The complete intersection theorem for systems of finite sets

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1 Historical background and the new theorem

We are concerned here with one of the oldest problems in combinatorial extremal theory.

It is readily described after we have made a few conventions. $\mathbb{N}$ denotes the set of positive integers and for $i, j \in \mathbb{N}$, $i < j$, the set $\{i, i+1, \dots, j\}$ is abbreviated as $[i, j]$. For $k, n \in \mathbb{N}$, $k \leq n$, we set

$$2^{[n]} = \{F : F \subset [1, n]\}, \qquad \binom{[n]}{k} = \{F \in 2^{[n]} : |F| = k\}. \tag{1.1}$$

A system of sets $\mathcal{A} \subset \binom{[n]}{k}$ is called t*–intersecting*, if

$$|A_1 \cap A_2| \geq t \ \text{ for all } \ A_1, A_2 \in \mathcal{A}, \tag{1.2}$$

and $I(n, k, t)$ denotes the set of all such systems.

The investigation of the function

$$M(n, k, t) = \max_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}|, \qquad 1 \leq t \leq k \leq n, \tag{1.3}$$

and the structure of maximal systems was initiated by Erdös, Ko, and Rado. According to [1] they proved already in the year 1938 the following theorem, which was published only in 1961 in the famous paper [2].

Theorem EKR.

For $1 \leq t \leq k$ and $n \geq n_o(k, t)$ (suitable)

$$M(n, k, t) = \binom{n-t}{k-t}. \tag{1.4}$$

Clearly, the system

$$\mathcal{A}(n, k, t) = \left\{ A \in \binom{[n]}{k} : [1, t] \subset A \right\} \tag{1.5}$$

has cardinality $\binom{n-t}{k-t}$ and is therefore optimal for $n \geq n_o(k, t)$.

The smallest $n_o(k, t)$ for which this is the case has been determined by Frankl [3] for $t \geq 15$ and subsequently by Wilson [4] for all t:

$$n_o(k, t) = (k - t + 1)(t + 1). \tag{1.6}$$

Moreover, for $n > (k - t + 1)(t + 1)$ there is — up to obvious permutations on the ground set $[1, n]$ — only one optimal system.In the present paper we settle all the remaining cases

$$n < (k - t + 1)(t + 1).$$

In particular we prove the so called $4m$–*Conjecture* (Erdös, Ko, Rado 1938, see also [9] and the survey [8])

$$M(4m,2m,2) = \left| \left\{ F \in \binom{[4m]}{2m} : F \cap [1,2m] \geq m+1 \right\} \right| . \tag{1.7}$$

Thus obviously

$$M(4m,2m,2) = \frac{1}{2} \left(\binom{4m}{2m} - \binom{2m}{m}^2 \right) . \tag{1.8}$$

The previously best upper bound on $M(4m,2m,2)$ is due to Calderbank and Frankl [5].

There is a natural extension of the $4m$–Conjecture in terms of the systems

$$\mathcal{F}_i = \left\{ F \in \binom{[n]}{k} : |F \cap [1, t+2i]| \geq t+i \right\} \quad \text{for} \quad 0 \leq i \leq \frac{n-t}{2} \tag{1.9}$$

to all possible parameters.

General Conjecture (Frankl 1978 [3]).

For $1 \leq t \leq k \leq n$

$$M(n,k,t) = \max_{0 \leq i \leq \frac{n-t}{2}} |\mathcal{F}_i|. \tag{1.10}$$

Notice that for $n = 4m$, $k = 2m$, $t = 2$ the maximum is assumed for $i = m-1$ and so the $4m$–Conjecture is covered. For $n \geq (k-t+1)(t+1)$ the maximum is assumed for $i = 0$. A further step towards proving the General Conjecture was taken in [3] for $t \geq 15$, where the cases $0.8(k-t+1)(t+1) < n < (k-t+1)(t+1)$ are settled. Here $\mathcal{F}_1$ is (up to permutations) the only optimal system. Some other cases have been settled in [6].

Our main result establishes the validity of the General Conjecture and provides an even more specific answer concerning uniqueness.

Theorem.

For $1 \leq t \leq k \leq n$ with

(i) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right)$ *for some* $r \in \mathbb{N} \cup \{0\}$
we have

$$M(n,k,t) = |\mathcal{F}_r|$$

and $\mathcal{F}_r$ is — up to permutations — the unique optimum.
(By convention $\frac{t-1}{r} = \infty$ for $r = 0$.)

(ii) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) = n$ *for* $r \in \mathbb{N} \cup \{0\}$
we have

$$M(n,k,t) = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$$

and an optimal system equals — up to permutations — either $\mathcal{F}_r$ or $\mathcal{F}_{r+1}$.

Convention:

In the sequel we write $\mathcal{A}_1 \equiv \mathcal{A}_2$, if the systems $\mathcal{A}_1, \mathcal{A}_2$ are equal up to permutations.

Remarks:

1. It suffices to treat the cases
$$n > 2k - t, \tag{1.11}$$
because for $n \leq 2k - t$ the whole system $\binom{[n]}{k}$ is t–intersecting.

2. Our method of proof follows ideas in our work in Number Theory (to appear in Acta Arithmetica). They led to the concept of generating sets in Section 2 and related pushing techniques in later sections, which go considerably beyond known techniques in Combinatorial Extremal Theory (see [7]).

3. The t–intersecting systems in $I(n,k,t)$ can be understood as systems with a *diameter* less than $2k - 2t$ in the Hamming distance. Instead of Intersection Theorems one can then speak of Diametric Theorems — a concept of geometrical meaning. This and relations to Isoperimetric Theorems have been discussed in [10].

4. Erdös mentions in [9] the $4m$–Conjecture as the last open problem from [2].

2 Generating sets and their properties

We begin with well–known notions.

Definition 2.1:

For $A_1 = \{i_1, i_2, \dots, i_s\} \in \binom{[n]}{s}, i_1 < i_2 < \dots < i_s$, and $A_2 = \{j_1, j_2, \dots, j_s\} \in \binom{[n]}{s}, j_1 < j_2 < \dots < j_s$, we write
$$A_1 \prec A_2 \ \text{ iff } \ i_\ell \leq j_\ell \ \text{ for all } \ 1 \leq \ell \leq s,$$
that is, A_1 can be obtained from A_2 by *left–pushing*. Furthermore let $\mathcal{L}(A_2)$ be the set of all sets obtained this way from A_2. Also set $\mathcal{L}(\mathcal{A}) = \bigcup\limits_{A \in \mathcal{A}} \mathcal{L}(A)$.

Definition 2.2:

$\mathcal{A} \subset 2^{[n]}$ is said to be *left compressed* iff $\mathcal{A} = \mathcal{L}(\mathcal{A})$.

Definition 2.3:

We denote by $LI(n,k,t) \subset I(n,k,t)$ the set of all left compressed systems from $I(n,k,t)$.

It is well–known and easily follows with the shifting technique of [2] that
$$M(n,k,t) = \max_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}| = \max_{\mathcal{A} \in LI(n,k,t)} |\mathcal{A}|. \tag{2.1}$$

Definition 2.4:

For any $B \in 2^{[n]}$ we define the upset $\mathcal{U}(B) = \{B' \in 2^{[n]} : B \subset B'\}$. More generally, for $\mathcal{B} \subset 2^{[n]}$ we define the upset

$$\mathcal{U}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{U}(B).$$

Now we introduce new concepts.

Definition 2.5:

For any $\mathcal{B} \subset \binom{[n]}{k}$ a set $g(\mathcal{B}) \subset 2^{[n]}$ is called a *generating set of* $\mathcal{B}$, if $\mathcal{U}\big(g(\mathcal{B})\big) \cap \binom{[n]}{k} = \mathcal{B}$. Furthermore, $G(\mathcal{B})$ is the set of all generating sets of $\mathcal{B}$. $\big(G(\mathcal{B}) \neq \varnothing$, because $\mathcal{B} \in G(\mathcal{B})\big)$.

A first auxiliary result is readily established.

Lemma 1.

For $\mathcal{A} \in I(n,k,t)$, $n > 2k - t$,

$$|E_1 \cap E_2| \geq t \quad \textit{for} \quad E_1, E_2 \in g(\mathcal{A}) \in G(\mathcal{A}).$$

Proof: Since $2k - t < n$, $|E_1 \cap E_2| < t$ would allow k–element extensions of E_1, E_2 which are not t–intersecting.

Next we introduce further basic concepts.

Definition 2.6:

For $B = \{b_1, b_2, \dots, b_{|B|}\} \subset [1,n], b_1 < b_2 < \dots < b_{|B|}$, write the biggest element $b_{|B|}$ as $s^+(B)$. Also for $\mathcal{B} \subset 2^{[n]}$ set

$$s^+(\mathcal{B}) = \max_{B \in \mathcal{B}} s^+(B).$$

Definition 2.7:

Let $\mathcal{A} \subset \binom{[n]}{k}$ be left compressed, i.e. $\mathcal{A} = \mathcal{L}(\mathcal{A})$. For any generating set $g(\mathcal{A}) \in G(\mathcal{A})$ consider $\mathcal{L}\big(g(\mathcal{A})\big)$ and introduce its set of minimal (in the sense of set–theoretical inclusion) elements $\mathcal{L}_*\big(g(\mathcal{A})\big)$.

Furthermore, define $G_*(\mathcal{A}) = \big\{g(\mathcal{A}) \in G(\mathcal{A}) : \mathcal{L}_*\big(g(\mathcal{A})\big) = g(\mathcal{A})\big\}$. (Notice that $\mathcal{A} \in G_*(\mathcal{A})$).

We continue with simple properties.

Lemma 2.

For a left compressed $\mathcal{A} \subset \binom{[n]}{k}$ and any $g(\mathcal{A}) \in G(\mathcal{A})$

(i) $\mathcal{L}_*\big(g(\mathcal{A})\big) \in G(\mathcal{A})$

(ii) $s^+\big(\mathcal{L}_*\big(g(\mathcal{A})\big) \leq s^+\big(g(\mathcal{A})\big)\big)$

(iii) *for $A \in \mathcal{L}_*\big(g(\mathcal{A})\big)$ and $B \prec A$ we have either $B \in \mathcal{L}_*\big(g(\mathcal{A})\big)$ or there exists a $B' \in \mathcal{L}_*\big(g(\mathcal{A})\big)$ with $B' \subset B$.*

The next important properties immediately follow from the definition of $G_*(\mathcal{A})$ and the left–compressedness of $\mathcal{A}$.

Lemma 3.

For a left compressed $\mathcal{A} \subset \binom{[n]}{k}$ and $g(\mathcal{A}) \in G_*(\mathcal{A})$ $\mathcal{A}$ is a disjoint union

$$\mathcal{A} = \dot{\to} \bigcup_{E \in g(\mathcal{A})} \mathcal{D}(E),$$

where

$$\mathcal{D}(E) = \left\{ B \in \binom{[n]}{k} : B = E \dot{\to} \cup B_1, B_1 \subset \left[s^+(E)+1, n\right], |B_1| = k - |E| \right\}.$$

Lemma 4.

For a left compressed $\mathcal{A} \subset \binom{[n]}{k}$ and $g(\mathcal{A}) \in G_*(\mathcal{A})$ choose $E \in g(\mathcal{A})$ such that $s^+(E) = s^+\big(g(\mathcal{A})\big)$ and consider the set of elements of $\mathcal{A}$, which are only generated by E, that is,

$$\mathcal{A}_E = \Big(\mathcal{U}(E) \smallsetminus \mathcal{U}\big(g(\mathcal{A}) \smallsetminus \{E\}\big)\Big) \cap \binom{[n]}{k}.$$

Then

$$\mathcal{A}_E = \mathcal{D}(E) \quad \textit{and} \quad |\mathcal{A}_E| = \binom{n - s^+(E)}{k - |E|}.$$

Lemma 5.

Let $\mathcal{A} \in LI(n,k,t)$, $g(\mathcal{A}) \in G_*(\mathcal{A})$, and let $E_1, E_2 \in g(\mathcal{A})$ have the properties

$$i \notin E_1 \cup E_2, \;\; j \in E_1 \cap E_2$$

for some $i, j \in [1,n]$ with $i < j$. Then

$$|E_1 \cap E_2| \geq t + 1.$$

Finally, we use the following convention.

Definition 2.8:

For $\mathcal{A} \in LI(n,k,t)$ we set

$$s_{\min}\big(G(\mathcal{A})\big) = \min_{g(\mathcal{A}) \in G(\mathcal{A})} s^+\big(g(\mathcal{A})\big).$$

3 The main two auxiliary results

Lemma 6.

Let $n > 2k-t$, $\mathcal{A} \in LI(n,k,t)$ with $|\mathcal{A}| = M(n,k,t)$, and let

$$n > \frac{(k-t+1)(t+2r+1)}{r+1} = (k-t+1)\left(2 + \frac{t-1}{r+1}\right) \tag{3.1}$$

for some $r \in \mathbb{N} \cup \{0\}$, then

$$s_{\min}\big(G(\mathcal{A})\big) \leq t + 2r. \tag{3.2}$$

Proof: We can assume, that $n \geq 2k-t+2$, because in the case $n = 2k-t+1$ we have from (3.1) $r \geq k-t+1$, and hence (3.2) trivially holds.

By Lemma 2 we have for some $g(\mathcal{A}) \in G_*(\mathcal{A})$

$$s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big).$$

Assume now that in the opposite to (3.2)

$$s^+\big(g(\mathcal{A})\big) = t + 2r + \delta \quad \text{for some} \quad \delta > 0. \tag{3.3}$$

We shall show that under the assumptions (3.1) and (3.3) there exists an $\mathcal{A}' \in I(n,k,t)$ with $|\mathcal{A}'| > |\mathcal{A}| = M(n,k,t)$, which is a contradiction.

For this we start with the partition $g(\mathcal{A}) = g_0(\mathcal{A}) \dot{\rightarrow} \cup g_1(\mathcal{A})$, where

$g_0(\mathcal{A}) = \big\{B \in g(\mathcal{A}) : s^+(B) = t + 2r + \delta\big\}$ and $g_1(\mathcal{A}) = g(\mathcal{A}) \smallsetminus g_0(\mathcal{A})$.

Obviously, for every $B_1 \in g_0(\mathcal{A})$ and $B_2 \in g_1(\mathcal{A})$ we have

$$\big|\big(B_1 \smallsetminus \{t + 2r + \delta\}\big) \cap B_2\big| \geq t. \tag{3.4}$$

The elements in $g_0(\mathcal{A})$ have an important property, which follows immediately from Lemma 5:

(P) for any $E_1, E_2 \in g_0(\mathcal{A})$ with $|E_1 \cap E_2| = t$ necessarily

$$|E_1| + |E_2| = 2t + 2r + \delta.$$

Now, we partition $g_0(\mathcal{A})$ according to the cardinalities of its members

$$g_0(\mathcal{A}) = \dot{\rightarrow} \bigcup_{t < i < t+2r+\delta} \mathcal{R}_i, \quad \mathcal{R}_i = g_0(\mathcal{A}) \cap \binom{[n]}{i}.$$

(Here we have used that in case $\mathcal{R}_t \neq \varnothing$, by Lemma 1 and left–compressedness, necessarily $\mathcal{R}_t = \big\{[1,t]\big\} = g(\mathcal{A})$ and (3.2) holds. Furthermore, if $\mathcal{R}_{t+2r+\delta} \neq \varnothing$, then $\mathcal{R}_{t+2r+\delta} = \big\{[1,t+2r+\delta]\big\} = g(\mathcal{A})$ and by optimality of $\mathcal{A}$ necessarily $r = \delta = 0$ and again (3.2) holds.)

Of course, some of the $\mathcal{R}_i$'s can be empty.

Next we omit the element $t+2r+\delta$, that is, we consider

$$\mathcal{R}'_i = \big\{E \subset [1, t+2r+\delta-1] : E \cup \{t+2r+\delta\} \in \mathcal{R}_i\big\}.$$

So $|\mathcal{R}_i| = |\mathcal{R}'_i|$ and for $E' \in \mathcal{R}'_i$ $|E'| = i-1$ $(i \in \mathbb{N})$.

From property (P) we know that for any $E'_1 \in \mathcal{R}'_i$, $E'_2 \in \mathcal{R}'_j$ with $i+j \neq 2t+2r+\delta$

$$|E'_1 \cap E'_2| \geq t. \tag{3.5}$$

We shall prove that (under conditions (3.1) and (3.3)) all $\mathcal{R}_i$-s are empty.

First we notice that the equation $|E_1|+|E_2| = 2t+2r+\delta$ for $E_1, E_2 \in g_0(\mathcal{A})$ implies that

$$|E_j| > k-(n-t-2r-\delta) \ \text{ for } \ j = 1,2,$$

because otherwise for instance $|E_1| \leq k-(n-t-2r-\delta)$ implies $|E_2| = 2t+2r+\delta-|E_1| \geq n-k+t$, which together with $|E_2| \leq k$ contradicts $n > 2k-t$.

Hence, if for all i, $i > k-(n-t-2r-\delta)$, we have $\mathcal{R}_i = \varnothing$, then

$$\mathcal{H} = \big(g(\mathcal{A}) \smallsetminus g_0(\mathcal{A})\big) \cup \left(\bigcup_{t<i<t+2r+\delta} \mathcal{R}'_i\right) \in I(n,t),$$

the set of all (unrestricted) t–intersecting systems in $2^{[n]}$, $|\mathcal{U}(\mathcal{H}) \cap \binom{[n]}{k}| \geq |\mathcal{A}|$, and $s^+(\mathcal{H}) < s^+\big(g(\mathcal{A})\big)$, which is a contradiction.

Suppose then that for some i, $k-(n-t-2r-\delta) < i < t+2r+\delta$ $\mathcal{R}_i \neq \varnothing$ or, equivalently, $\mathcal{R}'_i \neq \varnothing$. We distinguish two cases:

a) $i \neq \frac{2t+2r+\delta}{2}$ and b) $i = \frac{2t+2r+\delta}{2}$

Case a): We consider the sets

$$\begin{aligned} f_1 &= g_1(\mathcal{A}) \cup \big(g_0(\mathcal{A}) \smallsetminus (\mathcal{R}_i \cup \mathcal{R}_{2t+2r+\delta-i})\big) \cup \mathcal{R}'_i \\ f_2 &= g_1(\mathcal{A}) \cup \big(g_0(\mathcal{A}) \smallsetminus (\mathcal{R}_i \cup \mathcal{R}_{2t+2r+\delta-i})\big) \cup \mathcal{R}'_{2t+2r+\delta-i}. \end{aligned}$$

We know already (see property (P) and (3.5)) that

$$f_1, f_2 \in I(n,t)$$

and hence

$$\mathcal{B}_i = \mathcal{U}(f_i) \cap \binom{[n]}{k} \in I(n,k,t) \ \text{ for } \ i = 1,2.$$

The desired contradiction shall take the form

$$\max_{i=1,2} |\mathcal{B}_i| > |\mathcal{A}|. \tag{3.6}$$

We consider the set $\mathcal{A} \setminus \mathcal{B}_1$.

From the construction of f_1 and $\mathcal{R}_i'$ it follows that $\mathcal{A} \setminus \mathcal{B}_1$ consists of those elements of $\binom{[n]}{k}$, which are extensions only of the elements from $\mathcal{R}_{2t+2r+\delta-i}$. We determine their number.

$$|\mathcal{A} \setminus \mathcal{B}_1| = \left| \big(\mathcal{U}(\mathcal{R}_{2t+2r+\delta-i}) \setminus \mathcal{U}\big(g(\mathcal{A}) \setminus \mathcal{R}_{2t+2r+\delta-i}\big)\big) \cap \binom{[n]}{k} \right|$$

and by Lemma 4

$$|\mathcal{A} \setminus \mathcal{B}_1| = |\mathcal{R}_{2t+2r+\delta-i}| \cdot \binom{n-t-2r-\delta}{k-2t-2r-\delta+i}. \tag{3.7}$$

Symmetrically, we consider the set $\mathcal{B}_1 \setminus \mathcal{A}$.

Let B_1 be any element of $\mathcal{R}_i'$, so $|B_1| = i-1$. Clearly, $B_1 \notin g(\mathcal{A})$, because

$$B_1 \cup \{t+2r+\delta\} \in g(\mathcal{A}) \quad \text{and} \quad g(\mathcal{A}) \in G_*(\mathcal{A}).$$

Therefore for every $A \in \binom{[n]}{k}$ in the form $A = B_1 \cup B_2$, where $|B_2| = k-i+1$ and $B_2 \subset [t+2r+\delta+1, n]$, we have $A \in \mathcal{B}_1$ and $A \notin \mathcal{A}$, because $s^+\big(g(\mathcal{A})\big) = t+2r+\delta$.

We also notice that for $B_1, B_1' \in \mathcal{R}_i'$, $B_1 \neq B_1'$, we have

$$B_1 \cup B_2 \neq B_1' \cup B_2' \quad \text{for all} \quad B_2, B_2' \subset [t+2r+\delta+1, n].$$

Therefore

$$|\mathcal{B}_1 \setminus \mathcal{A}| \geq |\mathcal{R}_i'| \binom{n-t-2r-\delta}{k-i+1} = |\mathcal{R}_i| \binom{n-t-2r-\delta}{k-i+1}. \tag{3.8}$$

Analogously, we have

$$|\mathcal{A} \setminus \mathcal{B}_2| = |\mathcal{R}_i| \binom{n-t-2r-\delta}{k-i} \tag{3.9}$$

$$|\mathcal{B}_2 \setminus \mathcal{A}| \geq |\mathcal{R}_{2t+2r+\delta-i}| \binom{n-t-2r-\delta}{k-2t-2r-\delta+i+1}. \tag{3.10}$$

Actually, it is easy to show that there are equalities in (3.8) and in (3.10). However, that is not needed here.

Now (3.7) – (3.10) enable us to state the *negation* of (3.6) in the form

$$\left.\begin{aligned} |\mathcal{R}_i|\binom{n-t-2r-\delta}{k-i+1} &\leq |\mathcal{R}_{2t+2r+\delta-i}|\binom{n-t-2r-\delta}{k-2t-2r-\delta+i} \\ |\mathcal{R}_{2t+2r+\delta-i}|\binom{n-t-2r-\delta}{k-2t-2r-\delta+i+1} &\leq |\mathcal{R}_i|\binom{n-t-2r-\delta}{k-i} \end{aligned}\right\}. \quad \text{and} \tag{3.11}$$

Since by assumption $\mathcal{R}_i \neq \varnothing$, we can assume also $\mathcal{R}_{2t+2r+\delta-i} \neq \varnothing$, because otherwise the first inequality in (3.11) is false.

Furthermore, (3.11) implies

$$(n+t-k-i)(n-t-2r-\delta-k+i) \leq (k-i+1)(k-2t-2r-\delta+i+1).$$

However, this is false, because $n \geq 2k-t+2$ and consequently $n+t-k-i > k-i+1$ as well as $n-t-2r-\delta-k+i > k-2t-2r-\delta+i+1$.

Hence, (3.6) holds in contradiction to the optimality of $\mathcal{A}$. Therefore necessarily $\mathcal{R}_i = \varnothing$ for all $i \neq \frac{2t+2r+\delta}{2}$. (In this case a) we did not use the condition (3.1) and not even the condition $\delta > 0$).

Case b): $i = \frac{2t+2r+\delta}{2}$

Here necessarily $2 \mid \delta$. We consider the set $\mathcal{R}'_{t+r+\frac{\delta}{2}}$ and recall that for $B \in \mathcal{R}'_{t+r+\frac{\delta}{2}}$ $|B| = t+r+\frac{\delta}{2}-1$ and $B \subset [1, t+2r+\delta-1]$.

By the pigeon–hole principle there exists an $i \in [1, t+2r+\delta-1]$ and a $\mathcal{T} \subset \mathcal{R}'_{t+r+\frac{\delta}{2}}$ such that $i \notin B$ for all $B \in \mathcal{T}$ and

$$|\mathcal{T}| \geq \frac{r+\frac{\delta}{2}}{t+2r+\delta-1}|\mathcal{R}'_{t+r+\frac{\delta}{2}}| = \frac{r+\frac{\delta}{2}}{t+2r+\delta-1}|\mathcal{R}_{t+r+\frac{\delta}{2}}|. \tag{3.12}$$

By Lemma 5 we have $|B_1 \cap B_2| \geq t$ for all $B_1, B_2 \in \mathcal{T}$ and since by case a) $\mathcal{R}_i = \varnothing$ for $i \neq t+r+\frac{\delta}{2}$, we get

$$f' = \left(g(\mathcal{A}) \smallsetminus \mathcal{R}_{t+r+\frac{\delta}{2}}\right) \cup \mathcal{T} \in I(n,t).$$

We show now that under condition (3.1) we have

$$\left|\mathcal{U}(f') \cap \binom{[n]}{k}\right| > |\mathcal{A}|. \tag{3.13}$$

Indeed, let us write

$\mathcal{A} = \mathcal{U}\big(g(\mathcal{A})\big) \cap \binom{[n]}{k} = \mathcal{D}_1 \dot{\to} \cup \mathcal{D}_2$, where $\mathcal{D}_1 = \mathcal{U}\big(g(\mathcal{A}) \smallsetminus \mathcal{R}_{t+r+\frac{\delta}{2}}\big) \cap \binom{[n]}{k}$,

$\mathcal{D}_2 = \big(\mathcal{U}(\mathcal{R}_{t+r+\frac{\delta}{2}}) \smallsetminus \mathcal{U}\big(g(\mathcal{A}) \smallsetminus \mathcal{R}_{t+r+\frac{\delta}{2}}\big)\big) \cap \binom{[n]}{k}$

and

$\mathcal{U}(f') \cap \binom{[n]}{k} = \mathcal{D}_1 \dot{\to} \cup \mathcal{D}_3$, where $\mathcal{D}_3 = \big(\mathcal{U}(\mathcal{T}) \smallsetminus \mathcal{U}\big(g(\mathcal{A}) \smallsetminus \mathcal{R}_{t+r+\frac{\delta}{2}}\big)\big) \cap \binom{[n]}{k}$.

In this terminology, equivalent to (3.13) is

$$|\mathcal{D}_3| > |\mathcal{D}_2|. \tag{3.14}$$

We know (see Lemma 4) that

$$|\mathcal{D}_2| = |\mathcal{R}_{t+r+\frac{\delta}{2}}| \binom{n-t-2r-\delta}{k-t-r-\frac{\delta}{2}} \tag{3.15}$$

and lower estimate now $|\mathcal{D}_3|$.

Consider any $B \in \mathcal{T}$, $|B| = t + r + \frac{\delta}{2} - 1$. Clearly, $B \notin g(\mathcal{A})$, because

$$B \cup \{t + 2r + \delta\} \in g(\mathcal{A}) \ \text{ and } \ g(\mathcal{A}) \in G_*(\mathcal{A}).$$

Hence, for every $A \in \binom{[n]}{k}$ in the form $A = B \cup C_1$, where $|C_1| = k - |B|$ and $C_1 \subset [t + 2r + \delta, n]$, we have $A \in \mathcal{D}_3$.

We also notice that for all $B_1, B_2 \in \mathcal{T}$, $B_1 \neq B_2$, one has $B_1 \cup C_1 \neq B_2 \cup C_2$ for all $C_1, C_2 \subset [t + 2r + \delta, n]$.

Therefore

$$|\mathcal{D}_3| \geq |\mathcal{T}| \binom{n-t-2r-\delta+1}{k-t-r-\frac{\delta}{2}+1}. \tag{3.16}$$

In the light of (3.12), (3.14) – (3.16) sufficient for (3.13) is

$$\frac{r+\frac{\delta}{2}}{t+2r+\delta-1} \binom{n-t-2r-\delta+1}{k-t-r-\frac{\delta}{2}+1} > \binom{n-t-2r-\delta}{k-t-r-\frac{\delta}{2}} \tag{3.17}$$

or, equivalently, that

$$\left(r+\frac{\delta}{2}\right)(n-t-2r-\delta+1) > \left(k-t-r-\frac{\delta}{2}+1\right)(t+2r+\delta-1) \tag{3.18}$$

which in turn is equivalent to

$$n > \frac{(k-t+1)(t+2r+\delta-1)}{r+\frac{\delta}{2}}.$$

This is true by (3.1), because δ is even and

$$\frac{t+2r+1}{r+1} \geq \frac{t+2r+\delta-1}{r+\frac{\delta}{2}} \quad \text{for} \quad \delta \geq 2.$$

Hence $\delta = 0$ and the Lemma is proved.

Corollary. *(For the* $4m$*–Conjecture)*

Let $\mathcal{A} \in LI(4m, 2m, 2)$ *and* $|\mathcal{A}| = M(4m, 2m, 2)$*, then*

$$s_{\min}\big(G(\mathcal{A})\big) \leq 2m.$$

Proof:

Just choose $r = m - 1$ and notice that (3.1) in Lemma 6 holds:

$$4m > \frac{(2m-1)(2m+1)}{m}.$$

Finally, we present the second auxiliary result.

Lemma 7.

For any (not necessarily maximal) $\mathcal{A} \in I(n,k,t)$ *consider any generating set* $g(\mathcal{A}) \in G(\mathcal{A})$*. Further, for the complemented system* $\overline{\mathcal{A}} \in I(n, n-k, n-2k+t)$ *let* $f(\overline{\mathcal{A}})$ *be any generating set from* $G(\overline{\mathcal{A}})$*.*

Then for all $A \in g(\mathcal{A})$ and $B \in f(\overline{\mathcal{A}})$ we have

$$|A \cup B| \geq n - k + t. \tag{3.19}$$

Proof:

Assume that (3.19) does not hold:

$$|A \cup B| \leq n - k + t - 1.$$

Choose any $F \supset A \cup B$ with $|F| = n - k + t - 1$.

We know that

$$\mathcal{U}(A) \cap \binom{[n]}{k} \subset \mathcal{A} \quad \text{and} \quad \mathcal{U}(B) \cap \binom{[n]}{n-k} \subset \overline{\mathcal{A}}.$$

Since $n > 2k - t, t > 0$, and $n-k+t-1 \geq \max(k, n-k)$ we can choose $A^* \in \mathcal{U}(A) \cap \binom{[n]}{k}$ and $B^* \in \mathcal{U}(B) \cap \binom{[n]}{n-k}$ in such a way that $A^* \subset F$ and $B^* \subset F$.

Consider $\overline{B^*} \in \mathcal{A}$ and observe that $|A^* \cap \overline{B^*}| \leq (n-k+t-1) - (n-k) = t-1$, contradicting $\mathcal{A} \in I(n,k,t)$.

4 Proof of the 4m–Conjecture

We treat this famous case here separately, eventhough it is covered by the proof of the Theorem.

Let $\mathcal{A} \in LI(4m, 2m, 2)$, let $|\mathcal{A}| = M(4m, 2m, 2)$, and let $g(\mathcal{A}) \in G(\mathcal{A})$ such that $s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big)$.

From the Corollary we know that $s^+\big(g(\mathcal{A})\big) \leq 2m$, that is, for $B \in g(\mathcal{A})$ necessarily $B \subset [1, 2m]$. We consider the complemented $\mathcal{A}$:

$$\overline{\mathcal{A}} = \big\{A \subset [1, 4m] : [1, 4m] \smallsetminus A \in \mathcal{A}\big\}.$$

Clearly, $\overline{\mathcal{A}} \in I(4m, 2m, 2)$, $|\overline{\mathcal{A}}| = M(4m, 2m, 2)$, $\mathcal{A} \cap \overline{\mathcal{A}} = \varnothing$, and $\overline{\mathcal{A}}$ is right–compressed.

From the left–right symmetry and the Corollary we conclude that there exists a generating set $f(\overline{\mathcal{A}})$ such that for every $B \in f(\overline{\mathcal{A}})$ necessarily $B \subset [2m + 1, 4m]$.

Now, if for all $B_1 \in g(\mathcal{A})$ we have $|B_1| \geq m + 1$, then necessarily $\mathcal{A} = \mathcal{F}_{m-1}$ and $|\overline{\mathcal{A}}| = |\mathcal{A}| = |\mathcal{F}_{m-1}|$.

The same conclusion holds, if for all $B_2 \in f(\overline{\mathcal{A}})$ we have $|B_2| \geq m+1$. However, if none of these two cases occurs, then for some $B_1 \in g(\mathcal{A})$ and $B_2 \in f(\overline{\mathcal{A}})$ we have $|B_1|, |B_2| \leq m$.

Finally, for every $A \in \binom{[4m]}{2m}$ of the form $A \supset B_1 \cup B_2$ necessarily $A \in \mathcal{A} \cap \overline{\mathcal{A}}$, a contradiction to $\mathcal{A} \cap \overline{\mathcal{A}} = \varnothing$.

Remark 5:

Instead of the last step of the proof one can argue that $|B_1 \cup B_2| \leq 2m < 2m+2 = n-k+t$ is in contradiction to Lemma 7.

5 Proof of the Theorem

Case (i):

$$(k-t+1)\left(2 + \frac{t-1}{r+1}\right) < n < (k-t+1)\left(2 + \frac{t-1}{r}\right). \tag{5.1}$$

Choose $\mathcal{A} \in LI(n, k, t)$ with $|\mathcal{A}| = M(n, k, t)$ and let $g(\mathcal{A}) \in G(\mathcal{A})$ with $s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big)$.

We know from Lemma 6 that $s^+\big(g(\mathcal{A})\big) \leq t + 2r$ and we also know that $\overline{\mathcal{A}}$, the complemented $\mathcal{A}$, is right compressed, element of $I(n, n-k, n-2k+t)$, and satisfies $|\overline{\mathcal{A}}| = M(n, n-k, n-2k+t)$. An easy calculation leads from (5.1) to

$$(k-t+1)\left(2 + \frac{n-2k+t-1}{k-t-r+1}\right) < n < (k-t+1)\left(2 + \frac{n-2k+t-1}{k-t-r}\right) \quad \text{for } \ k > t+r \tag{5.2}$$

and to

$$(k-t+1)(n-2k+t+1) < n \quad \text{for} \quad k = t+r. \tag{5.3}$$

Since $n > 2k-t$, (5.1) implies that always $k \geq t+r$.

Therefore, for $k' = n-k$ and $t' = n-2k+t$ we get the *dual relations*

$$(k'-t'+1)\left(2+\frac{t'-1}{r'+1}\right) < n < (k'-t'+1)\left(2+\frac{t'-1}{r'}\right) \quad \text{for} \quad r' = k-t-r$$

and

$$(k'-t'+1)(t'+1) < n \quad \text{for} \quad k = t+r.$$

Using the dual version (with respect to right compressed sets) of Lemma 6 we get an $f(\overline{\mathcal{A}}) \in G(\overline{\mathcal{A}})$ with the property:

every $B \in f(\overline{\mathcal{A}})$ satisfies $B \subset \big[n-(t'+2r')+1, n\big] = [t+2r+1, n]$, because $t' = n-2k+t$ and $r' = k-t-r$.

Now, if for all $B_1 \in g(\mathcal{A})$ we have $|B_1| \geq t+r$, then clearly $\mathcal{A} = \mathcal{F}_r$, $\overline{\mathcal{A}} \equiv \mathcal{F}_{r'}$ (for $k' = n-k$, $t' = n-2k+t$) and the optimal sets are unique up to permutations.

The same sets are obtained if for all $B_2 \in f(\overline{\mathcal{A}})$ the inequality $|B_2| \geq t'+r'$ holds.

If none of the cases occurs, then for some $B_1^* \in g(\mathcal{A})$, $B_2^* \in f(\overline{\mathcal{A}})$ $|B_1^*| \leq t+r-1$ and $|B_2^*| \leq t'+r'-1 = n-k-r-1$ and by Lemma 7 we will have a contradiction, because $|B_1^* \cup B_2^*| \leq n-k+t-2$.

Case (ii):

$$n = (k-t+1)\left(2+\frac{t-1}{r+1}\right). \tag{5.4}$$

Let $\mathcal{A} \in LI(n,k,t)$ and $|\mathcal{A}| = M(n,k,t)$. Since $n > (k-t+1)\left(2+\frac{t-1}{r+2}\right)$, we conclude with Lemma 6 that

$$s_{\min}\big(G(\mathcal{A})\big) \leq t+2r+2.$$

Choose now $g(\mathcal{A}) \in G(\mathcal{A})$ with $s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big)$.

We consider again $\overline{\mathcal{A}}$, which is again right compressed and of maximal cardinality within $I(n, n-k, n-2k+t)$.

From (5.4) we derive

$n = (k-t+1)\left(2+\frac{t-1}{r+1}\right) = (k-t+1)\left(2+\frac{n-2k+t-1}{k-t-r}\right) = (k'-t'+1)\left(2+\frac{t'-1}{r'}\right)$, where $k' = n-k$, $t' = n-2k+t$, and $r' = k-t-r$.

Since $n = (k'-t'+1)\left(2+\frac{t'-1}{r'}\right) > (k'-t'+1)\left(2+\frac{t'-1}{r'+1}\right)$, from the dual version (with respect to right compressed sets) of Lemma 6 one has an $f(\overline{\mathcal{A}}) \in G(\overline{\mathcal{A}})$ whose elements B_1 satisfy $B_1 \subset [n-t'-2r'+1, n] = [t+2r+1, n]$.

Now, if for all $B_2 \in g(\mathcal{A})$ $|B_2| \geq t+r+1$, then uniquely $\mathcal{A} = \mathcal{F}_{r+1}$ and $\overline{\mathcal{A}} \equiv \mathcal{F}_{r'-1}$ (for $k' = n-k$, $t' = n-2k+t, r' = k-t-r$).

Also, if for all $B_1 \in f(\overline{\mathcal{A}})$ $|B_1| \geq t'+r'$, then uniquely $\overline{\mathcal{A}} \equiv \mathcal{F}_{r'}$ and $\mathcal{A} \equiv \mathcal{F}_r$. Otherwise, if for some $B_2 \in g(\mathcal{A})$ and $B_1 \in f(\overline{\mathcal{A}})$ $|B_2| \leq t+r$ and $|B_1| \leq t'+r'-1$, then $|B_1 \cup B_2| \leq t+r+t'+r'-1 = n-k+t-1$.

This again contradicts Lemma 7.

This finishes the proof, if we allow only left compressed systems as competitors. We thank K. Engel for asking for an argument for not left compressed competitors. We follow an idea of [3] to prove the uniqueness, stated in the Theorem.

We use the well–known exchange operation $S_{ij}, i < j$, defined for any family $\mathcal{A} \subset 2^{[n]}$ as follows: for $A \in \mathcal{A}$

$$S_{ij}(A) = \begin{cases} \{i\} \cup \big(A \smallsetminus \{j\}\big) & \text{if } \ i \notin A, j \in A, \{i\} \cup \big(A \smallsetminus \{j\}\big) \notin \mathcal{A} \\ A & \text{otherwise,} \end{cases}$$

$$S_{ij}(\mathcal{A}) = \big\{S_{ij}(A) : A \in \mathcal{A}\big\}.$$

Proposition:

Suppose that $\mathcal{A} \in I(n,k,t)$ and $\mathcal{A}$ gets transformed by finitely many exchange operations to the set $\mathcal{F}_r$ (see (1.9)) for some $0 \leq r \leq \frac{n-t}{2}$, then necessarily $\mathcal{A} \equiv \mathcal{F}_r$, provided that

$$\begin{aligned} n &\geq 2k-t+2 \ \text{ for } \ t \geq 2 \\ n &= 2k-t+1 \ \text{ for } \ t \geq 2 \ \text{ and } \ k = t+r \ \text{ or } \ k = t+r+1 \\ n &\geq 2k+1 \ \text{ for } \ t = 1 \ \text{ and } \ r = 0 \ \text{ or } \ r = 1. \end{aligned} \tag{$*$}$$

Proof:

Without loss of generality we can assume that

$$S_{ij}(\mathcal{A}) = \mathcal{F}_r. \tag{5.5}$$

It is clear that, if $i, j \in [1, t+2r]$ (or $i, j \notin [1, t+2r]$), then $\mathcal{A} = \mathcal{F}_r$ and the Proposition is true.

Assume then $i = t+2r$ and $j = n$. Let

$\mathcal{A}_1 = \{A \in \mathcal{A} : j \in A, i \notin A, ((A \smallsetminus \{j\}) \cup \{i\}) \notin \mathcal{A}\}$

$\mathcal{A}_2 = \{A \in \mathcal{A} : j \notin A, i \in A, ((A \smallsetminus \{i\}) \cup \{j\}) \notin \mathcal{A}\}$.

Clearly, if $\mathcal{A}_1 = \varnothing$, then $\mathcal{A} = \mathcal{F}_r$, and if $\mathcal{A}_2 = \varnothing$, then $\mathcal{A}$ is obtained from $\mathcal{F}_r$ by exchanging the coordinates $i = t + 2r$ and $j = n$, so the Proposition is true. Suppose then $\mathcal{A}_1 \neq \varnothing$, $\mathcal{A}_2 \neq \varnothing$ and let us show that $\mathcal{A} \notin I(n,k,t)$ (under conditions $(*)$).

We consider

$$\mathcal{H} = \left\{ H \in \binom{[n] \setminus \{i,j\}}{k-1} : |H \cap [1, t+2r-1]| = t + r - 1 \right\}.$$

We observe that for any $B \in \mathcal{A}_1 \dot{\to} \cup \mathcal{A}_2$

$|B \cap [1, t+2r-1]| = t + r - 1$ holds.

This fact follows from (5.5).

Moreover from the same assumption (5.5) we have:

for every $H \in \mathcal{H}$, either $H \cup \{j\} \in \mathcal{A}_1$ or $H \cup \{i\} \in \mathcal{A}_2$.

Now we form a graph $G(V,E)$ as follows:

$V = \mathcal{H}$ and $e(H_1, H_2) \in E$ iff $(H_1 \cap H_2) = t - 1$.

It can be verified, that the graph $G(V,E)$ is connected if and only if the conditions $(*)$ hold.

Hence under conditions $(*)$, if $\mathcal{A}_1 \neq \varnothing$, $\mathcal{A}_2 \neq \varnothing$, then there exist $B_1 \in \mathcal{A}_1$ and $B_2 \in \mathcal{A}_2$ with $|B_1 \cap B_2| = t - 1$, which is a contradiction to $\mathcal{A} \in I(n,k,t)$.

Proof of the uniqueness:

Let $n > 2k - t$, $\mathcal{A} \in I(n,k,t)$, $|\mathcal{A}| = M(n,k,t)$, and let after finitely many exchange operations $S_{ij}, i < j$, $\mathcal{A}$ be transformed to the left–compressed set $\mathcal{A}'$, $\mathcal{A}' \in LI(n,k,t)$, $|\mathcal{A}'| = |\mathcal{A}| = M(n,k,t)$.

We already know, that $\mathcal{A}' = \mathcal{F}_r$ for $r \in \mathbb{N} \cup \{0\}$, where r is defined by the conditions in the Theorem. It can be easily verified, that these r's satisfy conditions $(*)$ stated in the Proposition and hence $\mathcal{A} \equiv \mathcal{F}_r$.

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