and $n \ge n_0(k, t)$ (suitable)*

$$M(n, k, t) = \binom{n-t}{k-t}.$$

Clearly, the system

$$\mathcal{A}(n, k, t) = \left\{A \in \binom{[n]}{k} : [1, t] \subset A\right\}$$

is t–intersecting , has cardinality $\binom{n-t}{k-t}$, and is therefore optimal for $n \ge n_0(k, t)$.

The smallest $n_0(k, t)$, for which this is the case, has been determined by Frankl [6] for $t \ge 15$ and subsequently by Wilson [12] for all t :

$$n_0(k, t) = (k - t + 1)(t + 1).$$

In the recent paper [3] we have settled all the remaining cases: $n < (k - t + 1)(t + 1)$. The following result plays a key role in the present paper.

Complete Intersection Theorem AK [3].

Define $\mathcal{F}_i = \left\{F \in \binom{[n]}{k} : |F \cap [1, t+2i]| \geq t+i\right\}$ *for* $0 \leq i \leq \frac{n-t}{2}$.

For $1 \leq t \leq k \leq n$ *with*

(i) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right)$ *for some* $r \in \mathbb{N} \cup \{0\}$

we have

$$M(n,k,t) = |\mathcal{F}_r|$$

and $\mathcal{F}_r$ *is — up to permutations — the unique optimum. By convention* $\frac{t-1}{r} = \infty$ *for* $r = 0$.

(ii) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) = n$ *for* $r \in \mathbb{N} \cup \{0\}$

we have

$$M(n,k,t) = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$$

and an optimal system equals — up to permutations — either $\mathcal{F}_r$ *or* $\mathcal{F}_{r+1}$.

Erdös, Ko, and Rado also initiated the study of optimal systems in $I(n,t)$ and of the function

$$M(n,t) = \max_{\mathcal{A} \in I(n,t)} |\mathcal{A}|.$$

A complete description was given by Katona, who in particular obtained

Theorem Ka [8].

$$M(n,t) = \begin{cases} \sum\limits_{i=\frac{n+t}{2}}^{n} \binom{n}{i}, & \text{if } n+t \text{ is even} \\ \sum\limits_{i=\frac{n+t+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+t-1}{2}}, & \text{if } n+t \text{ is odd.} \end{cases}$$

His proof proceeds by estimating "shadows" of sets in $\binom{[n]}{k}$. Actually, it also can be proved by the method of the present paper.

Now we make a transition from $2^{[n]}$ to $\mathcal{X}_2^n = \{0,1\}^n$ and the more general $\mathcal{X}_\alpha^n = \{0, 1, \ldots, \alpha-1\}^n$.

Clearly, any set $A \in 2^{[n]}$ can be represented as word $a^n = (a_1, \ldots, a_n) \in \mathcal{X}_2^n$, where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \in [n] \setminus A, \end{cases}$$

and conversely.

Furthermore, in $\mathcal{X}_\alpha^n$ we have a second concept of intersection. We call $\mathcal{A} \subset \mathcal{X}_\alpha^n$ $t - \mathcal{X}_\alpha$–intersecting , if for all $a^n, b^n \in \mathcal{A}$

$$int(a^n, b^n) \triangleq |\{j \in [n] : a_j = b_j\}| \geq t. \tag{1.2}$$

Let $J_\alpha(n,t)$ denote the set of all such systems.

Since $d_H(a^n,b^n) = \big(\{j \in [1,n] : a_j \neq b_j\}\big) = n - int(a^n,b^n)$ we can equivalently say that $\mathcal{A}$ has a diameter

$$diam(\mathcal{A}) \triangleq \max_{a^n,b^n \in \mathcal{A}} d_H(a^n,b^n) \leq d = n-t$$

or that $\mathcal{A}$ is d–diametric .

It is important to notice, that the notions $t-\mathcal{X}_2$–intersecting in $\mathcal{X}_2^n$ and t–intersecting in $2^{[n]}$ are quite different!

We are concerned here with the function

$$N_\alpha(n,t) = \max_{\mathcal{A} \in J_\alpha(n,t)} |\mathcal{A}|. \tag{1.3}$$

There is already a well–known result for $\alpha = 2$ due to Kleitman

Theorem Kl [9].

$$N_2(n,t) = \begin{cases} \sum\limits_{i=0}^{\frac{n-t}{2}} \binom{n}{i}, & \textit{if } n-t \textit{ is even} \\ 2 \sum\limits_{i=0}^{\frac{n-t-1}{2}} \binom{n-1}{i}, & \textit{if } n-t \textit{ is odd.} \end{cases}$$

This result and Theorem Ka imply

$$N_2(n,t) = M(n,t). \tag{1.4}$$

Actually, it was shown in [2] that the two theorems can be easily derived from each other by passing through upsets.

Finally, we report on the results known (to us), which were obtained during the last three decades on $N_\alpha(n,t)$ for $\alpha > 2$.

Berge [4] proved that

$$N_\alpha(n,1) = \alpha^{n-1} \quad \text{for} \quad \alpha \geq 3. \tag{1.5}$$

Livingston [11] showed that the $\mathcal{A} \in J_\alpha(n,1)$ satisfying (1.5) are of the form

$$\mathcal{A} = \big\{a^n = (a_1,\dots,a_n) \in \mathcal{X}_\alpha^n : a_i = a\big\}$$

for some $i \in [n]$ and $a \in \{0,1,\dots,\alpha-1\}$.

Frankl and Füredi [7] conjectured that

$$N_\alpha(n,t) = \alpha^{n-t} \quad \text{iff} \quad n \leq t+1 \quad \text{or} \quad \alpha \geq t+1, \tag{1.6}$$

and they proved this for $t \geq 15$.

Ahlswede, Cai and Zhang [1] remarked that for $n \leq t + 1 + \frac{\log t}{\log(\alpha-1)}$

$$N_\alpha(n,t) = \begin{cases} \sum\limits_{i=0}^{\frac{n-t}{2}} \binom{n}{i}(\alpha-1)^i, & \text{if } n-t \text{ is even} \\ \alpha \sum\limits_{i=0}^{\frac{n-t-1}{2}} \binom{n-1}{i}(\alpha-1)^i, & \text{if } n-t \text{ is odd.} \end{cases} \tag{1.7}$$

Frankl and Füredi [7] (and in diametric formulation also Ahlswede, Cai, and Zhang [1]) have made the

General Conjecture

$$N_\alpha(n,t) = \max_{0 \leq i \leq \frac{n-t}{2}} |\mathcal{K}_i| \quad \text{for all } n, \alpha, t, \tag{1.8}$$

where with the convention $B(a^n) = \{j : a_j = \alpha - 1\}$

$$\mathcal{K}_i = \big\{ a^n \in \mathcal{X}_\alpha^n : B(a^n) \cap [1, t+2i]| \geq t + i \big\}. \tag{1.9}$$

Clearly, $\mathcal{K}_i \in J_\alpha(n,t)$ for $0 \leq i \leq \frac{n-t}{2}$.

We note that in this terminology the results (1.6) and (1.7) can be summarized in the form

$$N_\alpha(n,t) = \begin{cases} |\mathcal{K}_0|, & \text{if } n \leq t+1 \text{ or } \alpha \geq t+1, t \geq 15 \\ \left|\mathcal{K}_{\lfloor \frac{n-t}{2} \rfloor}\right|, & \text{if } n \leq t+1+\frac{\log t}{\log(\alpha-1)}. \end{cases}$$

This covers only very few values of the parameters n, α, t . We settle here all cases by establishing the General Conjecture.

Diametric Theorem. *For $\alpha \geq 2$ let $r \in \{0\} \cup \mathbb{N}$ be the largest integer such that*

$$t + 2r < \min\left\{ n+1, t + 2\frac{t-1}{\alpha-2} \right\},$$

then $N_\alpha(n,t) = |\mathcal{K}_r|$. (By convention $\frac{t-1}{s-2} = \infty$ for $\alpha = 2$.)

Remark 1: Actually, we also can prove that up to permutation of $\{1, 2, \dots, n\}$ and permutations of the alphabet in the components there is exactly one optimal configuration, unless $t > 1$, $t + 2\frac{t-1}{\alpha-2} \leq n$ and $\frac{t-1}{\alpha-2}$ is integral, in which case we have two optimal configurations, $\mathcal{K}_{\frac{t-1}{\alpha-2}}$ and $\mathcal{K}_{\frac{t-1}{\alpha-2}-1}$.

Remark 2: A generalization of Theorem Ka to every $\alpha \geq 2$ can be obtained with the intersection concept based on the quantity

$$\operatorname{int}^*(a^n, b^n) \triangleq |\{ j \in [n] : a_j = b_j = \alpha - 1 \}|$$

instead of the quantity $\operatorname{int}(a^n, b^n)$ defined in (1.2).

In analogy to $N_\alpha(n,t)$ we get now $M_\alpha(n,t)$, where $M_2(n,t)$ equals the familiar $M(n,t)$ and obviously $M_\alpha(n,t) \leq N_\alpha(n,t)$. The structure of the $\mathcal{K}_i$'s and the Diametric Theorem imply now $M_\alpha(n,t) = N_\alpha(n,t)$. In particular we have thus shown that Theorem Ka can also be proved by our methods.

2. Reduction to canonical and stable sets

We give combinatorial characterisations of $N_\alpha(n,t)$ of increasing precision. The first one is not new.

Proposition FF [7] $$N_\alpha(n,t) = \max_{\mathcal{G}\in I(n,t)} \sum_{i=0}^{n} |\mathcal{G}_i|(\alpha-1)^{n-i}, \tag{2.1}$$

where $\mathcal{G}_i = \mathcal{G} \cap \binom{[n]}{i}$.

We derive this result, because we want to start from first principles and at the same time introduce some concepts. Here it is more convenient to write A for a word $a^n = (a_1,\dots,a_n)$. It seems that the following transformation was first used by Kleitman [9].

For any $\mathcal{A} \subset \mathcal{X}_\alpha^n$, any $A = (a_1, a_2, \dots, a_n) \in \mathcal{A}$, and $1 \le j \le n$, $0 \le i \le \alpha - 1$ we define

$$T_{ji}(A) = \begin{cases} (a_1,\dots,a_{j-1},\alpha-1,a_{j+1},\dots,a_n), & \text{if this is not element of } \mathcal{A} \text{ and } a_j = i \\ A \ \text{ otherwise} \end{cases}$$

and
$$T_{ji}(\mathcal{A}) = \big\{T_{ji}(A) : A \in \mathcal{A}\big\}.$$

Repeated application of these transformations yields after finitely many steps an $\mathcal{A}' \subset \mathcal{X}_\alpha^n$, for which
$$T_{ji}(\mathcal{A}') = \mathcal{A}' \ \text{ for all } \ 1 \le j \le n, 0 \le i \le \alpha - 1. \tag{2.2}$$

Definition 2.1: A set $\mathcal{A} \subset \mathcal{X}_\alpha^n$ is said to be *canonical*, if

$$T_{ij}(\mathcal{A}) = \mathcal{A} \ \text{ for all } \ 1 \le j \le n, 0 \le i \le \alpha - 1.$$

The transformation T_{ji} has the important properties to keep the cardinality and the $t-\mathcal{X}_\alpha$–intersection property unchanged, i.e. $|T_{ji}(\mathcal{A})| = |\mathcal{A}|$ and $\mathcal{A} \in J_\alpha(n,t)$ implies $T_{ji}(\mathcal{A}) \in J_\alpha(n,t)$.

Hence
$$N_\alpha(n,t) = \max_{\mathcal{A}\in J_\alpha(n,t)} |\mathcal{A}| = \max_{\mathcal{A}\in CJ_\alpha(n,t)} |\mathcal{A}|, \tag{2.3}$$

where $CJ_\alpha(n,t) \subset J_\alpha(n,t)$ is the set of all canonical systems in $J_\alpha(n,t)$.

Now we make the transition to $2^{[n]}$.

Definition 2.2: To a system $\mathcal{A} \in CJ_\alpha(n,t)$ we associate the *set–theoretical image* $\mathcal{B}(\mathcal{A}) = \big\{B(A) : A \in \mathcal{A}\big\}$, where $B(A)$ is defined in (1.8).

We have an immediate consequence

Lemma 1 FF [7]. *For* $\mathcal{A} \in CJ_\alpha(n,t)$

$$\mathcal{B}(\mathcal{A}) \in I(n,t).$$

Definition 2.3: For any $D \in 2^{[n]}$ we define the upset $\mathcal{U}(D) = \left\{D' \in 2^{[n]} : D \subset D'\right\}$. More generally (with slight abuse of notation) for $\mathcal{D} \subset 2^{[n]}$ we define the upset

$$\mathcal{U}(\mathcal{D}) = \bigcup_{D \in \mathcal{D}} \mathcal{U}(D).$$

Again we have a direct consequence of the definitions.

Lemma 2. *Let* $\mathcal{A} \in CJ_\alpha(n,t)$ *satisfy* $|\mathcal{A}| = N_\alpha(n,t)$ *, and let* $\mathcal{B}(\mathcal{A})$ *be the set–theoretical image of* $\mathcal{A}$ *. Then*

(i) $\mathcal{B}(\mathcal{A})$ *is an upset.*

(ii)
$$|\mathcal{A}| = \sum_{B(A) \in \mathcal{B}(\mathcal{A})} (\alpha-1)^{n-|B(A)|} = \sum_{i=0}^{n} g_i (\alpha-1)^{n-i}, \tag{2.4}$$

where $g_i = \left| \mathcal{B}(\mathcal{A}) \cap \binom{[n]}{i} \right|$.

This yields the Proposition. We introduce another familiar concept.

Definition 2.4: A set of subsets $\mathcal{D} \subset 2^{[n]}$ is said to be *left–compressed* or *stable*, if for every $D \in \mathcal{D}$ and every $1 \leq i < j \leq n$ with $i \notin D$, $j \in D$ necessarily $D' = \big((D \smallsetminus \{j\}) \cup \{i\}\big) \in \mathcal{D}$.

We denote by $LI(n,t) \subset I(n,t)$ the set of all stable systems in $I(n,t)$ and by $LCJ_\alpha(n,t) \subset CJ_\alpha(n,t)$ the set of all systems $\mathcal{A} \in CJ_\alpha(n,t)$ with $\mathcal{B}(\mathcal{A}) \in LI(n,t)$.

From (2.2) and the left–pushing technique of [5] it readily follows that

$$N_\alpha(n,t) = \max_{\mathcal{A} \in J_\alpha(n,t)} |\mathcal{A}| = \max_{\mathcal{A} \in CJ_\alpha(n,t)} |\mathcal{A}| = \max_{\mathcal{A} \in LCJ_\alpha(n,t)} |\mathcal{A}|. \tag{2.5}$$

Next, for an $E \in 2^{[n]}$ we introduce

$$\mathcal{V}(E) = \left\{A \in \mathcal{X}_\alpha^n : B(A) \in \mathcal{U}(E)\right\}. \tag{2.6}$$

Clearly,

$$|\mathcal{V}(E)| = \alpha^{n-|E|}. \tag{2.7}$$

More generally, for $\mathcal{E} \subset 2^{[n]}$ we introduce

$$\mathcal{V}(\mathcal{E}) = \bigcup_{E \in \mathcal{E}} \mathcal{V}(E).$$

Definition 2.5: Let $\mathcal{A} \in CJ_\alpha(n,t)$ and let $\mathcal{B}(\mathcal{A})$ be the set–theoretical image of $\mathcal{A}$. Then $\mathcal{A}$ is called *α–upset* , if

$$\mathcal{A} = \mathcal{V}\big(\mathcal{B}(\mathcal{A})\big).$$

Definition 2.6: For $E = \{e_1, e_2, \ldots, e_{|E|}\} \subset [n]$, $e_1 < e_2 < \cdots < e_{|E|}$, write the biggest element $e_{|E|}$ as $s^+(E)$. Also for $\mathcal{E} \subset 2^{[n]}$ set

$$s^+(\mathcal{E}) = \max_{E \in \mathcal{E}} s^+(E).$$

The next important properties immediately follow from left–compressedness arguments (similar to those in [3]).

Lemma 3. *Let $\mathcal{A} \in LCJ_\alpha(n,t)$, let $\mathcal{A}$ be an α–upset , and let $\mathcal{B}(\mathcal{A})$ be the set–theoretical image of $\mathcal{A}$. Further, let $M(\mathcal{A})$ be the set of minimal elements of $\mathcal{B}(\mathcal{A})$ (in the sense of set–theoretical inclusion). Then $\mathcal{A}$ is a disjoint union*

$$\mathcal{A} = \dot{\bigcup_{E \in M(\mathcal{A})}} D(E),$$

where

$$D(E) = \left\{ \begin{array}{l} A = (a_1, \ldots, a_n) \in \mathcal{X}_\alpha^n : B(A) \cap \big[1, s^+(E)\big] = E \quad \text{and} \\ (a_{s^+(E)+1}, \ldots, a_n) \in \mathcal{X}_\alpha^{n-s^+(E)} \end{array} \right\}. \tag{2.8}$$

Lemma 4. *For an α–upset $\mathcal{A} \in LCJ_\alpha(n,t)$ choose $E \in M(\mathcal{A})$ such that $s^+(E) = s^+\big(M(\mathcal{A})\big)$ and consider the set of elements of $\mathcal{A}$, which are only generated by E , that is,*

$$\mathcal{A}_E = \mathcal{V}(E) \smallsetminus \mathcal{V}\big(M(\mathcal{A}) \smallsetminus E\big).$$

Then

$$\mathcal{A}_E = D(E) \qquad \big(D(E) \text{ is defined in (2.8)}\big)$$

and

$$|\mathcal{A}_E| = (\alpha - 1)^{s^+(E) - |E|} \cdot \alpha^{n - s^+(E)}. \tag{2.9}$$

Lemma 5. *Let $\mathcal{A} \in LCJ_\alpha(n,t)$, let $\mathcal{A}$ be an α–upset , and let $E_1, E_2 \in M(\mathcal{A})$ have the properties $i \notin E_1 \cup E_2$, $j \in E_1 \cap E_2$ for some $i, j \in [n]$ with $i < j$. Then*

$$|E_1 \cap E_2| \geq t + 1.$$

3. The main two auxiliary results

Lemma 6. *For* $\alpha > 2$ *let* $\mathcal{A} \in LCJ_\alpha(n,t)$ *with* $|\mathcal{A}| = N_\alpha(n,t)$, *let* $\mathcal{B}(\mathcal{A})$ *be the set–theoretical image of* $\mathcal{A}$, *and let* $M(\mathcal{A})$ *be the set of all minimal elements of* $\mathcal{B}(\mathcal{A})$. *Then*

$$s^+\big(M(\mathcal{A})\big) = t + 2r \le t + \frac{2(t-1)}{\alpha-2} \quad \textit{for some} \quad r \in \{0\} \cup \mathbb{N}. \tag{3.1}$$

Moreover, if $\frac{t-1}{\alpha-2}$ *is a positive integer, then there exists an* $\mathcal{A}' \in LCJ_\alpha(n,t)$ *with* $|\mathcal{A}'| = N_\alpha(n,t)$ *such that*

$$s^+\big(M(\mathcal{A}')\big) = t + 2r' < t + \frac{2(t-1)}{\alpha-2} \quad \textit{for some} \quad r' \in \{0\} \cup \mathbb{N}. \tag{3.2}$$

Remark 1: From this result easily follows the proof of Conjecture (1.6), stated in the introduction. The proof goes as follows:

for $n \le t+1$ or $\alpha \ge t+1$ we have from (3.2) $s^+\big(M(A')\big) = t$ and hence $|\mathcal{A}| = N_\alpha(n,t) = \alpha^{n-t} = |K_0|$ and if $n > t+1$ and $\alpha < t+1$ it is easily verified that

$$N_\alpha(n,t) \ge |K_1| > |K_0|.$$

Remark 2: From this result the following observation is immediate: the structure of the optimal families does not depend on n for $n \ge t + \frac{2(t-1)}{\alpha-2}$.

Remark 3: The proof of Lemma 6 is mainly based on the ideas and methods used in our previous paper [3].

Proof: First we prove (3.1). Assume to the opposite of (3.1) that

$$s^+\big(M(\mathcal{A})\big) = \ell > t + \frac{2(t-1)}{\alpha-2} \tag{3.3}$$

or

$$\ell \le t + \frac{2(t-1)}{\alpha-2}, \quad \text{but} \quad 2 \nmid \ell - t. \tag{3.4}$$

We shall show that under these assumptions there exists an $\mathcal{A}' \in J_\alpha(n,t)$ with $|\mathcal{A}'| > |\mathcal{A}|$, which is a contradiction to $|\mathcal{A}| = N_\alpha(n,t)$. For this we start with the partition

$$M(\mathcal{A}) = M_0(\mathcal{A}) \,\dot\cup\, M_1(\mathcal{A}), \quad \text{where}$$
$$M_0(\mathcal{A}) = \big\{E \in M(\mathcal{A}) : s^+(E) = s^+\big(M(\mathcal{A})\big) = \ell\big\} \quad \text{and} \quad M_1(\mathcal{A}) = M(\mathcal{A}) \smallsetminus M_0(\mathcal{A}).$$

Obviously, for every $E_1 \in M_0(\mathcal{A})$ and $E_2 \in M_1(\mathcal{A})$ we have

$$\big|\big(E_1 \smallsetminus \{\ell\}\big) \cap E_2\big| \ge t.$$

The elements in $M_0(\mathcal{A})$ have an important property, which follows immediately from Lemma 5:

(P) for any $E_1, E_2 \in M_0(\mathcal{A})$ with $|E_1 \cap E_2| = t$ necessarily

$$|E_1| + |E_2| = \ell + t.$$

Now we partition $M_0(\mathcal{A})$ according to the cardinalities of its members

$$M_0(\mathcal{A}) = \bigcup_i \mathcal{R}_i, \mathcal{R}_i = M_0(\mathcal{A}) \cap \binom{[n]}{i}.$$

Of course, some of the $\mathcal{R}_i$'s can be empty.

Next we omit the element $\{\ell\}$, that is, we consider

$$\mathcal{R}_i' = \big\{E \subset [1, \ell-1] : E \cup \{\ell\} \in \mathcal{R}_i\big\}.$$

So $|\mathcal{R}_i| = |\mathcal{R}_i'|$ and for $E' \in \mathcal{R}_i'$ $|E'| = i - 1$. From Property (P) we know that for $E_1' \in \mathcal{R}_i'$, $E_2' \in \mathcal{R}_j'$ with $i + j \neq \ell + t$ necessarily

$$|E_1' \cap E_2'| \geq t. \tag{3.5}$$

We shall prove that (under assumptions (3.3) or (3.4)) all $\mathcal{R}_i$'s are empty.

Suppose, that for some i , $\mathcal{R}_i \neq \varnothing$ or equivalently $\mathcal{R}_i' \neq \varnothing$. We distinguish two cases:

a) $i \neq \frac{\ell+t}{2}$ and b) $i = \frac{\ell+t}{2}$.

Case a): We consider the sets

$$f_1 = M_1(\mathcal{A}) \cup \big(M_0(\mathcal{A}) \smallsetminus (\mathcal{R}_i \cup \mathcal{R}_{\ell+t-i})\big) \cup \mathcal{R}_i' \quad \text{and}$$

$$f_2 = M_1(\mathcal{A}) \cup \big(M_0(\mathcal{A}) \smallsetminus (\mathcal{R}_i \cup \mathcal{R}_{\ell+t-i})\big) \cup \mathcal{R}_{\ell+t-i}'.$$

We know already (see Property (P) and (3.5)) that

$$f_1, f_2 \in I(n,t)$$

and hence

$$\mathcal{A}_i = \mathcal{V}(f_i) \in J_\alpha(n,t) \quad \text{for} \quad i = 1, 2.$$

The desired contradiction shall take the form

$$\max_{i=1,2} |\mathcal{A}_i| > |\mathcal{A}|. \tag{3.6}$$

We consider the set $\mathcal{A} \setminus \mathcal{A}_1$. From the construction of f_1 and the $\mathcal{R}_i$'s it follows that

$$\mathcal{A} \setminus \mathcal{A}_1 = \bigcup_{E \in \mathcal{R}_{\ell+t-i}} D(E),$$

where the $D(E)$'s are defined in (2.8). Using Lemma 4 we have

$$|\mathcal{A} \setminus \mathcal{A}_1| = |\mathcal{R}_{\ell+t-i}| \cdot (\alpha - 1)^{i-t} \cdot \alpha^{n-\ell}. \tag{3.7}$$

Now we consider $\mathcal{A}_1 \setminus \mathcal{A}$. Let E_1 be any element of $\mathcal{R}_i$'s , so $|E_1| = i - 1$. We consider

$$D'(E_1) = \left\{ \begin{array}{l} A = (a_1, \ldots, a_n) \in \mathcal{X}_\alpha^n : B(A) \cap [1, \ell] = E_1 \text{ and} \\ (a_{\ell+1}, \ldots, a_n) \in \mathcal{X}_\alpha^{n-\ell} \end{array} \right\} \tag{3.8}$$

(recalling that $B(A) \in 2^{[n]}$ is the set–theoretical image of A).

It can easily be seen that

$$D'(E_1) \in \mathcal{A}_1 \setminus \mathcal{A}, \tag{3.9}$$

and

$$|D'(E_1)| = (\alpha - 1)^{\ell-i+1} \cdot \alpha^{n-\ell}. \tag{3.10}$$

We also notice that for $E_1, E_2 \in \mathcal{R}_i'$, $E_1 \neq E_2$, one has

$$D'(E_1) \cap D'(E_2) = \varnothing. \tag{3.11}$$

Therefore

$$|\mathcal{A}_1 \setminus \mathcal{A}| \geq |\mathcal{R}_i| \cdot (\alpha - 1)^{\ell-i+1} \cdot \alpha^{n-\ell}. \tag{3.12}$$

Analogously, we have

$$|\mathcal{A} \setminus \mathcal{A}_2| = |\mathcal{R}_i| \cdot (\alpha - 1)^{\ell-i} \cdot \alpha^{n-\ell} \tag{3.13}$$

and

$$|\mathcal{A}_2 \setminus \mathcal{A}_1| \geq |\mathcal{R}_{\ell+t-i}| \cdot (\alpha - 1)^{i-t+1} \cdot \alpha^{n-\ell}. \tag{3.14}$$

Actually, it is easy to show that there are equalities in (3.12) and (3.14). However, that is not needed here.

Now (3.7) and (3.12) – (3.14) enable us to state the negation of (3.6) in the form

$$\left\{ \begin{array}{ll} |\mathcal{R}_i|(\alpha - 1)^{\ell-i+1} \cdot \alpha^{n-\ell} & \leq |\mathcal{R}_{\ell+t-i}| \cdot (\alpha - 1)^{i-t} \cdot \alpha^{n-\ell} \\ |\mathcal{R}_{\ell+t-i}|(\alpha - 1)^{i-t+1} \cdot \alpha^{n-\ell} & \leq |\mathcal{R}_i| \cdot (\alpha - 1)^{\ell-i} \cdot \alpha^{n-\ell}, \end{array} \right.$$

which is obviously false, because $\mathcal{R}_i \neq \varnothing$ and $\alpha > 2$.

The case a) $i \neq \frac{t+\ell}{2}$, which we have just considered, shows that $2|(t+\ell)$, $\ell = s^+\big(M(\mathcal{A})\big)$. This shows that assumption (3.4) is false. Moreover, if $2|\ell+t$, then necessarily

$$|E| = \frac{\ell+t}{2} \text{ for all } E \in M(\mathcal{A}) \text{ with } s^+(E) = s^+\big(M(\mathcal{A})\big) = \ell.$$

Case b): $i = \frac{\ell+t}{2}$.

Here necessarily $2|\ell+t$. We consider the set $\mathcal{R}'_{\frac{t+\ell}{2}}$ and recall that for $E \in \mathcal{R}'_{\frac{t+\ell}{2}}$

$$|E| = \frac{t+\ell}{2} - 1 \text{ and } E \subset [1, \ell-1].$$

By the pigeon–hole principle there exists an $i \in [1, \ell-1]$ and a $\mathcal{T} \subset \mathcal{R}'_{\frac{t+\ell}{2}}$ such that $i \notin E$ for all $E \in \mathcal{T}$ and

$$|\mathcal{T}| \geq \frac{\ell-t}{2(\ell-1)} \cdot \left|\mathcal{R}'_{\frac{t+\ell}{2}}\right|. \tag{3.15}$$

By Lemma 5 we have $|E_1 \cap E_2| \geq t$ for all $E_1, E_2 \in \mathcal{T}$ and since by case a) $\mathcal{R}_i = \varnothing$ for $i \neq \frac{t+\ell}{2}$, we get

$$f' = \left(M(\mathcal{A}) \smallsetminus \mathcal{R}_{\frac{t+\ell}{2}}\right) \cup \mathcal{T} \in I(n,t)$$

and hence

$$\mathcal{V}(f') \in J_\alpha(n,t).$$

We are going to show (under condition (3.3)) that

$$|\mathcal{V}(f')| > |\mathcal{A}|. \tag{3.16}$$

Indeed let us write

$$\begin{aligned}
\mathcal{A} = \mathcal{V}\big(M(\mathcal{A})\big) = D_1 \,\dot\cup\, D_2, \quad \text{where } D_1 = \mathcal{V}\left(M(\mathcal{A}) \smallsetminus \mathcal{R}_{\frac{t+\ell}{2}}\right), \\
D_2 = \mathcal{V}\left(\mathcal{R}_{\frac{t+\ell}{2}}\right) \smallsetminus \mathcal{V}\left(M(\mathcal{A}) \smallsetminus \mathcal{R}_{\frac{t+\ell}{2}}\right), \quad \text{and } \mathcal{V}(f') = D_1 \,\dot\cup\, D_3, \quad \text{where} \\
D_3 = \mathcal{V}(\mathcal{T}) \smallsetminus \mathcal{V}\left(M(\mathcal{A}) \smallsetminus \mathcal{R}_{\frac{t+\ell}{2}}\right).
\end{aligned}$$

In this terminology, equivalent to (3.16) is

$$|D_3| > |D_2|. \tag{3.17}$$

We know by Lemma 4 that

$$D_2 = \left|\mathcal{R}_{\frac{t+\ell}{2}}\right| \cdot (\alpha-1)^{\frac{\ell-t}{2}} \cdot \alpha^{n-\ell} \tag{3.18}$$

and estimate $|D_3|$ from below.

Let $E \in \mathcal{T}$, $E \subset [\ell-1]$, and $|E| = \frac{t+\ell}{2} - 1$. We consider

$$\mathcal{C}(E) = \{A = (a_1, \dots, a_n) \in \mathcal{X}_\alpha^n : B(A) \cap [\ell-1] = E \text{ and } (a_\ell, \dots, a_n) \in \mathcal{X}_\alpha^{n-\ell+1}\}.$$

Clearly, $\mathcal{C}(E) \in D_3$ and $|\mathcal{C}(E)| = (\alpha-1)^{\frac{\ell-t}{2}} \cdot \alpha^{n-\ell+1}$. We also notice that for all $E_1, E_2 \in \mathcal{T}$, $E_1 \neq E_2$, we have

$$\mathcal{C}(E_1) \cap \mathcal{C}(E_2) = \varnothing.$$

Therefore

$$|D_3| \geq |\mathcal{T}| \cdot (\alpha-1)^{\frac{\ell-t}{2}} \cdot \alpha^{n-\ell+1}. \tag{3.19}$$

Actually, as in the similar situation in case a), we have equality in (3.19), but again it is not needed here.

In the light of (3.15) and (3.17) – (3.19) sufficient for (3.16) is

$$\frac{\ell-t}{2(\ell-1)} \left|\mathcal{R}_{\frac{t+\ell}{2}}\right| \cdot (\alpha-1)^{\frac{\ell-t}{2}} \cdot \alpha^{n-\ell+1} > \left|\mathcal{R}_{\frac{t+\ell}{2}}\right| \cdot (\alpha-1)^{\frac{\ell-t}{2}} \cdot \alpha^{n-\ell}.$$

According to (3.3) and $\mathcal{R}_{\frac{t+\ell}{2}} \neq \varnothing$ this is true. Therefore (3.16) holds in contradiction to the optimality of $\mathcal{A}$. Hence the assumption (3.3) is false and the first part of Lemma 6 is proved.

Now let $\frac{t-1}{\alpha-2}$ be a positive integer and let

$$s^+\big(M(\mathcal{A})\big) = \ell = t + 2\frac{t-1}{\alpha-2}. \tag{3.20}$$

We already know from case a) that

$$|E| = \frac{\ell+t}{2} \quad \text{for all } E \in M(\mathcal{A}) \text{ with } s^+(E) = \ell.$$

We repeat the steps described in case b) and observe that instead of (3.16), under the assumption (3.20), a slightly weaker inequality $\mathcal{V}(f') \geq |\mathcal{A}|$ holds. Lemma 6 is proved.

We need the following "Comparison Lemma". It makes it possible to link the Theorem with Theorem AK via its Corollary in Section 4.

Let $S \subset 2^{2^{[m]}}$, i.e. $\mathcal{L} \in S \to \mathcal{L} \subset 2^{[m]}$.

For given $t \in \mathbb{N}$, $t \leq m$ and $\beta_t, \beta_{t+1}, \dots, \beta_m \in \mathbb{R}^+$ we consider

$$h(S, \beta_t, \dots, \beta_m) = \max_{\mathcal{L} \in S} \sum_{i=t}^{m} |\mathcal{L}_i| \cdot \beta_i, \tag{3.21}$$

where $\mathcal{L}_i = \mathcal{L} \cap \binom{[m]}{i}$.

Suppose there is an $\mathcal{L}^* \in S$ so that for some $r \in \mathbb{N}$

$$\mathcal{L}_i^* = \varnothing \text{ if } t \leq i < t+r \text{ and } |\mathcal{L}_i^*| \geq |\mathcal{L}_i| \tag{*}$$

for all $t+r \leq i \leq m$ and all $\mathcal{L} \in S$.

Lemma 7. *Let* $\mathcal{L} \subset 2^{2^{[m]}}$, $\beta_t, \beta_{t+1}, \ldots, \beta_m \in \mathbb{R}^+$ *and let* $h(S, \beta_t, \ldots, \beta_m)$ *be assumed at* $\mathcal{L}^* \in S$ *, which is described just above. Then for any* $\gamma_t, \ldots, \gamma_m \in \mathbb{R}^+$ *such that*

$$\frac{\beta_i}{\beta_{i+1}} \geq \frac{\gamma_i}{\gamma_{i+1}} \qquad i = t, \ldots, m-1 \tag{3.22}$$

still $h(S, \gamma_t, \ldots, \gamma_m)$ *is assumed at* $\mathcal{L}^*$ *, i.e.*

$$h(S, \gamma_t, \ldots, \gamma_m) = \max_{\mathcal{L} \in S} \sum_{i=t}^{m} |\mathcal{L}_i| \cdot \gamma_i = \sum_{i=t}^{m} |\mathcal{L}_i^*| \cdot \gamma_i.$$

Proof: Let $\gamma_t, \ldots, \gamma_m \in \mathbb{R}^+$, such that $\frac{\beta_i}{\beta_{i+1}} \geq \frac{\gamma_i}{\gamma_{i+1}}$ for $i = t, \ldots, m-1$ and let us prove that

$$\sum_{i=t}^{m} |\mathcal{L}_i^*| \cdot \gamma_i \geq \sum_{i=t}^{m} |\mathcal{L}_i| \gamma_i \tag{3.23}$$

for every $\mathcal{L} \in S$.

From the condition on $\mathcal{L}^*$ in the Lemma we know that

$$\sum_{i=t}^{m} |\mathcal{L}_i^*| \beta_i \geq \sum_{i=t}^{m} |\mathcal{L}_i| \cdot \beta_i. \tag{3.24}$$

Without loss of generality we can assume $\beta_m = \gamma_m = 1$. We write the numbers $\beta_t, \ldots, \beta_m, \gamma_t, \ldots, \gamma_m$ in the form:

$$\begin{array}{ll}
\beta_m = 1 & \gamma_m = 1 \\
\beta_{m-1} = \delta_{m-1} & \gamma_{m-1} = \varepsilon_{m-1} \\
\beta_{m-2} = \delta_{m-1} \cdot \delta_{m-2} & \gamma_{m-2} = \varepsilon_{m-1} \varepsilon_{m-2} \\
\vdots & \vdots \\
\beta_i = \delta_{m-1} \cdot \delta_{m-2} \ldots \delta_i & \gamma_i = \varepsilon_{m-1} \varepsilon_{m-2} \ldots \varepsilon_i \\
\ldots & \ldots \\
\beta_t = \delta_{m-1} \delta_{m-2} \ldots \delta_t & \gamma_t = \varepsilon_{m-1} \varepsilon_{m-2} \ldots \varepsilon_t.
\end{array}$$

Then the condition (3.22) is equivalent to the inequality

$$\delta_i \geq \varepsilon_i \qquad i = t, \ldots, m-1. \tag{3.25}$$

Let $\ell \in \mathbb{N}$ be the smallest integer for which $\delta_{m-1} = \varepsilon_{m-1}, \delta_{m-2} = \varepsilon_{m-2}, \ldots, \delta_{m-\ell+1} = \varepsilon_{m-\ell+1}$, but $\delta_{m-\ell} > \varepsilon_{m-\ell}$.

Now we consider $\beta'_t, \ldots, \beta'_m$, where $\beta'_m = \beta_m = 1, \ldots, \beta'_{m-\ell+1} = \beta_{m-\ell+1}$ and $\beta'_i = \beta_i \cdot \frac{\varepsilon_{m-\ell}}{\delta_{m-\ell}}$, $t \leq i \leq m - \ell$.

Let us show, that

$$\sum_{i=t}^{m} |\mathcal{L}_i^*| \beta'_i \geq \sum_{i=t}^{m} |\mathcal{L}_i| \cdot \beta'_i. \tag{3.26}$$

If $m-\ell+1\le t+r$, then the condition (*) and (3.24) imply

$$\sum_{i=t}^{m}|\mathcal{L}_i^*|\cdot\beta_i'=\sum_{i=t}^{m}|\mathcal{L}_i^*|\cdot\beta_i\ge\sum_{i=t}^{m}|\mathcal{L}_i|\cdot\beta_i\ge\sum_{i=t}^{m}|\mathcal{L}_i|\cdot\beta_i'.$$

If $m-\ell+1>t+r$, then equivalent to (3.26) is

$$\sum_{i=m-\ell+1}^{m}|\mathcal{L}_i^*|\cdot\beta_i+\sum_{i=t+r}^{m-\ell}|\mathcal{L}_i^*|\cdot\frac{\beta_i\varepsilon_{m-\ell}}{\delta_{m-\ell}}\ge\sum_{i=m-\ell+1}^{m}|\mathcal{L}_i|\cdot\beta_i+\sum_{i=t}^{m-\ell}|\mathcal{L}_i|\cdot\frac{\beta_i\varepsilon_{m-\ell}}{\delta_{m-\ell}}$$

which in turn is equivalent to

$$(\delta_{m-\ell}-\varepsilon_{m-\ell})\cdot\sum_{i=m-\ell+1}^{m}|\mathcal{L}_i^*|\cdot\beta_i+\varepsilon_{m-\ell}\cdot\sum_{i=t+r}^{m}|\mathcal{L}_i^*|\cdot\beta_i\ge(\delta_{m-\ell}-\varepsilon_{m-\ell})\cdot\sum_{i=m-\ell+1}^{m}|\mathcal{L}_i|\cdot\beta_i+\varepsilon_{m-\ell}\sum_{i=t}^{m}|\mathcal{L}_i|\cdot\beta_i$$

and finally to

$$(\delta_{m-\ell}-\varepsilon_{m-\ell})\cdot\sum_{i=m-\ell+1}^{m}(|\mathcal{L}_i^*|-|\mathcal{L}_i|)\beta_i+\varepsilon_{m-\ell}\left(\sum_{i=t}^{m}|\mathcal{L}_i^*|\cdot\beta_i-\sum_{i=t}^{m}|\mathcal{L}_i|\cdot\beta_i\right)\ge 0.$$

This is true, because $\delta_{m-\ell}>\varepsilon_{m-\ell}$, $|\mathcal{L}_i^*|\ge|\mathcal{L}_i|$ for $i\ge m-\ell+1>t+r$ (see condition (*) and (3.24)).

It is easily seen that, continuing this transformation, we will arrive at the coefficients $\gamma_t,\dots,\gamma_m$ and (3.23) holds.

Remark: Lemma 7 can be formulated for much more general structures. For instance, instead of $2^{2^{[m]}}$ one can take $R_+^m=\big\{L=(\ell_1,\dots,\ell_m):\ell_i\ge 0\big\}$, choose suitable $S\subset R_+^m$, $L^*\in S$ and the claim of Lemma 7 still holds.

4. Proof of the Theorem

At first let us recall the sets

$$\mathcal{F}_r=\left\{F\in\binom{[n]}{k}:|F\cap[1,t+2r]|\ge t+r\right\}\quad\text{and}$$

$$\mathcal{K}_r=\big\{A\in\mathcal{X}_\alpha^n:|B(A)\cap[1,t+2r]|\ge t+r\big\},$$

where $B(A)\in 2^{[n]}$ is the set–theoretical image of A .

Let us also define the set

$$\mathcal{D}(r,t)=\big\{D\in 2^{[t+2r]}:|D|\ge t+r\big\}\ \text{ and let }\ \mathcal{D}_i=\mathcal{D}\cap\binom{[t+2r]}{i}.$$

We note, that $|\mathcal{D}_i| = 0$, if $i < t + r$, and $|\mathcal{D}_i| = \binom{t+2r}{i}$, if $i \geq t + r$.
Clearly $\mathcal{D}(r,t) \in I(t+2r,t)$. We can write the cardinalities of $\mathcal{F}_r$ and $\mathcal{K}_r$ as follows

$$|\mathcal{F}_r| = \sum_{j=0}^{r} \binom{2r+t}{t+r+j} \cdot \binom{n-2r-t}{k-t-r-j} = \sum_{i=0}^{t+2r} |\mathcal{D}_i| \cdot \binom{n-2r-t}{k-i} \quad \text{and}$$

$$|\mathcal{K}_r| = \sum_{j=0}^{r} \binom{2r+t}{t+r+j} \cdot (\alpha-1)^{r-j} \cdot \alpha^{n-2r-t} = \alpha^{n-2r-t} \cdot \sum_{i=0}^{t+2r} |\mathcal{D}_i| \cdot (\alpha-1)^{2r+t-i}.$$

Now we present an easy but important consequence of Theorem AK stated in the Introduction.

Corollary. *Let*

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) < m < (k-t+1)\left(2+\frac{t-1}{r}\right), r \in \mathbb{N} \tag{4.1}$$

and let

$$\gamma_i = \binom{m-2r-t}{k-i} \quad \textit{for} \quad i \geq t. \tag{4.2}$$

Then

$$\max_{\mathcal{A} \in I(2r+t,t)} \sum_{i=t}^{2r+t} |\mathcal{A}_i| \cdot \gamma_i, \quad \textit{where} \quad \mathcal{A}_i = \mathcal{A} \cap \binom{[2r+t]}{i},$$

is assumed at $\mathcal{A} = \mathcal{D}(r,t)$ *.*

Now we are ready to prove our main result.

Proof of the Theorem:

Let $r \in \{0\} \cup \mathbb{N}$ be the biggest integer so that

$$t + 2r < \min\left\{n+1, t + 2\frac{t-1}{\alpha-2}\right\}. \tag{4.3}$$

From Lemma 6 immediately follows that for this r

$$\begin{aligned} N_\alpha(n,t) &= \max_{\mathcal{A} \in I(2r+t,t)} \sum_{i=t}^{t+2r} |\mathcal{A}_i| \cdot (\alpha-1)^{t+2r-i} \cdot \alpha^{n-t-2r} \\ &= c \cdot \max_{\mathcal{A} \in I(2r+t,t)} \sum_{i=t}^{t+2r} |\mathcal{A}_i| \cdot (\alpha-1)^{t+2r-i}, \end{aligned} \tag{4.4}$$

where $c = \alpha^{n-t-2r}$ is a constant. We note that in the case $n \leq t + 2\frac{t-1}{\alpha-2}$ we have $c = 1$ or $c = \alpha$.

Now we are going to apply Lemma 7 with respect to $m = t + 2r$, $S = I(t + 2r, t) \subset 2^{2^{[2t+r]}}$, $\gamma_i = \binom{m_0-2r-t}{k-i}$, $i = t, \ldots, t+2r$, where m_0 is an integer from the interval in (4.1), $\delta_i = (\alpha-1)^{t+2r-i}$ (see (4.4)). As a set $\mathcal{L}^*$ in Lemma 7 we take $\mathcal{L}^* = \mathcal{D}(r, t) \in I(t+2r, t)$, since $\mathcal{D}(r, t)$ has the properties (*) and according to the corollary satisfies the condition in Lemma 7 on the set $\mathcal{L}^*$.

To apply Lemma 7 it remains to show the existence of a suitable m_0 from the interval in (4.1) for which the condition (3.22) in Lemma 7 holds:

$$\frac{\gamma_i}{\gamma_{i+1}} \geq (\alpha - 1) = \frac{\delta_i}{\delta_{i+1}}, i = t, \ldots, t + 2r - 1. \tag{4.5}$$

For this, necessarily we must have

$$k \geq t + 2r, \tag{4.6}$$

and

$$m_0 \geq \alpha(k - t) + 2r + t - 1 \quad \text{(this follows from (4.5))}. \tag{4.7}$$

Hence, to apply Lemma 7 it remains to show the existence of an integer $k \in \mathbb{N}$ for which the system of inequalities

$$\begin{cases} (k-t+1)\left(2+\frac{t-1}{r+1}\right) < m_0 < (k-t+1)\left(2+\frac{t-1}{r}\right) \\ \alpha(k-t)+2r+t-1 \leq m_o \end{cases} \tag{4.8}$$

is solvable for $m_0 \in \mathbb{N}$ provided that the conditions

$$k \geq t + 2r \quad \text{and} \quad r < \frac{t-1}{\alpha - 2} \tag{4.9}$$

hold (see (4.6) and (4.3)).

The system (4.8) is equivalent to

$$\begin{cases} \dfrac{rm_0}{2r+t-1} + t - 1 < k < \dfrac{m_0(r+1)}{2r+t+1} + t - 1 \\ \qquad\qquad\qquad\qquad k \leq \dfrac{m_0}{\alpha} - \dfrac{2r+t-1}{\alpha} + t. \end{cases} \tag{4.10}$$

To guarantee the existence of a $k \in \mathbb{N}$ for the first inequality, it is sufficient to take m_0 so big that

$$\frac{rm_0}{2r+t-1} + t - 1 < \frac{m_0(r+1)}{2r+t+1} + t - 2,$$

which gives

$$m_0 > \frac{(2r+t+1)(2r+t-1)}{t-1}. \tag{4.11}$$

Now we consider the inequality

$$\frac{rm_0}{2r+t-1}+t-1<\frac{m_0}{\alpha}-\frac{2r+t-1}{\alpha}+t-1$$

or equivalently

$$(2r+t-1)^2<m_0(2r+t-1-\alpha r).$$

Since $r<\frac{t-1}{\alpha-2}$ (see condition (4.9)) and consequently $2r+t-1-\beta r>0$, we have

$$m_0>\frac{(2r+t-1)^2}{2r+t-1-\alpha r}. \tag{4.12}$$

Next we consider the inequality

$$2r+t<\frac{rm_0}{2r+t-1}+t-1,$$

which gives

$$m_0>\frac{(2r+1)(2r+t-1)}{r}. \tag{4.13}$$

Finally we take $m_0\in\mathbb{N}$ so big, that m_0 satisfies (4.11) – (4.13), and we take as $k\in\mathbb{N}$ the smallest integer such that $k>\frac{rm_0}{2r+t-1}+t-1$.

For these $m_0,k\in\mathbb{N}$ (4.10) and (4.9) hold. Consequently also (4.8) holds.

Hence $\frac{\gamma_i}{\gamma_i+1}\geq(\alpha-1)=\frac{\delta_i}{\delta_i+1}$ for all $i=t,\dots,t+2r-1$ and we can apply Lemma 7. This finishes the proof of the Theorem, because

$$N_\alpha(n,t)=|\mathcal{K}_r|=\alpha^{n-2r-t}\cdot\sum_{i=0}^{t+2r}|\mathcal{D}_i|\cdot(\alpha-1)^{2r+t-i},$$

where $\mathcal{D}_i=\mathcal{D}(t,r)\cap\binom{[t+2r]}{i}$.

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